Internet Appendix for "Fast and slow informed trading"

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This document includes supplementary material to the paper. Section 1 analyzes the general benchmark model $\mathcal{M}_m$ with $m \geq 0$ lags, in which speculators do not use their signals beyond lag m. Section 2 describes the particular benchmark model $\mathcal{M}_2$ with $m = 2$ lags, and also analyzes an extension in which the slowest traders are able to learn from the order flow. Section 3 describes the benchmark model $\mathcal{M}_1$ with fast and slow traders, but where certain signals about the increments of the fundamental value are made public with a delay of two periods. Section 4 verifies that the intuition of the benchmark model $\mathcal{M}_1$ in the paper extends to a setup in which the fundamental value has more than one component. In addition, Section 4 studies the decision of speculators to use "smooth" trading strategies as in Kyle (1985). Section 5 discusses the general equilibrium of the model with inventory management from Section 4 in the paper, and provides the proofs that have been left out of the paper. In addition, Section 5 extends the model with inventory management by introducing more general strategies, predictable order flow, multiple IFTs, or more links in the intermediation chain. Section 6 analyzes a partial equilibrium of the model with inventory management from Section 4 in the paper, in which the IFT chooses to trade in the "smooth regime." Section 7 considers a discrete-time version of the benchmark model $\mathcal{M}_1$ in the paper, and shows that the difference between the two models is small.

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Contents

1 Benchmark model with m lags

In this section, I analyze the benchmark model $\mathcal{M}_m$ with $m \geq 0$ lags, in which speculators do not use their signals beyond lag m. Two particular cases are already discussed in the paper: $m = 0$ (Proposition 3 in Subsection 3.2 in the paper) and $m = 1$ (Subsection 3.1 in the paper). In both these cases, the equilibrium is described in closed form.

For the case $m > 1$, I show in Subsection 1.2 of this Internet Appendix that the equilibrium reduces to system of non-linear equations in the coefficients. In Subsection 1.3, I discuss the particular case in which all speculators have the same speed: if N_ℓ is the number of ℓ-speculators, then $N_0 > 0$, and $N_1 = \cdots = N_m = 0$. The proofs are provided in Subsection 1.4.

1.1 Notation preliminaries

To simplify the presentation, I use matrix notation. All vectors are in column format, and I denote by X' the transpose of the vector X.

I normalize some variables by dividing them with the forecast variance, σ_w^2. I denote this by placing a tilde above the variable. For instance, I define the normalized instantaneous order flow variance $\tilde{\sigma}_y^2$, as well as the normalized instantaneous noise trader variance $\tilde{\sigma}_u^2$ as follows:

$$\tilde{\sigma}_{y,t}^2 \;=\; \frac{\mathsf{Var}(\mathrm{d}y_t)}{\sigma_w^2 \mathrm{d}t}, \qquad \tilde{\sigma}_u^2 \;=\; \frac{\mathsf{Var}(\mathrm{d}u_t)}{\sigma_w^2 \mathrm{d}t} \;=\; \frac{\sigma_u^2}{\sigma_w^2}, \tag{IA.1}$$

where both these ratios are limits of the corresponding discrete ratios when Δt converges to zero (see the notation preliminaries in the Appendix in the paper).

Recall that an ℓ-speculator in the model $\mathcal{M}_m$ observes the signals after $\ell = 0, 1, \ldots, m$ lags, and has a trading strategy of the form:

$$\mathrm{d}x_t \;=\; \gamma_{\ell,t}^{(\ell)}(\mathrm{d}w_{t-\ell} - z_{t-\ell,t}) \;+\; \gamma_{\ell+1,t}^{(\ell)}(\mathrm{d}w_{t-\ell-1} - z_{t-\ell-1,t}) \;+\; \cdots \;+\; \gamma_{m,t}^{(\ell)}(\mathrm{d}w_{t-m} - z_{t-m,t}), \tag{IA.2}$$

where $z_{t-j,m}$ is the dealer's expectation of the j-lagged signal $\mathrm{d}w_{t-j}$, and $t-j$ is notation for $t - j\mathrm{d}t$ (see the discussion following equation (11) in the paper).

I collect the coefficients of the trading strategy in (IA.2) in a vector:

$$\gamma^{(\ell)} \;=\; \left[\, \gamma_\ell^{(\ell)} \,,\; \ldots \,,\; \gamma_m^{(\ell)} \,\right]', \tag{IA.3}$$

which contains only the entries corresponding to $j = \ell, \ell+1, \ldots, m$. If one needs to sum $\gamma^{(\ell)}$ over different ℓ, I make a slight abuse of notation and use the notation $\gamma^{(\ell)}$ for the same vector as in (IA.3) but padded with zeros at the entries $j = 0, 1, \ldots, \ell - 1$:

$$\gamma^{(\ell)} \;=\; \left[\, 0 \,,\; \ldots \,,\; 0 \,,\; \gamma_\ell^{(\ell)} \,,\; \ldots \,,\; \gamma_m^{(\ell)} \,\right]'. \tag{IA.4}$$

In general, if A is a matrix with elements $A_{i,j}$ for $i, j = 0, \ldots, m$, I denote by $A_{\geq \ell}$ the matrix with elements $A_{i,j}$ for $i, j \geq \ell$; and similarly for the vectors $B_{\geq \ell}$ and $\rho_{\geq \ell}$. A sum of vectors $X_{\geq \ell}$ over different ℓ is carried by padding $X_{\geq \ell}$ with zeros for the first ℓ entries.

I follow the usual convention that sums from a larger index to a smaller index are equal to zero. For instance, if $m = 0$, for any variable X_i the sum from the index 1 to m is by convention equal to zero:

$$m \;=\; 0 \quad\Longrightarrow\quad \sum_{i=1}^{m} X_i \;=\; 0. \tag{IA.5}$$

Similarly, an enumeration from a larger index to a smaller index is by convention the empty set. For instance, if $m = 0$, saying that the condition $\mathcal{P}_i$ holds for $i = 1, \ldots, m$ is equivalent to imposing no condition at all.

1.2 General speed case

In this subsection, I solve for the equilibrium of the model $\mathcal{M}_m$ when the number of lags $m \geq 0$ is fixed. Under an additional assumption stated below, I show that the equilibrium reduces to the solution of a system of equations (see Theorem IA.1). This system can be solved in closed form in some particular cases of interest, and can in principle be solved numerically.

To proceed with the solution, one needs to be more specific about how the dealer sets her expectation $z_{t-i,t} = \mathsf{E}(\mathrm{d}w_{t-i} \mid \{\mathrm{d}y_\tau\}_{\tau<t})$. Since $\mathrm{d}w_{t-i}$ is the speculator's signal from i trading rounds before (corresponding to calendar time $t - i\,\mathrm{d}t$), it is plausible to expect that (i) $z_{t-i,t}$ only involves the order flow from at most i periods before, and (ii) $z_{t-i,t}$ is linear in the order flow. Thus, I assume (and show it to be true in equilibrium) that:

$$z_{t-i,t} \;=\; \rho_{0,t}\mathrm{d}y_{t-i} + \cdots + \rho_{i-1,t}\mathrm{d}y_{t-1}, \quad i = 0, 1, \ldots, m, \tag{IA.6}$$

where $\mathrm{d}y_{t-j}$ is the order flow from j trading rounds before. Define the "fresh signal"

$\mathrm{d}_t w_{t-i}$ to be the unanticipated part of the signal at t:

$$\mathrm{d}_t w_{t-i} \;=\; \mathrm{d}w_{t-i} - z_{t-i,t}. \tag{IA.7}$$

For all lags $i, j = 0, \ldots, m$, denote:

$$A_{i,j,t} \;=\; \frac{\mathsf{Cov}\big(\mathrm{d}_t w_{t-i}, \mathrm{d}_t w_{t-j}\big)}{\sigma_w^2 \,\mathrm{d}t}, \qquad B_{j,t} \;=\; \frac{\mathsf{Cov}\big(w_t, \mathrm{d}_t w_{t-j}\big)}{\sigma_w^2 \,\mathrm{d}t}. \tag{IA.8}$$

Since A measures the instantaneous covariance of fresh signals at the relevant lags, I call A the "fresh covariance matrix." The vector B measures the instantaneous contribution of each fresh signal to the profit, thus I call B the "benefit vector." In Section 2, it is assumed that the speculator takes A and B as fixed, and considers them as set by the dealer (just as $\rho_{j,t}$ and λ_t).

Theorem IA.1 shows that a linear equilibrium exists if a certain system of equations is satisfied.

Theorem IA.1. *Let $m \geq 0$ be fixed, and consider the model $\mathcal{M}_m$ with m lags, and N_ℓ speculators of type $\ell = 0, \ldots, m$. Suppose there exists a linear equilibrium of the model with constant coefficients, of the form:*

$$\begin{aligned} \mathrm{d}x_t^{(\ell)} \;&=\; \gamma_\ell^{(\ell)} \mathrm{d}_t w_{t-\ell} \;+\; \cdots \;+\; \gamma_m^{(\ell)} \mathrm{d}_t w_{t-m}, \quad \ell = 0, \ldots, m, \\ \mathrm{d}_t w_{t-i} \;&=\; \mathrm{d}w_{t-i} - z_{t-i,t}, \quad i = 0, 1, \ldots, m, \\ z_{t-i,t} \;&=\; \rho_0 \mathrm{d}y_{t-i} + \cdots + \rho_{i-1} \mathrm{d}y_{t-1}, \quad i = 0, 1, \ldots, m, \\ \mathrm{d}p_t \;&=\; \lambda \mathrm{d}y_t. \end{aligned} \tag{IA.9}$$

Then, the constants λ, ρ_i, $\gamma_i^{(\ell)}$ satisfy the following system of equations ($i = 0, \ldots, m$):

$$\begin{aligned} &\bar{\gamma} \;=\; \sum_{\ell=0}^{m} N_\ell \, \gamma^{(\ell)}, \quad \text{with} \quad \gamma^{(\ell)} \;=\; \big(A_{\geq \ell}\big)^{-1} \left(\frac{1}{\lambda} \, B_{\geq \ell} - \tilde{\sigma}_y^2 \rho_{\geq \ell} \right), \\ &\tilde{\sigma}_y^2 \rho \;=\; A \bar{\gamma}, \qquad \tilde{\sigma}_y^2 \lambda \;=\; B' \bar{\gamma}, \qquad \tilde{\sigma}_y^2 \;=\; \frac{\tilde{\sigma}_u^2}{1 - \bar{\gamma}' \rho}, \\ &A_{i,j} \;=\; \mathbf{1}_{i=j} - \tilde{\sigma}_y^2 \sum_{k=1}^{\min(i,j)} \rho_{i-k} \rho_{j-k}, \qquad B_i \;=\; 1 - \tilde{\sigma}_y^2 \lambda \sum_{k=1}^{i} \rho_{i-k}, \end{aligned} \tag{IA.10}$$

where $\mathbf{1}_{\mathcal{P}}$ is the indicator function, which equals 1 if $\mathcal{P}$ is true, and 0 otherwise.

Conversely, suppose the constants λ, ρ_i, $\gamma_i^{(\ell)}$ satisfy (IA.10), and in addition the following conditions are satisfied: (i) $\lambda > 0$; (ii) for all $\ell = 0, \ldots, m$, the matrix $A_{\geq \ell}$

is invertible; and (iii) the numbers $\beta_k = \sum_{i=0}^{m-k} \rho_i \bar{\gamma}_{k+i}$ *satisfy* $1 > \beta_1 > \cdots > \beta_m > 0$. *Then, the equations in (IA.9) provide an equilibrium of the model.*

Note that ρ_m, the last entry of the vector $\rho = \left[\rho_0 \ , \ \ldots \ , \ \rho_m \right]'$, is not part of the dealer's expectations $z_{t-i,t}$, but I introduce it in order to simplify notation. In particular, the last row of the equilibrium equation $\tilde{\sigma}_y^2 \rho = A\bar{\gamma}$ can be omitted.

In principle, the system of equations (IA.10) can be solved numerically as follows. To simplify notation, I make a change of variables and denote by $r_i = \tilde{\sigma}_y \rho_i$, $\Lambda = \tilde{\sigma}_y \lambda$, $g = \frac{\bar{\gamma}}{\tilde{\sigma}_y}$. Then, suppose I start with some values for r_i and Λ. Then, A can be expressed only in terms of r_i, and the equation $\tilde{\sigma}_y^2 \rho = A\bar{\gamma}$ implies that $g = A^{-1} r$ can also be expressed only in terms of r_i. Also, B can be expressed only in terms of r_i and Λ. Then, the first equation in (IA.10) and $\Lambda = B'g$ (which is the rescaled equation, $\tilde{\sigma}_y^2 \lambda = B'\bar{\gamma}$) become $m + 2$ equations in the variables r_i and Λ.

In practice, however, this procedure does not work well. Numerically, it turns out that the solution is badly behaved, especially when N or m are large.[1] Moreover, without more explicit formulas, it is difficult to study properties of the solution. In the next subsection, I provide a more explicit solution for the case when all speculators have the same speed, i.e., when there are only 0-speculators.

I now analyze the forecast error variance:

$$\Sigma_t \;=\; \mathsf{Var}\big((w_t - p_{t-1})^2\big). \tag{IA.11}$$

Note that Σ_t is inversely related to price informativeness. Indeed, when prices are informative, they stay close to the forecast w_t, which implies that the variance Σ_t is small. Define the instantaneous price variance:

$$\sigma_p^2 \;=\; \frac{\mathsf{Var}(\mathrm{d}p_t)}{\mathrm{d}t} \;=\; \frac{\lambda^2 \ \mathsf{Var}(\mathrm{d}y_t)}{\mathrm{d}t} \;=\; \lambda^2 \sigma_y^2. \tag{IA.12}$$

The next result shows that growth rate of Σ is constant, and it is equal to the difference between the forecast variance σ_w^2 and the price variance σ_p^2.

Proposition IA.1. *The growth in the forecast error variance is constant and satisfies the following formula:*

$$\Sigma_t' \;=\; \sigma_w^2 - \sigma_p^2. \tag{IA.13}$$

[1]This is because in that case the matrix A is almost singular, and thus the equation $g = A^{-1}r$ produces unreliable solutions. Indeed, equation (IA.82) from Subsection 1.4 shows that that the determinant of $(A^0)^{-1}$, a matrix close to A^{-1}, is equal to $\frac{(N+1)^{m+1}}{(m+1)N+1}$. This is a large number when m or N are large.

This result can be explained by the fact that competition among speculators increases price volatility, with an upper bound given by the forecast volatility σ_w^2. But competition also makes prices more informative, which implies that the forecast error variance grows more slowly. As shown in the next subsection, it is a feature of this equilibrium to have the forecast error variance grow at a positive rate. This result is in contrast to Kyle (1985), in which the forecast error variance decreases at a constant rate, so that it becomes zero at the end. The reason for this difference is that in my model traders in equilibrium only use their most recent signals, and thus do not trade on longer-lived information.[2]

1.3 Equal speed case

In this subsection, I search for an equilibrium of the model $\mathcal{M}_m$ with $m \geq 0$ lags in the simpler case when there is no speed difference among speculators. This translates into all speculators being 0-speculators, i.e., $N_0 > 0$ and $N_1 = N_2 = \cdots = N_m = 0$. Because there are only 0-speculators, I write their number simply as $N = N_0$.

Theorem IA.2 provides an efficient numerical procedure to solve for the equilibrium. When the number of speculators is large, I also obtain asymptotic formulas for the equilibrium trading strategies and pricing functions. Proposition IA.3 then shows that the value of information decays exponentially. This result is proved rigorously only asymptotically, when the number of speculators is large. However, I verify numerically that the result remains true for a large number of parameter values.

Proposition IA.2 is a restatement of Theorem IA.1 to the case when all speculators have the same speed.

Proposition IA.2. *Let $m \geq 0$ be fixed, and consider the model $\mathcal{M}_m$ with m lags, and N speculators with equal speed (of type $\ell = 0$). Suppose there exists a linear equilibrium of the model with constant coefficients, of the form:*

$$\begin{aligned} \mathrm{d}x_t &= \gamma_0 \mathrm{d}_t w_t + \gamma_1 \mathrm{d}_t w_{t-1} + \cdots + \gamma_m \mathrm{d}_t w_{t-m}, \\ \mathrm{d}_t w_{t-i} &= \mathrm{d}w_{t-i} - z_{t-i,t}, \quad i = 0, 1, \ldots, m, \\ z_{t-i,t} &= \rho_0 \mathrm{d}y_{t-i} + \cdots + \rho_{i-1} \mathrm{d}y_{t-1}, \quad i = 0, 1, \ldots, m, \\ \mathrm{d}p_t &= \lambda \mathrm{d}y_t. \end{aligned} \tag{IA.14}$$

Then, the constants λ, ρ_i, γ_i and $\bar{\gamma}_i = N\gamma_i$ satisfy the following system of equations

[2]For a discussion on why traders might not want to use longer-lived information, see Section 4 in this Internet Appendix.

($i=0,\dots,m$):

$$B_i = \frac{1}{(N+1)^i}, \qquad \tilde\sigma_y = \tilde\sigma_u\sqrt{N+1}, \qquad \tilde\sigma_y^2\rho_i = \frac{1}{\lambda}\frac{N}{(N+1)^{i+1}},$$
$$\tilde\sigma_y^2\rho = A\bar\gamma, \qquad \tilde\sigma_y^2\lambda = B'\bar\gamma, \qquad A_{i,j} = \mathbf{1}_{i=j} - \frac{1}{\tilde\sigma_y^2\lambda^2}\frac{N}{N+2}\frac{(N+1)^{2\min(i,j)}-1}{(N+1)^{i+j}}. \tag{IA.15}$$

Conversely, suppose the constants λ, ρ_i, γ_i satisfy (IA.15), and in addition the following conditions are satisfied: (i) $\lambda > 0$; (ii) the matrix A is invertible; and (iii) the numbers $\beta_k = \sum_{i=0}^{m-k}\rho_i\bar\gamma_{k+i}$ satisfy $1 > \beta_1 > \cdots > \beta_m > 0$. Then, the equations in (IA.14) provide an equilibrium of the model.

Note that the system of equations (IA.15) has a simpler form. Following the discussion after Theorem IA.1, I make a change of variables and denote by $r_i = \tilde\sigma_y\rho_i$, $\Lambda = \tilde\sigma_y\lambda$, $g = \frac{\bar\gamma}{\tilde\sigma_y}$. In this case, one sees that $r_i = \frac{1}{\Lambda}\frac{N}{(N+1)^{i+1}}$ can further be expressed in terms of Λ. This suggests the following procedure to search for a solution of (IA.15): Suppose one starts with some value for Λ. From (IA.15) one sees that all the constants of the model (g, A, B, r) can be expressed as a function of Λ. Then, the equation $\Lambda = g'B$ becomes the equation that determines Λ. In Subsection 1.4, I show that the equation in Λ is an infinite polynomial equation, which in practice can be solved very accurately. Then, the conditions (i) and (ii) from Proposition IA.2 follow from a certain condition (IA.77) on Λ from the proof of Theorem IA.2.

The next result uses the procedure outlined above to find approximations for the equilibrium coefficients, which use the "big-O" notation.[3]

Theorem IA.2. *Let $m \geq 0$ be fixed, and consider the model $\mathcal{M}_m$ with m lags, and N speculators with equal speed (of type $\ell = 0$). Define the following numbers:*

$$\gamma_i^0 = \frac{\sigma_u}{\sigma_w}\frac{1}{\sqrt{N+1}}\frac{m-i+1}{m+1}, \qquad i = 0,1,\dots,m,$$
$$\rho_i^0 = \frac{\sigma_w}{\sigma_u}\frac{N}{(N+1)^{i+1+1/2}}, \qquad i=0,1,\dots,m-1, \tag{IA.16}$$
$$\lambda^0 = \frac{\sigma_w}{\sigma_u}\frac{1}{\sqrt{N+1}}.$$

[3]For $\alpha \in \mathbb{R}$, I say that the expression x_N is of the order of N^α, and write $x_N = O_N(N^\alpha)$, if there exists an integer N_* and a real number M such that $|x_N| \leq M|N^\alpha|$ for all $N \geq N_*$. In other words, x_M is of order N^α if $\frac{x_N}{N^\alpha}$ is bounded when N is sufficiently large.

Then if conditions (IA.76) and (IA.77) from Subsection 1.4 are satisfied,[4] *there exists an equilibrium. In this equilibrium, the coefficients of the optimal strategy* (γ_i) *and of the pricing functions* (λ, ρ_i) *approximate the coefficients in (IA.16) as follows:*

$$\begin{aligned} \gamma_i &= \gamma_i^0\Big(1 + O_N(1)\Big), \quad i = 0, 1, \dots, m, \\ \rho_i &= \rho_i^0\Big(1 + O_N\big(\tfrac{1}{N}\big)\Big), \quad i = 0, 1, \dots, m-1, \\ \lambda &= \lambda^0\Big(1 - O_N\big(\tfrac{1}{N}\big)\Big). \end{aligned} \tag{IA.17}$$

Figure IA.1 shows the optimal weights for various numbers of speculators N and various maximum signal lags m. For all the parameter values considered, the weights decrease with the lag. However, while the approximate weights, $\gamma_i^0 = \frac{\sigma_u}{\sigma_w}\frac{1}{\sqrt{N+1}}\frac{m-i+1}{m+1}$, decrease at the same rate, the actual weights decrease less quickly for smaller lags, and then decrease faster for larger lags. When m is large, one can also see that the initial decrease in the actual weights is very small.[5]

Proposition IA.3 shows that the expected profit from each additional signal decays exponentially.

Proposition IA.3. *Let* π_0 *be the expected profit at* $t = 0$ *of a speculator in equilibrium, and let* γ_i *be his optimal trading weight on the signal with lag* $i = 0, \dots, m$. *Then the profit can be decomposed as follows:*

$$\pi_0 = \sigma_w^2 \sum_{j=0}^{m} \pi_{0,j} = \sigma_w^2 \left(\frac{\gamma_0}{N+1} + \frac{\gamma_1}{(N+1)^2} + \cdots \frac{\gamma_m}{(N+1)^{m+1}} \right). \tag{IA.18}$$

Moreover, the ratio of two consecutive components of the expected profit is:

$$\frac{\pi_{0,j+1}}{\pi_{0,j}} = \frac{\gamma_{j+1}}{\gamma_j}\frac{1}{N+1} = O_N\big(\tfrac{1}{N}\big). \tag{IA.19}$$

A graphic illustration of this result is in Figure IA.2, which shows the profits of a speculator who can trade on at most $m = 5$ lagged signals. The cases studied correspond to the number of speculators $N \in \{1, 2, 3, 5, 20, 100\}$. One sees that indeed, when N is

[4] Numerically, these conditions are satisfied for all the values of N and m considered.

[5] Thus, in the limit when m approaches infinity, I conjecture that the weights become approximately equal. In that case, the informed traders behave as in Kyle (1985), by trading a multiple of the sum $\int_0^t \mathrm{d}w_\tau = w_t - w_0$. However, one can see from Theorem IA.2 that in my model the weights do not become of the order of $\mathrm{d}t$, as in the Kyle model, but rather remain of the same order of magnitude as for the lower m. In my model therefore prices are very close to strong-form efficient when m is large. This equilibrium resembles that of Caldentey and Stacchetti (2010).

large, the profits coming even from a signal of lag 1 are small.

I next analyze price volatility. Proposition IA.4 shows that price volatility has an upper bound, which makes rigorous the intuition for the general case, discussed after Proposition IA.1. Moreover, Proposition IA.4 provides a more thorough understanding about how information is revealed over time by trading. For this purpose, I define "signal revelation" as the covariance of a signal $\mathrm{d}w_t$ with $\mathrm{d}p_{t+k}$, the price change from k trading periods later:

$$SR_k \;=\; \frac{\mathsf{Cov}(\mathrm{d}w_t, \mathrm{d}p_{t+k})}{\sigma_w^2 \mathrm{d}t} \;=\; \frac{\mathsf{Cov}(\mathrm{d}w_{t-k}, \mathrm{d}p_t)}{\sigma_w^2 \mathrm{d}t}, \quad k = 0, 1, \dots . \tag{IA.20}$$

Since $\sum_{k=0}^{\infty} \mathrm{d}w_{t-k} = w_t$ (speculator's initial forecast is $w_0 = 0$), the sum of all SR_k equals:

$$\sum_{k=0}^{\infty} SR_k \;=\; \frac{\mathsf{Cov}(w_t, \mathrm{d}p_t)}{\sigma_w^2 \mathrm{d}t} \;=\; \frac{\lambda\, \mathsf{Cov}(w_t, \mathrm{d}y_t)}{\sigma_w^2 \mathrm{d}t} \;=\; \frac{\lambda^2 \sigma_y^2}{\sigma_w^2} \;=\; \frac{\sigma_p^2}{\sigma_w^2}, \tag{IA.21}$$

where I use the formula $\mathsf{Cov}(w_t, \mathrm{d}y_t) = \lambda\, \mathsf{Var}(\mathrm{d}y_t) = \lambda \sigma_y^2$ from the dealer's pricing equation for λ, proved in (IA.46) in Subsection 1.4.

Proposition IA.4. *Price volatility is always smaller than the forecast volatility. Their difference is small when the number of speculators is large:*

$$\sigma_w^2 - \sigma_p^2 \;=\; O_N\big(\tfrac{1}{N}\big). \tag{IA.22}$$

The signal revelation measure satisfies:

$$SR_k \;=\; \frac{N}{(N+1)^{k+1}}, \quad k = 0, 1, \dots, m, \quad \Longrightarrow \quad \sum_{k=0}^{m} SR_k \;=\; 1 - \frac{1}{(N+1)^{m+1}}. \tag{IA.23}$$

Therefore, the difference $\sigma_w^2 - \sigma_p^2$ is also small when m is large.

Thus, an interesting implication of the Proposition is that, when the number of lags m is large, each signal $\mathrm{d}w_t$ gets revealed by trading almost entirely. From Proposition IA.1, this case coincides with the one in which the growth rate of Σ, the forecast error variance, is very small.

1.4 Proofs

Before I proceed with the proofs of the equilibrium results, I introduce more useful notation. As before, a tilde above a symbol denotes normalization by σ_w, while $\widetilde{\mathsf{Cov}}$ and $\widetilde{\mathsf{Var}}$ are the instantaneous covariance and variance (already normalized by $\mathrm{d}t$), normalized by σ_w^2. For instance:

$$\tilde{\sigma}_u \;=\; \frac{\sigma_u}{\sigma_w}, \qquad \sigma_y^2 \;=\; \widetilde{\mathsf{Var}}(\mathrm{d}y_t) \;=\; \frac{\mathsf{Var}(\mathrm{d}y_t)}{\sigma_w^2 \mathrm{d}t}. \tag{IA.24}$$

I denote by $M_{a,b}$ the set of matrices of real numbers with a rows and b columns, by $M_a = M_{a,a}$ the set of square matrices, and by $V_a = M_{a,1}$ the set of column vectors.

If $\ell \in \{0, 1, \ldots, m\}$, recall from Subsection 1.1 that the vector $\gamma^{(\ell)}$ collects the ℓ-speculator's weight on $\mathrm{d}w_{t-i} - z_{t-i,t}$. By a slight abuse of notation, I also write $\gamma^{(\ell)}$ as a vector in V_{m+1} by padding with zeros for the entries $j = 0, \ldots, \ell - 1$. Define the aggregate speculator weights, $\bar{\gamma} \in V_{m+1}$:

$$\bar{\gamma} \;=\; \sum_{\ell=0}^{m} N_\ell \, \gamma^{(\ell)}. \tag{IA.25}$$

Let $\rho \in V_{m+1}$ be the vector that collects the coefficients involved in the dealer's expectation $z_{t-i,t}$ of $\mathrm{d}w_{t-i}$:

$$\rho \;=\; \big[\, \rho_0 \,,\; \ldots \;,\; \rho_m \,\big]', \tag{IA.26}$$

where ρ_m is not part of the dealer's expectations $z_{t-i,t}$ ($i = 0, 1, \ldots, m$), but is introduced in order to simplify notation.

For $i = 0, \ldots, m$, let $\mathrm{d}_t w_{t-i}$ be the unpredictable part of the signal $\mathrm{d}w_{t-i}$ (computed before trading at t):

$$\mathrm{d}_t w_{t-i} \;=\; \mathrm{d}w_{t-i} - z_{t-i,t}. \tag{IA.27}$$

Define the matrices $A \in M_{m+1}$ and $B \in V_{m+1}$. For $i, j = 0, \ldots, m$, define:

$$\begin{aligned} A_{i,j} \;&=\; \widetilde{\mathsf{Cov}}(\mathrm{d}_t w_{t-i}, \mathrm{d}_t w_{t-j}) \;=\; \frac{1}{\sigma_w^2 \mathrm{d}t}\, \mathsf{Cov}(\mathrm{d}_t w_{t-i}, \mathrm{d}_t w_{t-j}), \\ B_j \;&=\; \widetilde{\mathsf{Cov}}(w_t, \mathrm{d}_t w_{t-j}) \;=\; \frac{1}{\sigma_w^2 \mathrm{d}t}\, \mathsf{Cov}(w_t, \mathrm{d}_t w_{t-j}). \end{aligned} \tag{IA.28}$$

I rescale $\bar{\gamma}$, ρ, λ, by defining $r, g \in V_{m+1}$ and $\Lambda \in \mathbb{R}$ as follows:

$$g \;=\; \frac{\bar{\gamma}}{\tilde{\sigma}_y}, \qquad r \;=\; \tilde{\sigma}_y \, \rho, \qquad \Lambda \;=\; \tilde{\sigma}_y \, \lambda. \tag{IA.29}$$

Proof of Theorem IA.1. I need to prove that a linear equilibrium exists if there is a solution $(g, r, \Lambda, \tilde{\sigma}_y, A, B)$ to the following system of equations:

$$
\begin{aligned}
g &= \sum_{\ell=0}^{m} N_\ell \left(A_{\geq \ell}\right)^{-1} \left(\frac{1}{\Lambda} B_{\geq \ell} - r_{\geq \ell}\right), \\
r &= Ag, \qquad \Lambda = g'B, \qquad \tilde{\sigma}_y^2 = \frac{\tilde{\sigma}_u^2}{1 - g'r}, \\
A_{i,j} &= \mathbf{1}_{i=j} - \sum_{k=1}^{\min(i,j)} r_{i-k} r_{j-k}, \qquad B_i = 1 - \Lambda \sum_{k=1}^{i} r_{i-k}.
\end{aligned}
\tag{IA.30}
$$

Recall that $\mathbf{1}_{\mathcal{P}}$ is the indicator function, which equals 1 if $\mathcal{P}$ is true, and 0 otherwise. Also, $A_{\geq \ell}$ is the matrix with elements $A_{i,j}$ for $i, j \geq \ell$; and similarly for the vectors $B_{\geq \ell}$ and $r_{\geq \ell}$. The sum of vectors $X_{\geq \ell}$ over different ℓ is carried by padding $X_{\geq \ell}$ with zeros for the first ℓ entries.

Speculators' optimal strategy (γ)

I begin by analyzing the optimal strategy of an ℓ-speculator, where $\ell \in \{0, \ldots, m\}$. This speculator takes as given (i) the dealer's pricing rules: $\mathrm{d}p_t = \lambda \mathrm{d}y_t$ and $z_{t-i,t} = \rho_0 \mathrm{d}y_{t-i} + \cdots + \rho_{i-1} \mathrm{d}y_{t-1}$ for $i = 0, 1, \ldots, m$; and (ii) the other speculators' trading strategies. For instance, if another speculator is of type k, he is assumed to trade according to $\mathrm{d}x_t^{(k)} = \sum_{j=k}^{m} \gamma_j^{(k)} \mathrm{d}_t w_{t-j}$. Also, the ℓ-speculator chooses among trading strategies of the form: $\mathrm{d}x_t = \gamma_{\ell,t} \mathrm{d}_t w_{t-\ell} + \cdots + \gamma_{m,t} \mathrm{d}_t w_{t-m}$. Therefore, the ℓ-speculator assumes that the total order flow at t satisfies:

$$
\mathrm{d}y_t = \mathrm{d}u_t + \sum_{j=0}^{\ell-1} \bar{\gamma}_j \mathrm{d}_t w_{t-j} + \sum_{j=\ell}^{m} \left(\gamma_{j,t} + \gamma_j^-\right) \mathrm{d}_t w_{t-j}.
\tag{IA.31}
$$

where:

$$
\bar{\gamma}_j = \sum_{k=0}^{j} N_k \gamma_j^{(k)}, \quad j = 0, \ldots, m, \qquad \gamma_j^- = (N_\ell - 1)\gamma_j^{(\ell)} + \sum_{\substack{k=0 \\ k \neq \ell}}^{j} N_k \gamma_j^{(k)}, \quad j = \ell, \ldots, m.
\tag{IA.32}
$$

At $t = 0$, equation (12) implies that his normalized expected profit is:

$$
\tilde{\pi}_0 = \frac{\pi_0}{\sigma_w^2} = \frac{1}{\sigma_w^2} \, \mathsf{E}\left(\int_0^T \left(w_t - p_{t-1} - \lambda \mathrm{d}y_t\right) \mathrm{d}x_t\right).
\tag{IA.33}
$$

By construction, the terms $\mathrm{d}_t w_{t-j}$ are orthogonal to $\mathcal{I}_t$, hence also to p_{t-1}. Hence, $\mathrm{d}x_t$ is also orthogonal to p_{t-1}. I now use (IA.31) and the definitions: $A_{i,j} = \widetilde{\mathsf{Cov}}(\mathrm{d}_t w_{t-i}, \mathrm{d}_t w_{t-j})$, $B_j = \widetilde{\mathsf{Cov}}(w_t, \mathrm{d}_t w_j)$ to compute:

$$\begin{aligned}\tilde{\pi}_0 &= \mathsf{E}\left(\int_0^T \Big(w_t - \lambda \sum_{i=0}^{\ell-1} \bar{\gamma}_i \mathrm{d}_t w_{t-i} - \lambda \sum_{i=\ell}^{m} \big(\gamma_{i,t} + \gamma_i^-\big) \mathrm{d}_t w_{t-i}\Big) \sum_{j=\ell}^{m} \gamma_{j,t} \mathrm{d}_t w_{t-j}\right) \\ &= \sum_{j=\ell}^{m} B_j \gamma_{j,t} - \lambda \sum_{i=0}^{\ell-1} \sum_{j=\ell}^{m} \bar{\gamma}_i A_{i,j} \gamma_{j,t} - \lambda \sum_{i,j=\ell}^{m} \big(\gamma_{i,t} + \gamma_i^-\big) A_{i,j} \gamma_{j,t}.\end{aligned} \tag{IA.34}$$

Thus, I have reduced the problem to a linear–quadratic optimization. The first order condition with respect to $\gamma_{k,t}$, for $k = \ell, \dots, m$, is:

$$B_k - \lambda \sum_{i=0}^{\ell-1} \bar{\gamma}_i A_{i,k} - \lambda \sum_{i=\ell}^{m} \big(2\gamma_{i,t} + \gamma_i^-\big) A_{i,k} \;=\; 0. \tag{IA.35}$$

Denote by γ_t the $(m-\ell+1)$-column vector of trading weights at t. I divide the matrix A into four blocks, by restricting indices to be either $< \ell$ or $\geq \ell$. With matrix notation, the first order condition (IA.36) becomes:

$$B_{\geq \ell} - \lambda\, A_{\geq \ell, < \ell}\, \bar{\gamma}_{<\ell} - \lambda\, A_{\geq \ell} \left(2\gamma_t + \gamma^-\right) \;=\; 0. \tag{IA.36}$$

Then, for any ℓ-speculator and any t, one has:

$$2\gamma_t + \gamma^- \;=\; \big(A_{\geq \ell}\big)^{-1} \left(\frac{1}{\lambda} B_{\geq \ell} - A_{\geq \ell, <\ell}\, \bar{\gamma}_{<\ell}\right). \tag{IA.37}$$

This equation implies that the ℓ-speculators have identical weights in equilibrium (previously denoted by $\gamma^{(\ell)}$), and these weights do not depend on t. I then have $\gamma^{(\ell)} + \gamma^- = \bar{\gamma}_{\geq \ell}$, hence:

$$\gamma^{(\ell)} \;=\; \big(A_{\geq \ell}\big)^{-1} \left(\frac{1}{\lambda} B_{\geq \ell} - A_{\geq \ell, <\ell}\, \bar{\gamma}_{<\ell}\right) - \bar{\gamma}_{\geq \ell}. \tag{IA.38}$$

Thus, equation (IA.38) reduces the computation of the optimal weights $\gamma^{(\ell)}$ to the computation of the aggregate weights $\bar{\gamma}$.

I now derive the equation that $\bar{\gamma}$ must satisfy in equilibrium. To simplify formulas, note from (IA.47) that $A\bar{\gamma} = \tilde{\sigma}_y^2 \rho$, or in block matrix notation $A_{\geq \ell, <\ell}\, \bar{\gamma}_{<\ell} + A_{\geq \ell}\, \bar{\gamma}_{\geq \ell} = \tilde{\sigma}_y^2\, \rho_{\geq \ell}$. Using this, equation (IA.39) becomes:

$$\gamma^{(\ell)} \;=\; \big(A_{\geq \ell}\big)^{-1} \left(\frac{1}{\lambda} B_{\geq \ell} - \tilde{\sigma}_y^2\, \rho_{\geq \ell}\right). \tag{IA.39}$$

To obtain the equation that determines $\bar{\gamma}$, I multiply (IA.39) by N_ℓ and sum over all $\ell = 0, \ldots, m$, padding with zeroes where necessary. One obtains:

$$\bar{\gamma} \;=\; \sum_{\ell=0}^{m} N_\ell \gamma^{(\ell)} = \sum_{\ell=0}^{m} N_\ell \big(A_{\geq \ell}\big)^{-1} \left(\frac{1}{\lambda} B_{\geq \ell} - \tilde{\sigma}_y^2 \, \rho_{\geq \ell} \right). \tag{IA.40}$$

After dividing this equation by $\tilde{\sigma}_y$, use $g = \frac{\bar{\gamma}}{\tilde{\sigma}_y}$, $r = \tilde{\sigma}_y \, \rho$, and $\Lambda = \tilde{\sigma}_y \, \lambda$ to obtain the corresponding equation in (IA.30).

So far, I have shown that equation (IA.39) is a necessary condition for equilibrium. I now prove that it is also sufficient condition for the speculator's problem, if one imposes two additional conditions: (i) $\lambda > 0$; and (ii) for all $\ell = 0, \ldots, m$, the matrix $A_{\geq \ell}$ is invertible. Indeed, the second order condition in the maximization problem above for the ℓ-speculator is:

$$\lambda \; \det(A_{\geq \ell}) \;>\; 0. \tag{IA.41}$$

Normally, one expects that $\det(A_{\geq \ell}) > 0$, since economically A is the covariance matrix of the fresh signals, and the signals $\mathrm{d}w_{t-i}$ are independent. But if A is just the solution of a system of equations, this condition needs to be checked. If $\det(A_{\geq \ell}) > 0$, then the second order condition from (IA.41) becomes $\lambda > 0$, which is just condition (i).

Dealer's pricing rules (λ, ρ, A, B)

The dealer takes as given that the aggregate order flow is of the form:

$$\mathrm{d}y_t \;=\; \mathrm{d}u_t + \bar{\gamma}_0 \mathrm{d}w_t + \bar{\gamma}_1 \mathrm{d}_t w_{t-1} + \cdots + \bar{\gamma}_m \mathrm{d}_t w_{t-m}, \tag{IA.42}$$

where, for $k = 0, \ldots, m$, the speculators set:

$$\mathrm{d}_t w_{t-k} \;=\; \mathrm{d}w_{t-k} - \big(\rho_0^* \mathrm{d}y_{t-k} + \cdots + \rho_{k-1}^* \mathrm{d}y_{t-1} \big), \tag{IA.43}$$

with ρ_i^* constant. (Of course, in equilibrium the dealer will eventually set $\rho_i = \rho_i^*$.) I combine the two equations:

$$\begin{aligned} \mathrm{d}y_t \;&=\; \mathrm{d}u_t + \sum_{k=0}^{m} \bar{\gamma}_k \mathrm{d}w_{t-k} - \sum_{k=0}^{m} \bar{\gamma}_k \sum_{i=0}^{k-1} \rho_i^* \mathrm{d}y_{t-k+i} \\ &=\; \mathrm{d}u_t + \sum_{k=0}^{m} \bar{\gamma}_k \mathrm{d}w_{t-k} - \sum_{k=1}^{m} \sum_{i=0}^{m-k} \rho_i^* \bar{\gamma}_{k+i} \mathrm{d}y_{t-k}. \end{aligned} \tag{IA.44}$$

Because each speculator only trades on the unpredictable part of his signal, $\mathrm{d}y_t$ are orthogonal to each other. Thus, the dealer computes:

$$
\begin{aligned}
z_{t-k,t} \;&=\; \mathsf{E}\big(\mathrm{d}w_{t-k} \mid \mathrm{d}y_{t-k}, \dots, \mathrm{d}y_{t-1}\big) \;=\; \sum_{i=0}^{k-1} \rho_{i,t-k} \mathrm{d}y_{t-k+i}, \quad k = 0, \dots, m, \\
\mathrm{d}p_t \;&=\; \lambda_t \mathrm{d}y_t,
\end{aligned}
\tag{IA.45}
$$

where the coefficients $\rho_{i,t-k}$ and λ_t are:[6]

$$
\begin{aligned}
\rho_{i,t-k} \;&=\; \frac{\mathsf{Cov}(\mathrm{d}w_{t-k}, \mathrm{d}y_{t-k+i})}{\mathsf{Var}(\mathrm{d}y_{t-k+i})}, \quad k = 0, \dots, m, \quad i = 0, \dots, k-1, \\
\lambda_t \;&=\; \frac{\mathsf{Cov}(v_1, \mathrm{d}y_t)}{\mathsf{Var}(\mathrm{d}y_t)} \;=\; \frac{\mathsf{Cov}(w_t, \mathrm{d}y_t)}{\mathsf{Var}(\mathrm{d}y_t)}.
\end{aligned}
\tag{IA.46}
$$

At the end of this proof, I show that the following numbers do not depend on t: λ_t, $\rho_{i,t}$, $A_{i,j,t} = \widetilde{\mathsf{Cov}}(\mathrm{d}_t w_{t-i}, \mathrm{d}_t w_{t-j})$, $B_{j,t} = \widetilde{\mathsf{Cov}}(w_t, \mathrm{d}_t w_j)$.

Taking these numbers as constant, I now prove the rest of the equations in (IA.30). First, note that in equilibrium $\rho_i^* = \rho_i$. I rewrite the equation for ρ in (IA.46) by taking $k = i$. Thus, $\rho_i = \frac{\widetilde{\mathsf{Cov}}(\mathrm{d}w_{t-i}, \mathrm{d}y_t)}{\widetilde{\mathsf{Var}}(\mathrm{d}y_t)}$, and note that $\mathrm{d}y_t$ is orthogonal on all other $\mathrm{d}y_{t-k}$ for $k > 0$. Hence, from (IA.42) and (IA.43), one obtains:

$$
\rho_i \;=\; \frac{\widetilde{\mathsf{Cov}}(\mathrm{d}_t w_{t-i}, \mathrm{d}y_t)}{\widetilde{\mathsf{Var}}(\mathrm{d}y_t)} \;=\; \frac{\widetilde{\mathsf{Cov}}\big(\mathrm{d}_t w_{t-i}, \sum_{j=0}^m \bar{\gamma}_j \mathrm{d}_t w_{t-j}\big)}{\widetilde{\mathsf{Var}}(\mathrm{d}y_t)} \;=\; \frac{\sum_{j=0}^m A_{i,j} \bar{\gamma}_j}{\tilde{\sigma}_y^2}. \tag{IA.47}
$$

Since $g = \frac{\bar{\gamma}}{\tilde{\sigma}_y}$ and $r = \tilde{\sigma}_y\, \rho$, one gets $r_i = (Ag)_i$, or in matrix notation $r = Ag$. This proves the corresponding equation in (IA.30). Also, from the equation from λ in (IA.46), one obtains:

$$
\lambda \;=\; \frac{\widetilde{\mathsf{Cov}}(w_t, \mathrm{d}y_t)}{\widetilde{\mathsf{Var}}(\mathrm{d}y_t)} \;=\; \frac{\widetilde{\mathsf{Cov}}\big(w_t, \sum_{j=0}^m \bar{\gamma}_j \mathrm{d}_t w_{t-j}\big)}{\widetilde{\mathsf{Var}}(\mathrm{d}y_t)} \;=\; \frac{\sum_{j=0}^m \bar{\gamma}_j B_j}{\tilde{\sigma}_y^2}. \tag{IA.48}
$$

Since $\Lambda = \lambda\, \tilde{\sigma}_y$, one gets $\Lambda = \sum_{j=0}^m g_j B_j$, or in matrix notation $\Lambda = g' B$. This proves the corresponding equation in (IA.30).

By computing $\widetilde{\mathsf{Cov}}(\mathrm{d}y_t, \mathrm{d}y_t)$ and using (IA.42), it follows that $\tilde{\sigma}_y^2 = \widetilde{\mathsf{Var}}(\mathrm{d}y_t)$ satisfies $\tilde{\sigma}_y^2 = \tilde{\sigma}_u^2 + \sum_{i,j=0}^m A_{i,j} \bar{\gamma}_i \bar{\gamma}_j$, or in matrix notation $\tilde{\sigma}_y^2 = \tilde{\sigma}_u^2 + \bar{\gamma}' A \bar{\gamma}$. Since $\bar{\gamma} = g\, \tilde{\sigma}_y$, one computes:

$$
\tilde{\sigma}_y^2 \;=\; \bar{\gamma}' A \bar{\gamma} + \tilde{\sigma}_u^2 \;=\; g' A g\, \tilde{\sigma}_y^2 + \tilde{\sigma}_u^2. \tag{IA.49}
$$

[6] Note that in principle $\rho_{i,t-k}$ might also depend on t, the time at which the expectation is computed. However, the formula shows that ρ only depends on i and $t - k$, and not on t separately.

Since $Ag = r$, one gets $\tilde{\sigma}_y^2 = g' r\, \tilde{\sigma}_y^2 + \tilde{\sigma}_u^2$, which implies $\tilde{\sigma}_y^2 = \frac{\tilde{\sigma}_u^2}{1-g'r}$. This proves the corresponding equation in (IA.30).

Now, consider the equation $A_{k,\ell} = \widetilde{\mathsf{Cov}}(\mathrm{d}_t w_{t-k}, \mathrm{d}_t w_{t-\ell})$. From (IA.43), $\mathrm{d}_t w_{t-k} = \mathrm{d}w_{t-k} - \big(\rho_0 \mathrm{d}y_{t-k} + \cdots + \rho_{k-1}\mathrm{d}y_{t-1}\big)$. But $\mathrm{d}_t w_{t-\ell}$ is orthogonal to the previous order flow, hence $A_{k,\ell} = \widetilde{\mathsf{Cov}}(\mathrm{d}w_{t-k}, \mathrm{d}_t w_{t-\ell})$. Because A is a symmetric matrix, without loss of generality assume $k \geq \ell$, which implies $\ell = \min(k,\ell)$. Since $\widetilde{\mathsf{Cov}}(\mathrm{d}w_{t-i}, \mathrm{d}y_t) = \rho_i \tilde{\sigma}_y^2\, \mathbf{1}_{i \geq 0}$, one obtains:

$$
\begin{aligned}
A_{k,\ell} &= \widetilde{\mathsf{Cov}}\Big(\mathrm{d}w_{t-k}, \mathrm{d}w_{t-\ell} - \big(\rho_0 \mathrm{d}y_{t-\ell} + \cdots + \rho_{\ell-1}\mathrm{d}y_{t-1}\big)\Big) \\
&= \mathbf{1}_{k=l} \; - \; \sum_{j=0}^{\ell-1} \rho_j \rho_{k-\ell+j}\, \tilde{\sigma}_y^2 \;=\; \mathbf{1}_{k=l} \; - \; \sum_{i=1}^{\ell} \rho_{k-i}\rho_{\ell-i}\, \tilde{\sigma}_y^2.
\end{aligned} \tag{IA.50}
$$

Since $r = \rho\, \tilde{\sigma}_y$, one gets $A_{k,\ell} = \mathbf{1}_{k=l} - \sum_{j=1}^{\ell} r_{k-i} r_{\ell-i}$, which proves the corresponding equation in (IA.30). I also compute $B_\ell = \widetilde{\mathsf{Cov}}(w_t, \mathrm{d}_t w_\ell)$. From (IA.43), one gets:

$$
\begin{aligned}
B_\ell &= \widetilde{\mathsf{Cov}}\Big(w_t, \mathrm{d}w_{t-\ell} - \big(\rho_0 \mathrm{d}y_{t-\ell} + \cdots + \rho_{\ell-1}\mathrm{d}y_{t-1}\big)\Big) \\
&= 1 - \lambda\big(\rho_0 + \cdots + \rho_{\ell-1}\big)
\end{aligned} \tag{IA.51}
$$

Since $\Lambda = \lambda\, \tilde{\sigma}_y$ and $r = \rho\, \tilde{\sigma}_y$, one gets $B_\ell = 1 - \Lambda \sum_{j=0}^{\ell-1} r_j = 1 - \Lambda \sum_{i=1}^{\ell} r_{i-k}$. This proves the corresponding equation in (IA.30).

I now prove that the various pricing coefficients do not depend on t. For this, I show that the following numbers are independent of t: $\mathsf{Cov}(\mathrm{d}w_t, \mathrm{d}y_{t+k})$ for all k; $\mathsf{Cov}(w_t, \mathrm{d}y_{t+k})$ for all k; $\mathsf{Var}(\mathrm{d}y_t)$; $\mathsf{Cov}(w_t, \mathrm{d}_t w_j)$ for $j = 0, \ldots, m$; and $\mathsf{Cov}(\mathrm{d}_t w_i, \mathrm{d}_t w_j)$ for $i, j = 0, \ldots, m$.

First, I prove by induction that $\widetilde{\mathsf{Cov}}(\mathrm{d}w_t, \mathrm{d}y_{t+k})$ does not depend on t for $k \geq 0$. (This is trivially true for $k < 0$.) The statement is true for $k = 0$, since equation (IA.42) implies $\widetilde{\mathsf{Cov}}(\mathrm{d}w_t, \mathrm{d}y_t) = \bar{\gamma}_0$. Assume that the statement is is true for all $i < k$. I now prove that $\widetilde{\mathsf{Cov}}(\mathrm{d}w_t, \mathrm{d}y_{t+k})$ does not depend on t. Equations (IA.44) implies that $\mathrm{d}y_{t+k}$ only involves three types of terms: (i) $\mathrm{d}u_{t+k}$, (ii) $\mathrm{d}w_{t+k-i}$ for $i = 0, \ldots, m$, and (iii) $\mathrm{d}y_{t+k-1-i}$ for $i = 0, \ldots, m-1$. Also, the coefficients ρ_i^* do not depend on time. Therefore, by the induction hypothesis all these terms have covariances with $\mathrm{d}w_t$ that do not depend on t.

Next, I prove that $a_t = \widetilde{\mathsf{Cov}}(w_t, \mathrm{d}y_t)$ does not depend on t. Equation (IA.44) implies the following recursive formula for all t:

$$
a_t \;=\; \sum_{k=0}^{m} \bar{\gamma}_k - \sum_{k=1}^{m} \sum_{i=0}^{m-k} \rho_i^* \bar{\gamma}_{k+i} a_{t-k}. \tag{IA.52}
$$

But Lemma IA.1 below implies that a_t does not depend on t, provided that:

$$1 \;>\; \beta_1 \;>\; \cdots \;>\; \beta_m \;>\; 0, \quad \text{with} \quad \beta_k \;=\; \sum_{i=0}^{m-k} \rho_i^* \bar{\gamma}_{k+i}. \tag{IA.53}$$

Therefore, $\widetilde{\mathsf{Cov}}(w_t, \mathrm{d}y_t)$ does not depend on t. This result also implies that $\widetilde{\mathsf{Cov}}(w_t, \mathrm{d}y_{t+k})$ does not depend on t for any integer k. To see this, note first that the case $k > 0$ reduces to the case $k = 0$. Indeed, $\widetilde{\mathsf{Cov}}(w_t, \mathrm{d}y_{t+k}) = \widetilde{\mathsf{Cov}}(w_{t+k}, \mathrm{d}y_{t+k}) - \sum_{i=1}^{k} \widetilde{\mathsf{Cov}}(\mathrm{d}w_{t+i}, \mathrm{d}y_{t+k})$, and I have already proved that all these terms are independent of t. Also, the case $k < 0$ reduces to the case $k = 0$, since $\widetilde{\mathsf{Cov}}(w_t, \mathrm{d}y_{t-i}) = \widetilde{\mathsf{Cov}}(w_{t-i}, \mathrm{d}y_{t-i})$ if $i \geq 0$.

I now prove that $\widetilde{\mathsf{Var}}(\mathrm{d}y_t) = \sum_{k=0}^{m} \bar{\gamma}_k \,\mathsf{Cov}\big(\mathrm{d}_t w_{t-k}, \mathrm{d}y_t\big)$ does not depend on t. Since $\mathrm{d}y_t$ is orthogonal to previous order flow, $\widetilde{\mathsf{Var}}(\mathrm{d}y_t) = \sum_{k=0}^{m} \bar{\gamma}_k \,\mathsf{Cov}\big(\mathrm{d}w_{t-k}, \mathrm{d}y_t\big)$. But these terms have already been proved to be independent of t.

Finally, one uses the results proved above to show that $B_{j,t} = \widetilde{\mathsf{Cov}}(w_t, \mathrm{d}_t w_j)$ and $A_{i,j,t} = \widetilde{\mathsf{Cov}}(\mathrm{d}_t w_i, \mathrm{d}_t w_j)$ do not depend on t. Indeed, I have shown that $\widetilde{\mathsf{Cov}}(\mathrm{d}w_t, \mathrm{d}y_{t-k})$ and $\widetilde{\mathsf{Cov}}(w_t, \mathrm{d}y_{t-k})$ are independent of t, and all is left to do is to use the fact that $\mathrm{d}y_{t-k}$ are orthogonal to each other.

So far, I have provided necessary equations for the equilibrium. I now prove that the conditions in (IA.30) except for the first one are also sufficient to justify the dealer's pricing equations, if one imposes an additional condition: (iii) the numbers $\beta_k = \sum_{i=0}^{m-k} \rho_i \bar{\gamma}_{k+i}$ satisfy $1 > \beta_1 > \cdots > \beta_m > 0$. But, as shown before, condition (iii) ensures that the equilibrium pricing coefficients are well-defined and constant. More generally, Lemma IA.1 can be used to replace condition (iii) with the condition that the (complex) roots of the polynomial $Q(z) = z^m + \beta_1 z^{m-1} + \cdots + \beta_{m-1} z + \beta_m$ lie in the open unit disk $D = \{z \in \mathbb{C} \mid |z| < 1\}$. The proof is now complete. ■

Lemma IA.1. *Let $X_1, \ldots, X_m \in \mathbb{R}$, and consider a sequence $X_t \in \mathbb{R}$ which satisfies the following recursive equation:*

$$X_t + \beta_1 X_{t-1} + \cdots + \beta_m X_{t-m} \;=\; \alpha, \quad t \geq m+1. \tag{IA.54}$$

Then the sequence X_t converges to $\bar{X} = \frac{\alpha}{1+(\beta_1+\cdots+\beta_m)}$, regardless of the initial values $X_1, \ldots, X_m$, if and only if all the (complex) roots of the polynomial $Q(z) = z^m + \beta_1 z^{m-1} + \cdots + \beta_{m-1} z + \beta_m$ lie in the open unit disk $D = \{z \in \mathbb{C} \mid |z| < 1\}$. For this, a sufficient condition is that the coefficients β_i satisfy:

$$1 \;>\; \beta_1 \;>\; \cdots \;>\; \beta_m \;>\; 0. \tag{IA.55}$$

Furthermore, if $\alpha = \alpha_t$ is not constant, then under the same conditions on β_i, the difference $X_t - \frac{\alpha_t}{1+(\beta_1+\cdots+\beta_m)}$ converges to zero, regardless of the initial values for X.

Proof. First, note that $\bar{X}$ is well-defined as long as $1 + \beta_1 + \cdots + \beta_m \neq 0$. Indeed, if $1 + \beta_1 + \cdots + \beta_m = 0$, then $Q(z)$ would have $z = 1$ as a root, which does not lie in the open unit disk D. Denote by $q_1, \ldots, q_m$ the roots of $Q(z)$. Let $Y_t = X_t - \bar{X}$. Then, the new sequence Y_t satisfies the recursive equation $Y_t + \beta_1 Y_{t-1} + \cdots + \beta_m Y_{t-m} = 0$, which has the following general solution:

$$Y_t \;=\; C_1 q_1^t + \cdots C_m q_m^t, \quad t \geq 1, \tag{IA.56}$$

where $C_1, \ldots, C_m$ are arbitrary complex constants.[7] Thus, Y_t is convergent for any values of C_i if and only if, for all $i = 1, \ldots, m$, either $q_i \in D$ or $q_i = 1$. But when $q_i = 1$ for some $i = 1, \ldots, m$, one has $0 = Q(1) = 1 + \beta_1 + \cdots + \beta_m$, hence the value of $\bar{X}$ is not defined. This completes the proof of the "if and only if" statement.

The statement that (IA.55) implies that all roots of Q lie in D is known as the Eneström–Kakeya theorem. For completeness, I include the proof here. First, I prove that all roots of Q must lie in $\bar{D} = \{z \in \mathbb{C} \mid |z| \leq 1\}$. By contradiction, suppose that there exists z_* a root of Q with $|z_*| > 1$. Then, I also have $(1 - z_*)Q(z_*) = 0$ which implies $z_*^{m+1} = \beta_m + \sum_{i=0}^{m-1}(\beta_i - \beta_{i+1}) z_*^{m-i}$, where $\beta_0 = 1$. After taking absolute values, one gets $|z_*^{m+1}| \leq \beta_m + \sum_{i=0}^{m-1}(\beta_i - \beta_{i+1})|z_*^{m-i}| < \beta_m |z_*^m| + \sum_{i=0}^{m-1}(\beta_i - \beta_{i+1})|z_*^m| = \big(\beta_m + \sum_{i=0}^{m-1}(\beta_i - \beta_{i+1})\big)|z_*^m| = |z_*^m|$. Thus, $|z_*^{m+1}| < |z_*^m|$, which is a contradiction. I have just proved that all the roots of Q must lie in $\bar{D}$. Finally, I show that the roots of any $Q(z) = z^m + \beta_1 z^{m-1} + \cdots + \beta_{m-1} z + \beta_m$ satisfying (IA.55) must actually lie in D. Let $r < 1$ be sufficiently close to 1 so that I have $r^m > \beta_1 r^{m-1} > \ldots > \beta_{m-1} r > \beta_m > 0$. Then, the polynomial $Q_r(z) = Q(rz)$ must have all roots in $\bar{D}$. Let z_* be a root of Q. Then, $Q_r\big(\frac{z_*}{r}\big) = Q(z_*) = 0$, which implies that $\frac{z_*}{r} \in \bar{D}$, or equivalently $z_* \in r\bar{D}$. But $r\bar{D} \subset D$, and the proof is now complete. ■

Proof of Proposition IA.1. Since the forecast error variance equals $\Sigma_t = \mathsf{Var}(w_t - p_{t-1}) = \mathsf{E}\big((w_t - p_{t-1})^2\big)$, one computes the derivative of Σ_t as follows:

$$\begin{aligned} \Sigma_t' \;&=\; \frac{1}{\mathrm{d}t}\,\mathsf{E}\Big(2(\mathrm{d}w_{t+1} - \mathrm{d}p_t)(w_t - p_{t-1}) + (\mathrm{d}w_{t+1} - \mathrm{d}p_t)^2\Big), \\ &=\; -2\frac{\mathsf{Cov}(w_t, \mathrm{d}p_t)}{\mathrm{d}t} + \sigma_w^2 + \frac{\mathsf{Var}(\mathrm{d}p_t)}{\mathrm{d}t}. \end{aligned} \tag{IA.57}$$

[7] To obtain real values of Y_t, one needs to impose the following conditions: (i) if q_i is real, then so is C_i; and (ii) if q_i and q_j are complex conjugate, then so are C_i and C_j.

The pricing equation (IA.46) from the proof of Theorem IA.1 shows that $\mathsf{Cov}(w_t, \mathrm{d}y_t) = \lambda\, \mathsf{Var}(\mathrm{d}y_t) = \lambda\sigma_y^2$, therefore using $\mathrm{d}p_t = \lambda \mathrm{d}y_t$ one obtains:

$$\frac{\mathsf{Var}(\mathrm{d}p_t)}{\mathrm{d}t} \;=\; \frac{\mathsf{Cov}(w_t, \mathrm{d}p_t)}{\mathrm{d}t} \;=\; \lambda^2\sigma_y^2 \;=\; \sigma_p^2. \tag{IA.58}$$

The equation (IA.57) now implies that $\Sigma_t' = \sigma_w^2 - \sigma_p^2$, which finishes the proof. ■

Proof of Proposition IA.2. Compared to the setup of Theorem IA.1, here there are only speculators with zero lag ($\ell = 0$). Therefore, to finish the proof of this Proposition, I need to show that the system in (IA.10) reduces to the system in (IA.15). With the usual notation, the system in (IA.10) translates into (IA.30).

When all speculators are of type $\ell = 0$, and there are $N_0 = N$ of them, equation (IA.30) becomes:[8]

$$\begin{gathered} A\left(\frac{1}{N}\,g + g\right) \;=\; \frac{1}{\Lambda}B, \qquad r \;=\; Ag, \qquad \Lambda \;=\; g'B, \qquad \tilde{\sigma}_y^2 \;=\; \frac{\tilde{\sigma}_u^2}{1 - g'r}, \\ A_{i,j} \;=\; \mathbf{1}_{i=j} - \sum_{k=1}^{\min(i,j)} r_{i-k} r_{j-k}, \qquad B_i \;=\; 1 - \Lambda \sum_{k=1}^{i} r_{i-k}. \end{gathered} \tag{IA.59}$$

The first two equations imply $Ag = r = \frac{N}{N+1}\frac{1}{\Lambda}B$. When $i = 0$, this equation implies $r_0 = \frac{1}{\Lambda}\frac{N}{N+1}$. When $i = 1$, one gets $r_1 = \frac{1}{\Lambda}\frac{N}{N+1}(1 - \Lambda\, r_0) = \frac{1}{\Lambda}\frac{N}{(N+1)^2}$. By induction, one gets ($i = 0, \dots, m$):

$$r_i \;=\; \frac{1}{\Lambda}\,\frac{N}{(N+1)^{i+1}}. \tag{IA.60}$$

This proves the equation for ρ_i in (IA.15). One also obtains:

$$B_i \;=\; \frac{1}{(N+1)^i}, \tag{IA.61}$$

which proves the equation for B_i in (IA.15). Moreover, one computes $g'r = g'B\,\frac{N}{N+1}\,\frac{1}{\Lambda}$. But $g'B = \Lambda$, hence:

$$g'r \;=\; \frac{N}{N+1}. \tag{IA.62}$$

This implies:

$$\tilde{\sigma}_y^2 \;=\; \frac{\tilde{\sigma}_u^2}{1 - g'r} \;=\; (N+1)\tilde{\sigma}_u^2, \qquad \text{or} \qquad \tilde{\sigma}_y \;=\; \sqrt{N+1}\,\tilde{\sigma}_u, \tag{IA.63}$$

[8]Note that except for the first equation, the other equations in (IA.59) are the same as in (IA.30). I also provide a direct derivation of the first equation in the proof of Proposition IA.3.

which proves the equation for $\tilde{\sigma}_y$ in (IA.15). Using the formula (IA.60) for r, I use (IA.59) to compute $(i, j = 0, \dots, m)$:

$$A_{i,j} \;=\; \mathbf{1}_{i=j} - \frac{1}{\Lambda^2} \frac{N}{N+2} \frac{(N+1)^{2\min(i,j)} - 1}{(N+1)^{i+j}}, \tag{IA.64}$$

which proves the equation for $A_{i,j}$ in (IA.15). Finally, the two equations, $r = Ag$ and $\Lambda = g'B$, after rescaling are equivalent to $\tilde{\sigma}_y^2 \rho \;=\; A\bar{\gamma}$ and $\tilde{\sigma}_y^2 \lambda \;=\; B'\bar{\gamma}$, which finishes the proof. ■

Proof of Theorem IA.2. I use the notations from the proofs of Theorem IA.1 and Proposition IA.2. The idea is to study the behavior of (A, Λ, r, g) around $\Lambda = 1$, but without yet imposing the condition $\Lambda = g'B$. To do that, define the following numbers:

$$\begin{aligned} A^0_{i,j} \;&=\; \mathbf{1}_{i=j} - \frac{N}{N+2} \frac{(N+1)^{2\min(i,j)} - 1}{(N+1)^{i+j}}, \qquad \Lambda_0 \;=\; 1, \\ r^0_i \;&=\; \frac{N}{(N+1)^{i+1}}, \qquad g^0_i \;=\; \frac{N}{N+1} \frac{(m-i+1)N+1}{(m+1)N+1}. \end{aligned} \tag{IA.65}$$

One verifies the formula:[9]

$$A^0 g^0 \;=\; r^0, \quad \text{or, equivalently,} \quad g^0 \;=\; \big(A^0\big)^{-1} r^0. \tag{IA.66}$$

For $\varepsilon < 1$, define the following variables:

$$\begin{aligned} A^\varepsilon_{i,j} \;&=\; \mathbf{1}_{i=j} - \frac{1}{1-\varepsilon} \frac{N}{N+2} \frac{(N+1)^{2\min(i,j)} - 1}{(N+1)^{i+j}}, \qquad \Lambda^\varepsilon \;=\; \sqrt{1-\varepsilon} \\ r^\varepsilon_i \;&=\; \frac{1}{\sqrt{1-\varepsilon}} \frac{N}{(N+1)^{i+1}}, \qquad g^\varepsilon \;=\; \left(A^\varepsilon\right)^{-1} r^\varepsilon, \end{aligned} \tag{IA.67}$$

whenever A^ε is invertible. Using (IA.66), it follows that the variables defined in (IA.65) are the same as the variables in (IA.67) in the particular case when $\varepsilon = 0$. In other words, given a solution $(A, B, \Lambda, r, g, \tilde{\sigma}_y)$ to the system (IA.30), if one defines:

$$\varepsilon \;=\; 1 - \Lambda^2, \tag{IA.68}$$

it follows that the variables (Λ, A, r, g) satisfy:

$$\Lambda \;=\; \Lambda^\varepsilon, \quad A \;=\; A^\varepsilon, \quad r \;=\; r^\varepsilon, \quad g \;=\; g^\varepsilon. \tag{IA.69}$$

[9]This can be done either directly, or by using the method described below, which involves recursive computation of the inverse matrix, $(A^0)^{-1}$. Then, one verifies by induction that $(A^0)^{-1} r^0 = g^0$.

Multiplying the equation for $A = A^\varepsilon$ from (IA.67) by $\Lambda^2 = 1 - \varepsilon$, one obtains:

$$\Lambda^2 A \;=\; A^0 - \varepsilon I, \tag{IA.70}$$

where I is the identity matrix ($I_{i,j} = \mathbf{1}_{i=j}$). This implies $\frac{1}{\Lambda^2} A^{-1} = \big(A^0 - \varepsilon I\big)^{-1}$. Multiplying this equation to the right with $r = \frac{1}{\Lambda} r^0$, one obtains:

$$\frac{g}{\Lambda} \;=\; \big(A^0 - \varepsilon I\big)^{-1} r^0. \tag{IA.71}$$

Multiplying this equation to the left with B', and using $B'g = \Lambda$, one obtains:

$$1 \;=\; B'\big(A^0 - \varepsilon I\big)^{-1} r^0. \tag{IA.72}$$

This equation determines ε, or equivalently $\Lambda = \sqrt{1-\varepsilon}$. I make this equation more explicit by observing that the inverse matrix $\big(A^0 - \varepsilon I\big)^{-1}$ has the following series expansion:

$$\big(A^0 - \varepsilon I\big)^{-1} \;=\; (A^0)^{-1} + \varepsilon\,(A^0)^{-2} + \varepsilon^2 (A^0)^{-3} + \cdots. \tag{IA.73}$$

Multiplying this equation to the left by B' and to the right by r^0, one obtains:

$$1 \;=\; B'\big(A^0 - \varepsilon I\big)^{-1} r^0 \;=\; B'(A^0)^{-1} r^0 + \varepsilon\, B'(A^0)^{-2} r^0 + \varepsilon^2 B'(A^0)^{-3} r^0 + \cdots. \tag{IA.74}$$

One computes $1 - B'g^0 = \frac{1}{(m+1)N+1}$. Since $\big(A^0\big)^{-1} r^0 = g^0$, one obtains:

$$\frac{1}{(m+1)N+1} \;=\; \varepsilon\, B'\big(A^0\big)^{-1} g^0 + \varepsilon^2 B'\big(A^0\big)^{-2} g^0 + \varepsilon^3 B'\big(A^0\big)^{-3} g^0 + \cdots. \tag{IA.75}$$

I next determine sufficient conditions for the existence of an equilibrium. From the previous discussion, one needs the following conditions: (i) $\varepsilon < 1$ (Λ is well-defined and $\Lambda > 0$); (ii) $(A^0 - \varepsilon I)$ is invertible or equivalently $A = A^\varepsilon$ is invertible (g is well-defined); and (iii) the numbers β_k from the proof of Theorem IA.1 satisfy (IA.53) for $k = 1, \ldots, m$ (which implies $\mathsf{Cov}(w_t, \mathrm{d}y_t)$ is independent of t). With the current notation, condition (iii) requires that:[10]

$$1 \;>\; \beta_1 \;>\; \cdots \;>\; \beta_m \;>\; 0, \quad \text{with} \quad \beta_k \;=\; \sum_{i=0}^{m-k} r_i g_{k+i}. \tag{IA.76}$$

[10]One can check that condition (iii) is implied by the condition $g_1 > g_2 > \cdots > g_m > 0$.

I also introduce the following condition that implies (i) and (ii):

$$\text{Equation (IA.75) has a solution } \varepsilon \in \left(0\ ,\ \frac{1}{\frac{N(m+2)^2}{8} + \frac{(m+2)^2}{m+1}}\right). \tag{IA.77}$$

Since $\frac{N(m+2)^2}{8} + \frac{(m+2)^2}{m+1} > 1$, clearly (IA.77) implies $\varepsilon < 1$, which proves (i). The difficult part is to show that (IA.77) also implies that $(A^0 - \varepsilon I)$ is invertible, which proves (ii). For this, one needs a better understanding of the inverse matrix $\left(A^0\right)^{-1}$. Denote by $A^0_{(m)} \in M_{m+1}$ the matrix A^0 from (IA.65) by making explicit the dependence on m. Since A^0 satisfies $A^0_{i,j} = \mathbf{1}_{i=j} - \sum_{k=1}^{\min(i,j)} r_{i-k} r_{j-k}$, it follows that the block $\left(A^0_{(m)}\right)^{11}$, which is obtained $A^0_{(m)}$ by removing the last row and the last column, is the same as $A^0_{(m-1)}$. One then obtains:

$$A^0_{(m)} = \begin{bmatrix} A^0_{(m-1)} & a_{(m)} \\ a'_{(m)} & \alpha_{(m)} \end{bmatrix}, \tag{IA.78}$$

for some m-column vector $a_{(m)}$, and scalar $\alpha_{(m)}$. Write the inverse matrix $H_{(m)} = (A^0_{(m)})^{-1}$ also in block format:

$$H_{(m)} = \begin{bmatrix} H^{11}_{(m)} & h_{(m)} \\ h'_{(m)} & \eta_{(m)} \end{bmatrix}. \tag{IA.79}$$

From the theory of block matrices, one has the following formulas:

$$\begin{aligned} \eta_{(m)} &= \frac{1}{\alpha_{(m)} - a'_{(m)}\left(A^0_{(m-1)}\right)^{-1} a_{(m)}}, \\ h_{(m)} &= -\eta_{(m)}\left(A^0_{(m-1)}\right)^{-1} a_{(m)}, \\ H^{11}_{(m)} &= \left(A^0_{(m-1)}\right)^{-1} + \frac{h_{(m)} h'_{(m)}}{\eta_{(m)}}. \end{aligned} \tag{IA.80}$$

By induction, one verifies that:

$$\begin{aligned} \eta_{(m)} &= \frac{(mN+1)(N+1)}{(m+1)N+1}, \\ h_{(m)} &= \frac{N^2}{(m+1)N+1} \left[\, 0\ ,\ 1\ ,\ \cdots\ ,\ m-1 \,\right]'. \end{aligned} \tag{IA.81}$$

Using the equations above, one can now prove various useful formulas. As a first result, I prove by induction that:

$$\det\big(A^0_{(m)}\big) \;=\; \frac{(m+1)N+1}{(N+1)^{m+1}}. \tag{IA.82}$$

For $m=0$, the equality is true, since in this case $A^0_{(0)}=1$. Suppose it is true for $m-1$. From the theory of block matrices,

$$\det\big(A^0_{(m)}\big) \;=\; \det\big(A^0_{(m-1)}\big)\,\det\Big(\alpha_{(m)} - a'_{(m)}\big(A^0_{(m-1)}\big)^{-1}a_{(m)}\Big) \;=\; \frac{\det\Big(A^0_{(m-1)}\Big)}{\eta_{(m)}}, \tag{IA.83}$$

which together with the formula for $\eta_{(m)}$ from (IA.81) proves the induction step. Another useful result is:

$$H_{i,j} \;\geq\; 0, \quad i,j=0,\ldots,m. \tag{IA.84}$$

Indeed, one uses the recursive formula $H^{11}_{(m)} = H_{(m-1)} + \frac{h_{(m)}h'_{(m)}}{\eta_{(m)}}$ and the explicit formulas in (IA.81) to verify by induction that all entries of $H=H_{(m)}$ are positive.[11]

In order to prove that the matrix $(A^0-\varepsilon I)$ is invertible, I rewrite equation (IA.85):

$$\big(A^0-\varepsilon I\big)^{-1} \;=\; H\,\Big(1+\varepsilon H+\varepsilon^2H^2+\varepsilon^3H^3+\cdots\Big). \tag{IA.85}$$

Thus, if one can show that the right-hand side is a convergent series (in the space of matrices), then its limit is a matrix that coincides with the matrix inverse $\big(A^0-\varepsilon I\big)^{-1}$. To prove convergence, I use the infinity norm, $\|H\|_\infty$, which is the maximum absolute row sum of the matrix, i.e., $H = \max\limits_{i=0,\ldots,m}\sum_{j=0}^m |H_{i,j}|,$. Thus, if one can show that $\|\varepsilon H\|_\infty < 1$, this proves condition (ii).

I now search for an upper bound for $\|H\|_\infty$. For instance, I show that $\big\|\frac{H}{N(m+2)^2}\big\|_\infty \leq \frac{1}{4}\big(1+O_N\frac{1}{N})\big)$. For this, define $\bar{h}_{(m)}$ the $(m+1)$-column vector given by $\big(\bar{h}_{(m)}\big)_i = (N+1)\big((m+1)N+1\big)\sum_{j=0}^m\big(H_{(m)}\big)_{i,j}$. This is proved by induction to be a polynomial in N of degree 3. Denote by $C_{(m)}$ the vector of coefficients of N^3 in $\bar{h}_{(m)}$. Note that $\max\limits_{i=0,\ldots,m}\bar{h}_{(m)} = N^2(m+1)\|H\|_\infty\big(1+O_N\frac{1}{N})\big)$. At the same time, one has $\max\limits_{i=0,\ldots,m}\bar{h}_{(m)} = N^3\max\limits_{i=0,\ldots,m}C_{(m)}$, which implies $\big\|\frac{H}{N(m+2)^2}\big\|_\infty = \frac{1}{(m+1)(m+2)^2}\max\limits_{i=0,\ldots,m}C_{(m)}\,\big(1+O_N\frac{1}{N})\big)$. Now one computes $C_{(0)}=0$, and for $m>1$ one uses the recursive formulas above for H to get a recursive formula for C. More precisely, $\Big(\frac{C_{(m)}}{m+1}\Big)_i = \Big(\frac{C_{(m-1)}}{m}\Big)_i + \frac{i}{2}$ for $i=0,\ldots,m-1$, and $\Big(\frac{C_{(m)}}{m+1}\Big)_m = \frac{m}{2}$. By induction then, one shows that $\max\limits_i\big(C_{(m)}\big)_i \leq \frac{(m+1)^2(m+2)}{4}$, which implies the upper bound stated above for $\|H\|_\infty$. By similar methods, one verifies a

[11] The inequality is strict except that $H_{i,0}=H_{0,i}=0$ for $i>0$.

sharper estimate:

$$\left\| \frac{H}{N(m+2)^2} \right\|_\infty \;\le\; \frac{1}{8} + \frac{1}{(m+1)N}. \tag{IA.86}$$

Note that condition (IA.77) implies $\varepsilon\, N(m+2)^2 < \frac{1}{\frac{1}{8}+\frac{1}{(m+1)N}}$, which together with (IA.86) implies:

$$\varepsilon\, \|H\|_\infty \;<\; 1. \tag{IA.87}$$

This proves that the series $1+\varepsilon H+\varepsilon^2 H^2+\varepsilon^3 H^3+\cdots$ is convergent, and that the limit coincides with $\big(A^0-\varepsilon I\big)^{-1}$.

Next, I analyze how well (Λ, r, g) approximate (Λ^0, r^0, g^0). Recall that:

$$\gamma \;=\; \frac{\bar{\gamma}}{N} \;=\; \frac{g\tilde{\sigma}_y}{N}, \qquad \rho \;=\; \frac{r}{\tilde{\sigma}_y}, \qquad \lambda \;=\; \frac{\Lambda}{\tilde{\sigma}_y}, \tag{IA.88}$$

where from (IA.63) one obtains:

$$\tilde{\sigma}_y \;=\; \sqrt{N+1}\,\tilde{\sigma}_u \;=\; \sqrt{N+1}\,\frac{\sigma_u}{\sigma_w}. \tag{IA.89}$$

Note that in the statement of Theorem IA.2, I have defined $(i=0,\ldots,m)$:

$$\gamma_i^0 \;=\; \frac{\tilde{\sigma}_y}{N+1}\,\frac{m-i+1}{m+1}, \qquad \rho_i^0 \;=\; \frac{1}{\tilde{\sigma}_y}\,\frac{N}{(N+1)^{i+1}}, \qquad \lambda^0 \;=\; \frac{1}{\tilde{\sigma}_y}. \tag{IA.90}$$

Condition (IA.77) implies $\varepsilon < \frac{1}{\frac{N(m+2)^2}{8}+\frac{(m+2)^2}{m+1}}$, which shows that $\varepsilon = O_N\big(\frac{1}{N}\big)$. Also, since $\Lambda = \sqrt{1-\varepsilon}$, it follows that $\Lambda = 1 - O_N\big(\frac{1}{N}\big)$. Thus, one obtains:

$$\varepsilon \;=\; O_N\big(\tfrac{1}{N}\big), \qquad \Lambda \;=\; \sqrt{1-\varepsilon} \;=\; 1 - O_N\big(\tfrac{1}{N}\big). \tag{IA.91}$$

Now, from (IA.88) and (IA.90), one gets $\frac{\lambda}{\lambda^0} = \Lambda = 1 - O_N\big(\frac{1}{N}\big)$. This proves the approximate equation for λ in (IA.17). From (IA.88) and (IA.90), I also compute $\frac{\rho_i}{\rho_i^0} = \frac{r_i}{r_i^0}$, since $r_i^0 = \frac{N}{(N+1)^{i+1}}$. But $r = \frac{1}{\Lambda} r^0$, which implies $\frac{\rho_i}{\rho_i^0} = \frac{1}{\Lambda} = 1 + O_N\big(\frac{1}{N}\big)$. This proves the approximate equation for ρ_i in (IA.17). Finally, from (IA.88) and (IA.90), one gets $\frac{\gamma_i}{\gamma_i^0} = \frac{g_i}{\frac{N}{N+1}\frac{m-i+1}{m+1}} = \frac{g_i}{g_i^0}\big(1+O_N(\frac{1}{N})\big)$. I now show that $\frac{g_i}{g_i^0} = O_N(1)$, which proves the approximate equation for γ_i in (IA.17). Since $\gamma_i^0 = O_N(1)$, it is enough to show that $g_i - g_i^0 = O_N(1)$, or from (IA.91) it is enough to show $\frac{g_i}{\Lambda} - g_i^0 = O_N(1)$. If I combine equations (IA.71) and (IA.85), and use $\big(A^0\big)^{-1} r^0 = g^0$, one obtains:

$$\frac{g}{\Lambda} \;=\; \Big(1+\varepsilon H+\varepsilon^2 H^2+\varepsilon^3 H^3+\cdots\Big)\, g^0. \tag{IA.92}$$

Therefore, one gets $\frac{g}{\Lambda} - g^0 = \left(\varepsilon H + \varepsilon^2 H^2 + \varepsilon^3 H^3 + \cdots\right) g^0$. But (IA.87) implies that the convergent series of matrices is of the order $O_N(1)$, hence it remains of order $O_N(1)$ when multiplied with $g^0 = O_N(1)$. ■

Proof of Proposition IA.3. Following the proof of Theorem IA.1, consider a speculator who must choose the weights $\gamma_{i,t}$ on $\mathrm{d}_t w_{t-i}$. He assumes that all the other speculators use γ_i^*, hence with an aggregate weight of $(N-1)\gamma_i^*$ on $\mathrm{d}_t w_{t-i}$. Then, equation (IA.33) for the speculator's normalized expected profit at $t = 0$ becomes:

$$\begin{aligned}
\tilde{\pi}_0 &= \frac{\pi_0}{\sigma_w^2} = \frac{1}{\sigma_w^2} \mathsf{E}\left(\int_0^T (w_t - p_{t-1} - \lambda \mathrm{d}y_t)\mathrm{d}x_t\right) \\
&= \mathsf{E}\left(\int_0^T \left(w_t - \lambda \sum_{i=0}^m \left(\gamma_{i,t} + (N-1)\gamma_i^*\right)\mathrm{d}_t w_{t-i}\right) \sum_{j=0}^m \gamma_{j,t} \mathrm{d}_t w_{t-j}\right) \qquad \text{(IA.93)} \\
&= \sum_{j=0}^m B_j \gamma_{j,t} - \lambda \sum_{i,j=0}^m \left(\gamma_{i,t} + (N-1)\gamma_i^*\right) A_{i,j} \gamma_{j,t}.
\end{aligned}$$

The first order condition with respect to $\gamma_{k,t}$, for $k = 0, \ldots, m$, is:

$$B_k - \lambda \sum_{i=0}^m \left(2\gamma_{i,t} + (N-1)\gamma_i^*\right) A_{i,k} = 0. \qquad \text{(IA.94)}$$

Since this equation is true for all speculators, one obtains that all $\gamma_{i,t}$ are equal and independent on t, i.e., $\gamma_i = \gamma_i^* = \frac{\bar{\gamma}}{N}$. Using matrix notation, $B = \lambda(N+1)A\gamma$, hence $B = \lambda \frac{N+1}{N} A\bar{\gamma}$.[12] Thus, $\frac{N}{N+1} B = \lambda A \bar{\gamma}$, which implies $B - \lambda A \bar{\gamma} = \frac{B}{N+1}$. Thus, in equilibrium the normalized expected profit is equal to:

$$\tilde{\pi}_0 = \sum_{j=0}^m \left(B_j - \lambda \sum_{i=0}^m A_{j,i} \bar{\gamma}_i\right) \gamma_j = \sum_{j=0} \frac{B_j}{N+1} \gamma_j. \qquad \text{(IA.95)}$$

From equation (IA.61), $B_j = \frac{1}{(N+1)^j}$. One computes:

$$\tilde{\pi}_0 = \sum_{j=0}^m \tilde{\pi}_{0,j} = \sum_{j=0}^m \frac{\gamma_j}{(N+1)^{j+1}}, \qquad \text{(IA.96)}$$

[12] Since $\Lambda = \lambda \tilde{\sigma}_y$ and $g = \frac{\bar{\gamma}}{\tilde{\sigma}_y}$, one gets $\frac{1}{\Lambda} B = \frac{N+1}{N} A g$, which provides a direct proof of the first equation in (IA.59).

which proves (IA.18). The ratio of two consecutive components is:

$$\frac{\tilde{\pi}_{0,j+1}}{\tilde{\pi}_{0,j}} = \frac{\gamma_{j+1}}{\gamma_j} \frac{1}{N+1} = \frac{g_{j+1}}{g_j} \frac{1}{N+1}. \tag{IA.97}$$

But in the proof of Theorem IA.2, I show that $g_j = O_N(1)$. Thus, $\frac{\tilde{\pi}_{0,j+1}}{\tilde{\pi}_{0,j}} = O_N\left(\frac{1}{N}\right)$, which finishes the proof. ■

Proof of Proposition IA.4. Let $\sigma_w^2 - \sigma_p^2 = \sigma_w^2(1 - \lambda^2\tilde{\sigma}_y^2) = \sigma_w^2(1 - \Lambda^2)$. In the proof of Theorem IA.2, one has $1 - \Lambda^2 = \varepsilon$, and this is strictly positive according to the condition (IA.77). Moreover, from (IA.91), one has $\varepsilon = O_N\left(\frac{1}{N}\right)$, which finishes the first part of the Proposition.

For the second part, one only needs to prove the equation $SR_k = \frac{N}{(N+1)^{k+1}}$ when $k = 0, \dots, m$. From the definition of SR_k, one obtains:

$$SR_k = \frac{\lambda \, \mathsf{Cov}(\mathrm{d}w_{t-k}, \mathrm{d}y_t)}{\sigma_w^2 \mathrm{d}t} = \frac{\lambda \rho_k \, \mathsf{Var}(\mathrm{d}y_t)}{\sigma_w^2 \mathrm{d}t} = \lambda \rho_k \tilde{\sigma}_y^2 = \frac{N}{(N+1)^{k+1}}, \tag{IA.98}$$

where the second equality comes from (IA.46), and the last equation comes from (IA.15). ■

Figure IA.1: Optimal trading weights

Consider the model with $m \in \{1, 2, 5, 20\}$ lags and $N \in \{1, 5, 100\}$ identical speculators. The figure shows the rescaled aggregate weight $g_i = N\gamma_i \frac{1}{\sqrt{N+1}} \frac{\sigma_w}{\sigma_u}$ (continuous line) against the lag $i = 0, \ldots, m$, and compares it with the value $g_i^0 = \frac{N}{N+1} \frac{(m-i+1)N+1}{(m+1)N+1}$ (dashed line).

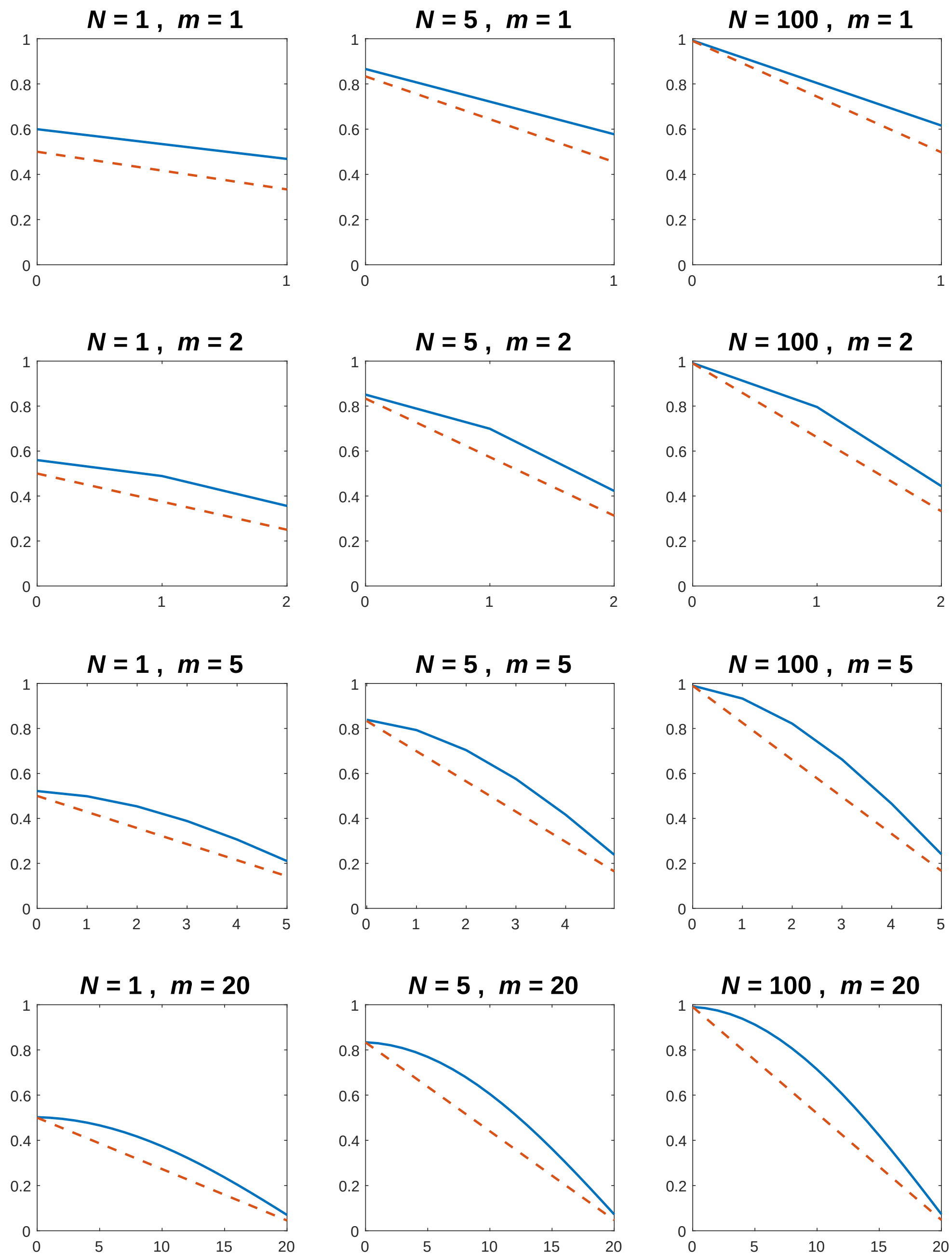
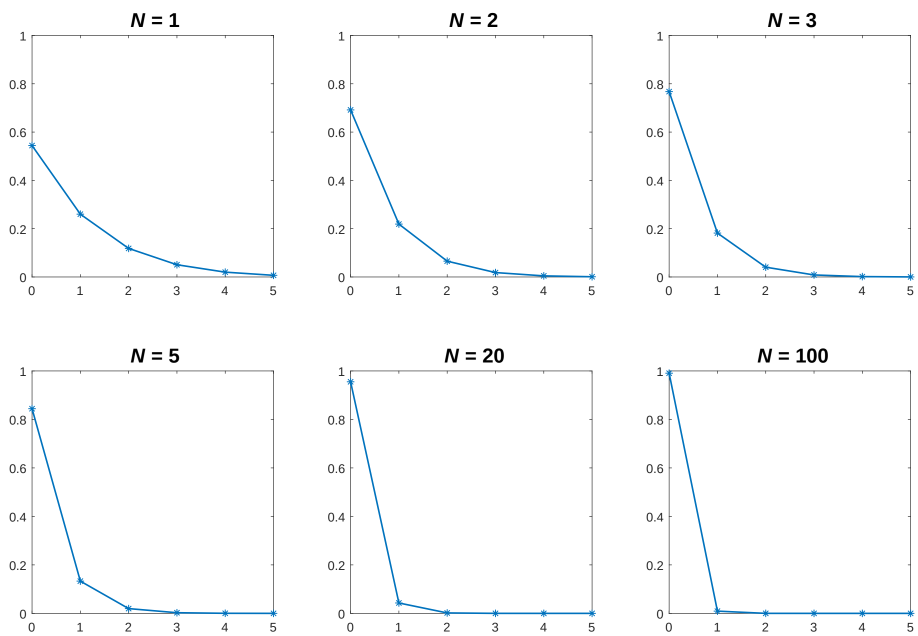


Figure IA.2: Profit from lagged signals

The figure shows the percentage of a speculator's profit from each his lagged signals when there is competition among $N \in \{1, 2, 3, 5, 20, 100\}$ identical speculators. In these examples, the speculators can trade up to $m = 5$ lagged signals.



2 Fast, medium, and slow informed trading

2.1 Benchmark model with $m = 2$ lags

I consider the benchmark model in which speculators can use signals with $m = 2$ lags. Thus, there are three types of speculators: fast trader (FT), medium trader (MT), and slow trader (ST). Recall that $t - k$ denotes $t - k\mathrm{d}t$.

I look for an equilibrium with the following properties: (i) the equilibrium is symmetric, in the sense that the FTs have identical trading strategies, and the same for the MTs and STs; and (ii) the equilibrium coefficients are constant with respect to time. For simplicity, I assume that all coefficients are constant, but the analysis carries through with non-constant coefficients as well.

To solve for the equilibrium, in the first step the speculators' trading strategies are taken as given, and I compute the dealer's pricing functions. In the second step, the dealer's pricing functions are taken as given, and I solve for the optimal trading strategies for the FTs and STs.

Dealer's pricing rules (λ, ρ, A, B)

According to the model timeline, before trading at t the dealer observes $\mathrm{d}w_{t-2}$ (or equivalently w_{t-2}). Then, she observes the order flow $\mathrm{d}y_t$ and sets the price p_t at which trading takes place:

$$p_t \;=\; \mathsf{E}\big(w_t \mid \mathcal{I}_t, \mathrm{d}y_t\big), \quad \text{with} \quad \mathcal{I}_t \;=\; \{\mathrm{d}y_{t-1}, \mathrm{d}y_{t-2}, \ldots\}. \tag{IA.99}$$

The aggregate order flow satisfies:

$$\mathrm{d}y_t \;=\; \bar{\gamma}\mathrm{d}w_t + \bar{\mu}\widetilde{\mathrm{d}w}_{t-1} + \bar{\nu}\widetilde{\widetilde{\mathrm{d}w}}_{t-2} + \mathrm{d}u_t, \tag{IA.100}$$

where $\bar{\gamma}$, $\bar{\mu}$, and $\bar{\nu}$ are the aggregate trading coefficients, $\widetilde{\mathrm{d}w}_{t-1}$ is the unexpected part of $\mathrm{d}w_{t-1}$ (the component orthogonal on $\mathcal{I}_t$), and $\widetilde{\mathrm{d}w}_{t-2}$ is the unexpected part of $\mathrm{d}w_{t-2}$:

$$\begin{aligned} \widetilde{\mathrm{d}w}_{t-1} \;&=\; \mathrm{d}w_{t-1} - \widehat{\mathrm{d}w}_{t-1}, \qquad & \widetilde{\widetilde{\mathrm{d}w}}_{t} \;&=\; \mathrm{d}w_{t-2} - \widehat{\widehat{\mathrm{d}w}}_{t-2}, \quad \text{with} \\ \widehat{\mathrm{d}w}_{t-1} \;&=\; \mathsf{E}\big(\mathrm{d}w_{t-1} \mid \mathcal{I}_t\big), \qquad & \widehat{\widehat{\mathrm{d}w}}_{t-2} \;&=\; \mathsf{E}\big(\mathrm{d}w_{t-2} \mid \mathcal{I}_t\big). \end{aligned} \tag{IA.101}$$

I introduce the following notation, where for simplicity, the t subscript is omitted:

$$\begin{aligned} A_{11} &= \frac{\mathsf{E}\big[(\widetilde{\mathrm{d}w_t})^2\big]}{\sigma_w^2 \mathrm{d}t}, & A_{12} &= \frac{\mathsf{E}\big[\widetilde{\mathrm{d}w_t}\widetilde{\widetilde{\mathrm{d}w}}_{t-1}\big]}{\sigma_w^2 \mathrm{d}t}, & A_{22} &= \frac{\mathsf{E}\big[(\widetilde{\widetilde{\mathrm{d}w}}_{t-1})^2\big]}{\sigma_w^2 \mathrm{d}t}, \\ B_1 &= \frac{\mathsf{E}\big[w_t\widetilde{\mathrm{d}w_t}\big]}{\sigma_w^2 \mathrm{d}t}, & B_2 &= \frac{\mathsf{E}\big[w_t\widetilde{\widetilde{\mathrm{d}w}}_{t-1}\big]}{\sigma_w^2 \mathrm{d}t}, & Y &= \frac{\mathsf{E}\big[(\mathrm{d}y_t)^2\big]}{\sigma_w^2 \mathrm{d}t}. \end{aligned} \tag{IA.102}$$

Note that $\mathrm{d}y_{t-1}$ is orthogonal on $\mathcal{I}_{t-1}$, hence it is also orthogonal on $\mathrm{d}y_{t-2}$. Also, $\mathrm{d}w_{t-1}$ is orthogonal on $\mathcal{I}_{t-1}$. The dealer computes:

$$\begin{aligned} \widehat{\mathrm{d}w}_{t-1} &= \mathsf{E}\big(\mathrm{d}w_{t-1} \mid \mathrm{d}y_{t-1}\big) = \rho \mathrm{d}y_{t-1}, \quad \text{with} \quad \rho = \frac{\mathsf{Cov}(\mathrm{d}w_{t-1}, \mathrm{d}y_{t-1})}{\mathsf{Var}(\mathrm{d}y_{t-1})}, \\ \widehat{\widehat{\mathrm{d}w}}_{t-2} &= \mathsf{E}\big(\mathrm{d}w_{t-2} \mid \mathrm{d}y_{t-1}, \mathrm{d}y_{t-2}\big) = \rho' \mathrm{d}y_{t-1} + \rho \mathrm{d}y_{t-2}, \quad \text{with} \quad \rho' = \frac{\mathsf{Cov}(\mathrm{d}w_{t-2}, \mathrm{d}y_{t-1})}{\mathsf{Var}(\mathrm{d}y_{t-1})}. \end{aligned} \tag{IA.103}$$

Using (IA.100) and (IA.102), one obtains:

$$\begin{aligned} Y &= \bar{\gamma}^2 + \bar{\mu}^2 A_{11} + \bar{\nu}^2 A_{22} + 2\bar{\mu}\bar{\nu} A_{12} + \tilde{\sigma}_u^2, \qquad \rho = \frac{\mathsf{Cov}(\mathrm{d}w_t, \mathrm{d}y_t)}{\mathsf{Var}(\mathrm{d}y_t)} = \frac{\bar{\gamma}}{Y}, \\ \rho' &= \frac{\mathsf{Cov}(\mathrm{d}w_{t-1}, \mathrm{d}y_t)}{\mathsf{Var}(\mathrm{d}y_t)} = \frac{\mathsf{Cov}(\widetilde{\mathrm{d}w}_{t-1}, \mathrm{d}y_t)}{\mathsf{Var}(\mathrm{d}y_t)} = \frac{\bar{\mu} A_{11} + \bar{\nu} A_{12}}{Y}. \end{aligned} \tag{IA.104}$$

Next, one computes the price $p_t = \mathsf{E}(w_t | \mathcal{I}_t, \mathrm{d}y_t) = p_{t-1} + \mathsf{E}(w_t - p_{t-1} | \mathcal{I}_t, \mathrm{d}y_t)$. But $\mathrm{d}y_t$ and $w_t - p_{t-1}$ are orthogonal on $\mathcal{I}_t$, which includes p_{t-1}, therefore one obtains:

$$p_t = p_{t-1} + \lambda \mathrm{d}y_t, \tag{IA.105}$$

where:

$$\lambda = \frac{\mathsf{Cov}(w_t, \mathrm{d}y_t)}{\mathsf{Var}(\mathrm{d}y_t)} = \frac{\bar{\gamma} + \bar{\mu} B_1 + \bar{\nu} B_2}{Y}. \tag{IA.106}$$

Note that (IA.104) and (IA.106) imply that:

$$\frac{\mathsf{E}\big[\mathrm{d}w_t \mathrm{d}y_t\big]}{\sigma_w^2 \mathrm{d}t} = \rho Y, \qquad \frac{\mathsf{E}\big[\mathrm{d}w_{t-1} \mathrm{d}y_t\big]}{\sigma_w^2 \mathrm{d}t} = \rho' Y, \qquad \frac{\mathsf{E}\big[w_t \mathrm{d}y_t\big]}{\sigma_w^2 \mathrm{d}t} = \lambda Y. \tag{IA.107}$$

Using (IA.107), one obtains the following formulas for A_{ij} and B_i:

$$
\begin{aligned}
A_{11} &= \frac{\mathsf{E}\big[(\mathrm{d}w_t - \rho \mathrm{d}y_t)^2\big]}{\sigma_w^2 \mathrm{d}t} \;=\; 1 - 2\rho^2 Y + \rho^2 Y \;=\; 1 - \rho^2 Y, \\
A_{12} &= \frac{\mathsf{E}\big[(\mathrm{d}w_t - \rho \mathrm{d}y_t)(\mathrm{d}w_{t-1} - \rho' \mathrm{d}y_t - \rho \mathrm{d}y_{t-1})\big]}{\sigma_w^2 \mathrm{d}t} \;=\; -\rho'\rho Y - \rho\rho' Y + \rho\rho' Y \;=\; -\rho\rho' Y, \\
A_{22} &= \frac{\mathsf{E}\big[(\mathrm{d}w_{t-1} - \rho' \mathrm{d}y_t - \rho \mathrm{d}y_{t-1})^2\big]}{\sigma_w^2 \mathrm{d}t} \;=\; 1 + \rho'^2 Y + \rho^2 Y - 2\rho'^2 Y - 2\rho^2 Y \;=\; 1 - \rho^2 Y - \rho'^2 Y, \\
B_1 &= \frac{\mathsf{E}\big[w_t(\mathrm{d}w_t - \rho \mathrm{d}y_t)\big]}{\sigma_w^2 \mathrm{d}t} \;=\; 1 - \rho\lambda Y, \\
B_2 &= \frac{\mathsf{E}\big[w_t(\mathrm{d}w_{t-1} - \rho' \mathrm{d}y_t - \rho \mathrm{d}y_{t-1})\big]}{\sigma_w^2 \mathrm{d}t} \;=\; 1 - \rho'\lambda Y - \rho\lambda Y.
\end{aligned}
\tag{IA.108}
$$

I introduce the following notation:

$$
a \;=\; \rho\bar{\gamma}, \qquad b \;=\; \rho\bar{\mu}, \qquad c \;=\; \rho\bar{\nu}, \qquad \delta \;=\; \rho^2 \tilde{\sigma}_u^2, \qquad r \;=\; \frac{\rho'}{\rho}, \qquad R \;=\; \frac{\lambda}{\rho}. \tag{IA.109}
$$

Using (IA.104) and (IA.108) one computes:

$$
\begin{aligned}
&\rho^2 Y \;=\; a, \qquad \rho\rho' Y \;=\; ra, \qquad \rho'^2 Y \;=\; r^2 a, \\
&A_{11} \;=\; 1 - a, \qquad A_{12} \;=\; -ra, \qquad A_{22} \;=\; 1 - a - r^2 a, \\
&B_1 \;=\; 1 - Ra, \qquad B_2 \;=\; 1 - rRa - Ra, \\
&ra \;=\; \rho\rho' Y \;=\; bA_{11} + cA_{12} \;=\; b(1-a) - cra \;\Longrightarrow\; r \;=\; \frac{b(1-a)}{a(1+c)}, \\
&Ra \;=\; \rho\lambda Y \;=\; a + bB_1 + cB_2 \;=\; a + b + c - Ra(b + c + rc) \\
&\qquad \Longrightarrow\; R \;=\; \frac{a+b+c}{a(1+b+c+rc)} \;=\; \frac{(1+c)(a+b+c)}{a(1+c)^2 + b(a+c)}, \\
&a = \rho^2 Y \;=\; a^2 + b^2(1-a) + c^2(1-a-r^2 a) + 2bc(-ra) + \delta \\
&\qquad \Longrightarrow\; \delta \;=\; \frac{(1-a)(a-c^2)\big(a - (\frac{b}{1+c})^2\big)}{a}.
\end{aligned}
\tag{IA.110}
$$

One also computes:

$$
\begin{aligned}
&A_{11} \;=\; 1 - a, \qquad A_{12} \;=\; -\frac{b(1-a)}{1+c}, \qquad A_{22} \;=\; \frac{(1-a)\big(a(1+c)^2 - b^2(1-a)\big)}{a(1+c)^2}, \\
&B_1 \;=\; \frac{1-a+rc}{1+b+c+rc}, \qquad B_2 \;=\; \frac{1-a-r(a+b)}{1+b+c+rc}.
\end{aligned}
\tag{IA.111}
$$

One needs the variances $\rho^2 Y = a$, δ, $A_{11} = 1 - a$ and A_{22} to be positive, which is equivalent to:

$$\max\Big\{\big(\tfrac{b}{1+c}\big)^2, c^2\Big\} \;<\; a \;<\; 1. \tag{IA.112}$$

Speculators' optimal strategy (γ, μ, ν)

Consider a FT, indexed by $i = 1, \ldots, N_F$. At $t = 0$ he chooses a trading strategy of the form:

$$\mathrm{d}x_t^i \;=\; \gamma^i \mathrm{d}w_t + \mu^i \widetilde{\mathrm{d}w}_{t-1} + \nu^i \widetilde{\widetilde{\mathrm{d}w}}_{t-2}, \tag{IA.113}$$

where $\widetilde{\mathrm{d}w}$ and $\widetilde{\widetilde{\mathrm{d}w}}$ satisfy:

$$\widetilde{\mathrm{d}w}_{t-1} \;=\; \mathrm{d}w_{t-1} - \rho \mathrm{d}y_{t-1}, \qquad \widetilde{\widetilde{\mathrm{d}w}}_{t-2} \;=\; \mathrm{d}w_{t-2} - \rho' \mathrm{d}y_{t-1} - \rho \mathrm{d}y_{t-2}, \tag{IA.114}$$

with fixed coefficients ρ and ρ'.[13] The aggregate order flow is of the form:

$$\begin{aligned} \mathrm{d}y_t \;&=\; \bar{\gamma}\mathrm{d}w_t + \bar{\mu}\widetilde{\mathrm{d}w}_{t-1} + \bar{\nu}\widetilde{\widetilde{\mathrm{d}w}}_{t-2} + \mathrm{d}u_t, \quad \text{with} \\ \bar{\gamma} \;&=\; \gamma^i + \gamma^{-i}, \qquad \bar{\mu} \;=\; \mu^i + \mu^{-i}, \qquad \bar{\nu} \;=\; \nu^i + \nu^{-i}. \end{aligned} \tag{IA.115}$$

FT i takes as given the coefficients γ^{-i} and μ^{-i}, and assumes the following functional form for the price:

$$p_t \;=\; p_{t-1} + \lambda \mathrm{d}y_t, \tag{IA.116}$$

with fixed coefficient λ. The normalized expected profit of FT i at $t = 0$ is $\frac{1}{\sigma_w^2}\,\mathsf{E}\int_0^T (w_t - p_t)\mathrm{d}x_t^i$:

$$\begin{aligned} \tilde{\pi}_F^i \;&=\; \frac{1}{\sigma_w^2}\,\mathsf{E}\int_0^T \Big[w_t - p_{t-1} - \lambda\big(\bar{\gamma}\mathrm{d}w_t + \bar{\mu}\widetilde{\mathrm{d}w}_{t-1} + \bar{\nu}\widetilde{\widetilde{\mathrm{d}w}}_{t-2} + \mathrm{d}u_t\big)\Big]\Big(\gamma^i \mathrm{d}w_t + \mu^i \widetilde{\mathrm{d}w}_{t-1} + \nu^i \widetilde{\widetilde{\mathrm{d}w}}_{t-2}\Big) \\ &=\; \gamma^i - \lambda\gamma^i\bar{\gamma} + \mu^i B_1 - \lambda\mu^i\bar{\mu}A_{11} - \lambda(\mu^i\bar{\nu} + \nu^i\bar{\mu})A_{12} + \nu^i B_2 - \lambda\nu^i\bar{\nu}A_{22}, \end{aligned} \tag{IA.117}$$

[13] In equilibrium, $\widetilde{\mathrm{d}w}_{t-1}$ and $\widetilde{\widetilde{\mathrm{d}w}}_{t-2}$ are, respectively, the components of $\mathrm{d}w_{t-1}$ and $\mathrm{d}w_{t-2}$ that are orthogonal to the information set $\mathcal{I}_t$ before trading at t.

where the coefficients A_{ij} and B_i are computed by the dealer. The first order conditions to maximize $\tilde{\pi}_F^i$ are:

$$\begin{cases} 1-\lambda(\gamma^i+\bar{\gamma}) \;=\; 0, \\ B_1-\lambda(\mu^i+\bar{\mu})A_{11}-\lambda(\nu^i+\bar{\nu})A_{12} \;=\; 0, \\ B_2-\lambda(\mu^i+\bar{\mu})A_{12}-\lambda(\nu^i+\bar{\nu})A_{22} \;=\; 0. \end{cases} \tag{IA.118}$$

The second order condition with respect to γ^i is that $\lambda > 0$. The Hessian matrix for the variables other than γ is $-A$, where:

$$A \;=\; \begin{bmatrix} A_{11} & A_{12} \\ A_{12} & A_{22} \end{bmatrix}. \tag{IA.119}$$

The conditions in (IA.112) must hold in equilibrium, hence one obtains the following inequalities:

$$A_{11} \;>\; 0, \qquad A_{22} \;>\; 0, \qquad \det(A) \;=\; \frac{(1-a)^2\big(a(1+c)^2-b^2\big)}{a(1+c)^2} \;>\; 0. \tag{IA.120}$$

These inequalities show that $-A$ is negative definite, hence the second order condition is satisfied.

Note that the first order conditions for MT i are the same as the last two equations from (IA.118), as γ_i does not appear in these last two equations. This implies that the MTs trade with the same coefficients μ^i and ν^i as the FTs. The situation is different for the STs, however, because the last equation in (IA.118) changes when $\mu^i = 0$. The first order condition for ST i is:

$$B_2-\lambda\bar{\mu}A_{12}-\lambda(\nu^i+\bar{\nu})A_{22} \;=\; 0. \tag{IA.121}$$

I search for a symmetric equilibrium in the sense that each type of speculator has the same optimal strategy. Define by N_L the number of speculators who trade on lagged signals (with lag one), and by N_D the number of speculators who trade on double-lagged signals (with lag two):

$$N_L \;=\; N_F+N_M, \qquad N_D \;=\; N_F+N_M+N_S. \tag{IA.122}$$

As explained above, one expects the coefficients μ and ν to be the same for the FTs and MTs, but the coefficient ν is not the same for the MTs and STs. I keep the notation ν

for the FTs and MTs, but denote by ν' the coefficient for the STs. One then has:

$$\bar{\gamma} \;=\; N_F\gamma, \qquad \bar{\mu} \;=\; N_L\mu, \qquad \bar{\nu} \;=\; N_L\nu + N_S\nu'. \tag{IA.123}$$

The first order conditions in (IA.118) and (IA.121) imply the following system:

$$\begin{cases} (N_F+1)\gamma \;=\; \frac{1}{\lambda}, \\ (N_L+1)\mu A_{11} + \big((N_L+1)\nu + N_S\nu'\big)A_{12} \;=\; \frac{B_1}{\lambda}, \\ (N_L+1)\mu A_{12} + \big((N_L+1)\nu + N_S\nu'\big)A_{22} \;=\; \frac{B_2}{\lambda}, \\ N_L\mu A_{12} + \big(N_L\nu + (N_S+1)\nu'\big)A_{22} \;=\; \frac{B_2}{\lambda}. \end{cases} \tag{IA.124}$$

Taking the difference between the last two equations, one obtains:

$$\nu' - \nu \;=\; \frac{A_{12}}{A_{22}}\mu. \tag{IA.125}$$

I now express the first three equations in (IA.124) in terms of $\bar{\gamma}$, $\bar{\mu}$, and $\bar{\nu}$. Using (IA.125), one has $\bar{\nu} = N_L\nu + N_S\nu' = N_D\gamma + N_S\frac{A_{12}}{A_{22}}\mu$, hence one computes:

$$\begin{aligned} \nu \;&=\; \frac{1}{N_D}\bar{\nu} - \frac{N_S}{N_D}\frac{A_{12}}{A_{22}}\mu, \qquad \nu' \;=\; \frac{1}{N_D}\bar{\nu} + \frac{N_L}{N_D}\frac{A_{12}}{A_{22}}\mu, \\ \Longrightarrow\;\; (N_L+1)\nu + N_S\nu' \;&=\; \frac{N_D+1}{N_D}\bar{\nu} - \frac{N_S}{N_D}\frac{A_{12}}{A_{22}}\mu, \\ \Longrightarrow\;\; (N_L+1)\mu A_{12} + \big((N_L+1)\nu + N_S\nu'\big)A_{22} \;&=\; \frac{N_D+1}{N_D}\big(N_L\mu A_{12} + \bar{\nu}A_{22}\big). \end{aligned} \tag{IA.126}$$

Taking the difference in (IA.124) between the second equation multiplied by A_{22} and the third equation multiplied by A_{12}, one obtains:

$$(N_L+1)\big(A_{11}A_{22} - A_{12}^2\big)\mu \;=\; \frac{B_1A_{22}}{\lambda} - \frac{B_2A_{12}}{\lambda}. \tag{IA.127}$$

Using (IA.126) and (IA.127), the system (IA.128) implies (with the notation in (IA.109)):

$$\begin{cases} \frac{N_F+1}{N_F}\,a \;=\; \frac{1}{R}, \\ \frac{N_L+1}{N_L}\,\big(A_{11}A_{22} - A_{12}^2\big)\,b \;=\; \frac{B_1A_{22}}{R} - \frac{B_2A_{12}}{R}, \\ \frac{N_D+1}{N_D}\big(A_{12}b + A_{22}c\big) \;=\; \frac{B_2}{R}. \end{cases} \tag{IA.128}$$

This system can be solved numerically.

2.2 Learning from order flow

I now assume that the STs are able to learn about the lagged signals by watching the order flow. Thus, the STs observe the aggregate order flow $\mathrm{d}y_{t-1}$ after trading at $t-1$ and receives the double-lagged signal $\mathrm{d}w_{t-2}$. Using all this information, before trading at t the last ST can observe the following component of the lagged order flow:

$$\mathrm{d}y^0_{t-1} \;=\; \bar{\gamma}\mathrm{d}w_{t-1} + \mathrm{d}u_{t-1}. \tag{IA.129}$$

The information set of STs before trading at t is:

$$\mathcal{K}_t \;=\; \{\mathrm{d}w_{t-2}, \mathrm{d}w_{t-3}, \ldots, \mathrm{d}y_{t-1}, \mathrm{d}y_{t-2}, \ldots, \mathrm{d}y^0_{t-1}, \mathrm{d}y^0_{t-2}, \ldots\}, \tag{IA.130}$$

Using this information, STs form a more precise forecast about $\mathrm{d}w_{t-1}$ than the dealer's forecast $\widehat{\mathrm{d}w}_{t-1}$. The dealer's forecast satisfies:

$$\widehat{\mathrm{d}w}_{t-1} \;=\; \mathsf{E}\big(\mathrm{d}w_{t-1} \mid \mathrm{d}y_{t-1}\big) \;=\; \rho\mathrm{d}y_{t-1}, \quad \text{with} \quad \rho \;=\; \frac{\mathsf{Cov}(\mathrm{d}w_{t-1}, \mathrm{d}y_{t-1})}{\mathsf{Var}(\mathrm{d}y_{t-1})}. \tag{IA.131}$$

Denote by $\widehat{\mathrm{d}w}^0_{t-1}$ the last forecast of the ST, and by $\widetilde{\mathrm{d}w}^0_{t-1}$ the part of this forecast that is unexpected by the dealer:

$$\widehat{\mathrm{d}w}^0_{t-1} \;=\; \mathsf{E}\big(\mathrm{d}w_{t-1} \mid \mathcal{K}_t\big), \qquad \widetilde{\mathrm{d}w}^0_{t-1} \;=\; \widehat{\mathrm{d}w}^0_{t-1} - \widehat{\mathrm{d}w}_{t-1}. \tag{IA.132}$$

One computes:

$$\widehat{\mathrm{d}w}^0_{t-1} \;=\; \mathsf{E}\big(\mathrm{d}w_{t-1} \mid \mathrm{d}y^0_{t-1}\big) \;=\; \rho_0\mathrm{d}y^0_{t-1}, \quad \text{with} \quad \rho_0 \;=\; \frac{\mathsf{Cov}(\mathrm{d}w_{t-1}, \mathrm{d}y^0_{t-1})}{\mathsf{Var}(\mathrm{d}y^0_{t-1})} \;=\; \frac{\bar{\gamma}}{\bar{\gamma}^2 + \tilde{\sigma}_u^2}. \tag{IA.133}$$

It follows that $\widetilde{\mathrm{d}w}^0_{t-1}$ satisfies:

$$\widetilde{\mathrm{d}w}^0_{t-1} \;=\; \rho_0\mathrm{d}y^0_{t-1} - \rho\mathrm{d}y_{t-1}. \tag{IA.134}$$

Thus, I assume that ST j has a trading strategy of the form:

$$\mathrm{d}x_t \;=\; \mu^j_0\widetilde{\mathrm{d}w}^0_{t-1} + \nu^j\widetilde{\widetilde{\mathrm{d}w}}_{t-2}. \tag{IA.135}$$

Since the STs trade on $\widetilde{\mathrm{d}w}^0_{t-1} = \widehat{\mathrm{d}w}^0_{t-1} - \widehat{\mathrm{d}w}_{t-1}$, the optimal trading strategy of the FTs and MTs must also include a term proportional to the difference between

$\mathrm{d}w_{t-1} - \widehat{\mathrm{d}w}^0_{t-1}$, which is the difference between the lagged signal (which they observe precisely) and the STs' forecast. But this last difference is the same as $\widetilde{\mathrm{d}w}_{t-1} - \widetilde{\mathrm{d}w}^0_{t-1}$, therefore it is equivalent to assume that the trading strategy of the FTs and MTs includes, besides a term proportional to $\widetilde{\mathrm{d}w}_{t-1}$, also a term proportional to $\widetilde{\mathrm{d}w}^0_{t-1}$. Thus, FT i has a trading strategy of the form:

$$\mathrm{d}x_t \;=\; \gamma^i \mathrm{d}w_t + \mu^i \widetilde{\mathrm{d}w}_{t-1} + \mu^i_0 \widetilde{\mathrm{d}w}^0_{t-1} + \nu^i \widetilde{\widetilde{\mathrm{d}w}}_{t-2}. \tag{IA.136}$$

Dealer's pricing rules (λ, ρ, A, B)

The dealer regards the order flow as being of the form:

$$\mathrm{d}y_t \;=\; \bar{\gamma}\mathrm{d}w_t + \bar{\mu}\widetilde{\mathrm{d}w}_{t-1} + \bar{\mu}_0 \widetilde{\mathrm{d}w}^0_{t-1} + \bar{\nu}\widetilde{\widetilde{\mathrm{d}w}}_{t-2} + \mathrm{d}u_t, \tag{IA.137}$$

where $\bar{\gamma}$, $\bar{\mu}$, $\bar{\mu}_0$, and $\bar{\nu}$ are the aggregate coefficients of all speculators. As in (IA.102), I introduce the following covariances:

$$\begin{aligned}
A_{11} &= \frac{\mathsf{E}\big[(\widetilde{\mathrm{d}w}_t)^2\big]}{\sigma_w^2 \mathrm{d}t}, \qquad A_{12} = \frac{\mathsf{E}\big[\widetilde{\mathrm{d}w}_t \widetilde{\widetilde{\mathrm{d}w}}_{t-1}\big]}{\sigma_w^2 \mathrm{d}t}, \qquad A_{22} = \frac{\mathsf{E}\big[(\widetilde{\widetilde{\mathrm{d}w}}_{t-1})^2\big]}{\sigma_w^2 \mathrm{d}t}, \\
B_1 &= \frac{\mathsf{E}\big[w_t \widetilde{\mathrm{d}w}_t\big]}{\sigma_w^2 \mathrm{d}t}, \qquad B_2 = \frac{\mathsf{E}\big[w_t \widetilde{\widetilde{\mathrm{d}w}}_{t-1}\big]}{\sigma_w^2 \mathrm{d}t}, \qquad Y = \frac{\mathsf{E}\big[(\mathrm{d}y_t)^2\big]}{\sigma_w^2 \mathrm{d}t}, \\
A_{00} &= \frac{\mathsf{E}\big[\big(\widetilde{\mathrm{d}w}^0_t\big)^2\big]}{\sigma_w^2 \mathrm{d}t}, \qquad A_{01} = \frac{\mathsf{E}\big[\widetilde{\mathrm{d}w}^0_t \widetilde{\mathrm{d}w}_t\big]}{\sigma_w^2 \mathrm{d}t}, \qquad A_{02} = \frac{\mathsf{E}\big[\widetilde{\mathrm{d}w}^0_t \widetilde{\widetilde{\mathrm{d}w}}_{t-1}\big]}{\sigma_w^2 \mathrm{d}t}, \\
B_0 &= \frac{\mathsf{E}\big[w_t \widetilde{\mathrm{d}w}^0_t\big]}{\sigma_w^2 \mathrm{d}t}, \qquad Y_0 = \frac{\mathsf{E}\big[\big(\mathrm{d}y^0_t\big)^2\big]}{\sigma_w^2 \mathrm{d}t}, \qquad Z_0 = \frac{\mathsf{E}\big[\mathrm{d}y_t \mathrm{d}y^0_t\big]}{\sigma_w^2 \mathrm{d}t}.
\end{aligned} \tag{IA.138}$$

Equation (IA.137) implies new formulas for Y, ρ, ρ' and λ:

$$\begin{aligned}
Y &= \bar{\gamma}^2 + \bar{\mu}^2 A_{11} + \bar{\mu}_0^2 A_{00} + \bar{\nu}^2 A_{22} + \tilde{\sigma}_u^2 + 2\bar{\mu}\bar{\mu}_0 A_{01} + 2\bar{\mu}\bar{\nu} A_{12} + 2\bar{\mu}_0 \bar{\nu} A_{02}, \\
\rho &= \frac{\mathsf{Cov}(\mathrm{d}w_t, \mathrm{d}y_t)}{\mathsf{Var}(\mathrm{d}y_t)} = \frac{\bar{\gamma}}{Y}, \\
\rho' &= \frac{\mathsf{Cov}(\mathrm{d}w_{t-1}, \mathrm{d}y_t)}{\mathsf{Var}(\mathrm{d}y_t)} = \frac{\mathsf{Cov}(\widetilde{\mathrm{d}w}_{t-1}, \mathrm{d}y_t)}{\mathsf{Var}(\mathrm{d}y_t)} = \frac{\bar{\mu} A_{11} + \bar{\mu}_0 A_{01} + \bar{\nu} A_{12}}{Y}, \\
\lambda &= \frac{\mathsf{Cov}(w_t, \mathrm{d}y_t)}{\mathsf{Var}(\mathrm{d}y_t)} = \frac{\bar{\gamma} + \bar{\mu} B_1 + \bar{\mu}_0 B_0 + \bar{\nu} B_2}{Y}.
\end{aligned} \tag{IA.139}$$

Similar to (IA.109), define:

$$a = \rho\bar{\gamma}, \quad b = \rho\bar{\mu}, \quad b_0 = \rho\bar{\mu}_0, \quad c = \rho\bar{\nu}, \quad \delta = \rho^2\tilde{\sigma}_u^2, \quad r = \frac{\rho'}{\rho}, \quad R = \frac{\lambda}{\rho}. \tag{IA.140}$$

As in (IA.108) and (IA.110), one computes:

$$\begin{aligned}
\rho^2 Y &= a, \qquad \delta = a - a^2 - b^2 A_{11} - b_0^2 A_{00} - c^2 A_{22} - 2bb_0 A_{01} - 2bcA_{12} - 2b_0 c A_{02},\\
A_{11} &= 1 - a, \qquad A_{12} = -ra, \qquad A_{22} = 1 - a - r^2 a,\\
B_1 &= 1 - Ra, \qquad B_2 = 1 - Ra - Rra,\\
r &= \frac{bA_{11} + b_0 A_{01} + cA_{12}}{a}, \qquad R = \frac{a + bB_1 + b_0 B_0 + cB_2}{a}.
\end{aligned} \tag{IA.141}$$

I now compute the covariances that involve $\widetilde{\mathrm{d}w}_t^0$ and $\mathrm{d}y_t^0$. First, note that $Y_0 = Z_0$. From (IA.129) and (IA.133) it follows that:

$$\begin{aligned}
Y_0 &= Z_0 = \bar{\gamma}^2 + \tilde{\sigma}_u^2, \qquad \rho^2 Y_0 = a^2 + \delta, \qquad \rho_0 = \frac{\bar{\gamma}}{Y_0} = \rho\frac{a}{a^2+\delta},\\
\rho_0 Y_0 &= \bar{\gamma}, \qquad \rho\rho_0 Y_0 = a, \qquad \rho_0^2 Y_0 = \rho_0\bar{\gamma} = \frac{a^2}{a^2+\delta}.
\end{aligned} \tag{IA.142}$$

Using (IA.142) one computes:

$$\begin{aligned}
A_{00} &= \frac{\mathsf{E}\left[\left(\rho_0 \mathrm{d}y_t^0 - \rho \mathrm{d}y_t\right)^2\right]}{\sigma_w^2 \mathrm{d}t} = \rho_0^2 Y_0 - 2\rho_0\rho Z_0 + \rho^2 Y = \frac{a^2}{a^2+\delta} - a,\\
A_{01} &= \frac{\mathsf{E}\left[\left(\rho_0 \mathrm{d}y_t^0 - \rho \mathrm{d}y_t\right)\left(\mathrm{d}w_t - \rho \mathrm{d}y_t\right)\right]}{\sigma_w^2 \mathrm{d}t} = \rho_0\bar{\gamma} - \rho_0\rho Z_0 - \rho\bar{\gamma} + \rho^2 Y = \frac{a^2}{a^2+\delta} - a,\\
A_{02} &= \frac{\mathsf{E}\left[\left(\rho_0 \mathrm{d}y_t^0 - \rho \mathrm{d}y_t\right)\left(\mathrm{d}w_{t-1} - \rho' \mathrm{d}y_t - \rho \mathrm{d}y_{t-1}\right)\right]}{\sigma_w^2 \mathrm{d}t} = -\rho_0\rho' Z_0 - \rho\rho' Y + \rho\rho' Y, = -ra,\\
B_0 &= \frac{\mathsf{E}\left[w_t\left(\rho_0 \mathrm{d}y_t^0 - \rho \mathrm{d}y_t\right)\right]}{\sigma_w^2 \mathrm{d}t} = \rho_0\bar{\gamma} - \rho\lambda Y = \frac{a^2}{a^2+\delta} - Ra.
\end{aligned} \tag{IA.143}$$

Speculators' optimal strategy (γ, μ, μ_0, ν)

The normalized expected profit of FT i at $t = 0$ is $\frac{1}{\sigma_w^2} \mathsf{E} \int_0^T (w_t - p_t) \mathrm{d}x_t$:

$$\begin{aligned} \tilde{\pi}_F^i &= \frac{1}{\sigma_w^2} \mathsf{E} \int_0^T \Big[w_t - p_{t-1} - \lambda\big(\bar{\gamma}\mathrm{d}w_t + \bar{\mu}\widetilde{\mathrm{d}w}_{t-1} + \bar{\mu}_0 \widetilde{\mathrm{d}w}^0_{t-1} + \bar{\nu}\widetilde{\widetilde{\mathrm{d}w}}_{t-2} + \mathrm{d}u_t\big)\Big] \cdot \\ &\quad \cdot \Big(\gamma^i \mathrm{d}w_t + \mu^i \widetilde{\mathrm{d}w}_{t-1} + \mu_0^i \widetilde{\mathrm{d}w}^0_{t-1} + \nu^i \widetilde{\widetilde{\mathrm{d}w}}_{t-2}\Big) \\ &= \gamma^i - \lambda\gamma^i\bar{\gamma} + \mu^i B_1 - \lambda\mu^i\bar{\mu}A_{11} - \lambda(\mu^i\bar{\mu}_0 + \mu_0^i\bar{\mu})A_{01} - \lambda(\mu^i\bar{\nu} + \nu^i\bar{\mu})A_{12} \\ &\quad + \mu_0^i B_0 - \lambda\mu_0^i\bar{\mu}_0 A_{00} - \lambda(\mu_0^i\bar{\nu} + \nu^i\bar{\mu}_0)A_{02} + \nu^i B_2 - \lambda\nu^i\bar{\nu}A_{22}, \end{aligned} \tag{IA.144}$$

where the coefficients A_{ij} and B_i are as in (IA.138). Note that FT i regards the aggregate coefficients as functions of his own coefficients: $\bar{\gamma} = \gamma^i + \gamma^{-i}$, $\bar{\mu} = \mu^i + \mu^{-i}$, $\bar{\mu}_0 = \mu_0^i + \mu_0^{-i}$, and $\bar{\nu} = \nu^i + \nu^{-i}$.

The first order conditions to maximize $\tilde{\pi}_F^i$ with respect to γ^i, μ^i, μ_0^i and ν^i are:

$$\begin{cases} 1 - \lambda(\gamma^i + \bar{\gamma}) \;=\; 0, \\ B_1 - \lambda(\mu^i + \bar{\mu})A_{11} - \lambda(\mu_0^i + \bar{\mu}_0)A_{01} - \lambda(\nu^i + \bar{\nu})A_{12} \;=\; 0, \\ B_0 - \lambda(\mu^i + \bar{\mu})A_{01} - \lambda(\mu_0^i + \bar{\mu}_0)A_{00} - \lambda(\nu^i + \bar{\nu})A_{02} \;=\; 0, \\ B_2 - \lambda(\mu^i + \bar{\mu})A_{12} - \lambda(\mu_0^i + \bar{\mu}_0)A_{02} - \lambda(\nu^i + \bar{\nu})A_{22} \;=\; 0. \end{cases} \tag{IA.145}$$

The first order conditions for MT i are the same as in (IA.145), except for the first equation. The first order conditions for ST i are similar to the last two equations in (IA.145), except that the coefficient $\mu^i = 0$:

$$\begin{cases} B_0 - \lambda\bar{\mu}A_{01} - \lambda(\mu_0^i + \bar{\mu}_0)A_{00} - \lambda(\nu^i + \bar{\nu})A_{02} \;=\; 0, \\ B_2 - \lambda\bar{\mu}A_{12} - \lambda(\mu_0^i + \bar{\mu}_0)A_{02} - \lambda(\nu^i + \bar{\nu})A_{22} \;=\; 0. \end{cases} \tag{IA.146}$$

Note that the Hessian matrix with for the coefficients other than γ^i is $-A$, where:

$$A \;=\; \begin{bmatrix} A_{11} & A_{01} & A_{12} \\ A_{01} & A_{00} & A_{02} \\ A_{12} & A_{02} & A_{22} \end{bmatrix}. \tag{IA.147}$$

Recall that in a symmetric equilibrium speculators of the same type have identical coefficients in their trading strategies. Denote by γ, μ, μ_0, and ν the coefficients of the FTs, which are the same as for the MTs (except for γ). Denote also by μ_0' and ν' the coefficients of the STs. The aggregate coefficients then satisfy:

$$\bar{\gamma} = N_F \gamma, \qquad \bar{\mu} = N_L \mu, \qquad \bar{\mu}_0 = N_L \mu_0 + N_S \mu_0', \qquad \bar{\nu} = N_L \nu + N_S \nu'. \quad \text{(IA.148)}$$

Putting together the first order conditions for all speculators, it follows that in a symmetric equilibrium:

$$\begin{cases} \frac{N_F+1}{N_F}\bar{\gamma} = \frac{1}{\lambda}, \\ \frac{N_L+1}{N_L}\bar{\mu}A_{11} + (\mu_0+\bar{\mu}_0)A_{01} + (\nu+\bar{\nu})A_{12} = \frac{B_1}{\lambda}, \\ \frac{N_L+1}{N_L}\bar{\mu}A_{01} + (\mu_0+\bar{\mu}_0)A_{00} + (\nu+\bar{\nu})A_{02} = \frac{B_0}{\lambda}, \\ \frac{N_L+1}{N_L}\bar{\mu}A_{12} + (\mu_0+\bar{\mu}_0)A_{02} + (\nu+\bar{\nu})A_{22} = \frac{B_2}{\lambda}, \\ \bar{\mu}A_{01} + (\mu_0'+\bar{\mu}_0)A_{00} + (\nu'+\bar{\nu})A_{02} = \frac{B_0}{\lambda}, \\ \bar{\mu}A_{12} + (\mu_0'+\bar{\mu}_0)A_{02} + (\nu'+\bar{\nu})A_{22} = \frac{B_2}{\lambda}. \end{cases} \quad \text{(IA.149)}$$

In this system take the difference between the fifth and third equations, and the sixth and fourth equations, to obtain:

$$\begin{aligned} &\mu_0' = \mu_0 + \alpha_0 \mu, \qquad \nu' = \nu + \alpha_2 \mu, \\ &A^{11} = \begin{bmatrix} A_{00} & A_{02} \\ A_{02} & A_{22} \end{bmatrix}, \qquad \begin{bmatrix} \alpha_0 \\ \alpha_2 \end{bmatrix} = \left(A^{11}\right)^{-1} \begin{bmatrix} A_{01} \\ A_{12} \end{bmatrix} \end{aligned} \quad \text{(IA.150)}$$

The equations in (IA.150) replace the last two equations in (IA.149). Using the aggregate coefficient formulas in (IA.148), it follows that the first four equations in (IA.149) are equivalent to:

$$\begin{cases} \frac{N_F+1}{N_F}\bar{\gamma} = \frac{1}{\lambda}, \\ \frac{N_L+1}{N_L}\,\frac{\det(A)}{\det(A^{11})}\,\bar{\mu} = \frac{B_1 - \alpha_0 B_0 - \alpha_2 B_2}{\lambda}, \\ \frac{N_D+1}{N_D}\Big(A_{01}\bar{\mu} + A_{00}\bar{\mu}_0 + A_{02}\bar{\nu}\Big) = \frac{B_0}{\lambda}, \\ \frac{N_D+1}{N_D}\Big(A_{12}\bar{\mu} + A_{02}\bar{\mu}_0 + A_{22}\bar{\nu}\Big) = \frac{B_2}{\lambda}. \end{cases} \quad \text{(IA.151)}$$

Using the formulas in (IA.141) and (IA.143) for A_{ij}, one verifies that:

$$\alpha_0 = 1, \qquad \alpha_2 = 1, \qquad \frac{\det(A)}{\det(A^{11})} = \frac{\delta}{a^2+\delta}. \quad \text{(IA.152)}$$

Therefore with the notation in (IA.140), the system of first order conditions implies:

$$\begin{cases} \frac{N_F+1}{N_F} a \ = \ \frac{1}{R}, \\ \frac{N_L+1}{N_L} \frac{\delta}{a^2+\delta} b \ = \ \frac{B_1-B_0}{R}, \\ \frac{N_D+1}{N_D}\Big(A_{01}b + A_{00}b_0 + A_{02}c\Big) \ = \ \frac{B_0}{R}, \\ \frac{N_D+1}{N_D}\Big(A_{12}b + A_{02}b_0 + A_{22}c\Big) \ = \ \frac{B_2}{R}. \end{cases} \tag{IA.153}$$

Note that equations (IA.150) and (IA.152) imply that:

$$\mu_0' \ = \ \mu_0 + \mu, \qquad \nu' \ = \ \nu. \tag{IA.154}$$

Thus, the trading strategies of the FTs, MTs, and STs are, respectively,

$$\begin{aligned} \mathrm{d}x_t^F \ &= \ \gamma \mathrm{d}w_t + \mu \widetilde{\mathrm{d}w}_{t-1} + \mu_0 \widetilde{\mathrm{d}w}_{t-1}^0 + \nu \widetilde{\widetilde{\mathrm{d}w}}_{t-2}, \\ \mathrm{d}x_t^M \ &= \ \mu \widetilde{\mathrm{d}w}_{t-1} + \mu_0 \widetilde{\mathrm{d}w}_{t-1}^0 + \nu \widetilde{\widetilde{\mathrm{d}w}}_{t-2}, \\ \mathrm{d}x_t^S \ &= \ (\mu + \mu_0) \widetilde{\mathrm{d}w}_{t-1}^0 + \nu \widetilde{\widetilde{\mathrm{d}w}}_{t-2}. \end{aligned} \tag{IA.155}$$

Numerically, it turns out that the optimal coefficients are positive except for μ_0, which is negative. To understand this sign, note that the benefits of trading on $\widetilde{\mathrm{d}w}_{t-1}^0$ are given by B_0, which is negative. But B_0 is by definition the (normalized) instantaneous covariance of w_t with $\widetilde{\mathrm{d}w}_t^0 = \rho_0 \mathrm{d}y_t^0 - \rho \mathrm{d}y_t$, and $\mathrm{d}y_t^0$ only contains the increment $\mathrm{d}w_t$, while $\mathrm{d}y_t$ also contains the lagged increments $\mathrm{d}w_{t-1}$ and $\mathrm{d}w_{t-2}$ which positively covary with w_t.[14]

Now suppose that one slow trader, ST i, stops learning from the order flow. I want to compare his expected profit with that of a regular ST. In general, the equilibrium

[14] A simple example illustrates the negative sign of B_0. Suppose in a one-period model a speculator wants to exploit information about $v = v_1 + v_2$ with IID components $v_i \sim \mathcal{N}(0, \Sigma_i)$, $i = 1, 2$. The speculator observes $v_1 + u$, while the dealer observes $v_1 + v_2 + u$. Suppose the speculator's strategy is to trade on a multiple of the unexpected part $\tilde{v}_1 = \mathsf{E}(v_1 | v_1 + u) - \mathsf{E}(v_1 | v_1 + v_2 + u) = \rho_0(v_1 + u) - \rho(v_1 + v_2 + u)$, where $\rho_0 = \frac{\Sigma_1}{\Sigma_1 + \Sigma_u}$ and $\rho = \frac{\Sigma_1}{\Sigma_1 + \Sigma_2 + \Sigma_u}$. Then, the equivalent of B_0 is the covariance $\mathsf{Cov}(v_1 + v_2, \tilde{v}_1) = -\frac{\Sigma_1 \Sigma_2 \Sigma_u}{(\Sigma_1 + \Sigma_u)(\Sigma_1 + \Sigma_2 + \Sigma_u)} < 0$, and the speculator's optimal strategy has a negative coefficient on $\tilde{v}_1$.

normalized expected profits corresponding to the strategies in (IA.155) are:

$$
\begin{aligned}
\tilde{\pi}_F &= \gamma - \lambda\gamma\bar{\gamma} + \mu B_1 - \lambda\mu\bar{\mu}A_{11} - \lambda(\mu\bar{\mu}_0 + \mu_0\bar{\mu})A_{01} - \lambda(\mu\bar{\nu} + \nu\bar{\mu})A_{12} \\
&\quad + \mu_0 B_0 - \lambda\mu_0\bar{\mu}_0 A_{00} - \lambda(\mu_0\bar{\nu} + \nu\bar{\mu}_0)A_{02} + \nu B_2 - \lambda\nu\bar{\nu}A_{22}, \\
\tilde{\pi}_M &= \mu B_1 - \lambda\mu\bar{\mu}A_{11} - \lambda(\mu\bar{\mu}_0 + \mu_0\bar{\mu})A_{01} - \lambda(\mu\bar{\nu} + \nu\bar{\mu})A_{12} \\
&\quad + \mu_0 B_0 - \lambda\mu_0\bar{\mu}_0 A_{00} - \lambda(\mu_0\bar{\nu} + \nu\bar{\mu}_0)A_{02} + \nu B_2 - \lambda\nu\bar{\nu}A_{22}, \\
\tilde{\pi}_S &= \mu_0 B_0 - \lambda\mu_0\bar{\mu}A_{01} - \lambda\nu\bar{\mu}A_{12} - \lambda\mu_0\bar{\mu}_0 A_{00} - \lambda(\mu_0\bar{\nu} + \nu\bar{\mu}_0)A_{02} + \nu B_2 - \lambda\nu\bar{\nu}A_{22}.
\end{aligned}
\tag{IA.156}
$$

If ST i does not learn from the order flow, then his trading strategy is of the form $\mathrm{d}x_t^i = \nu^i \widetilde{\mathrm{d}w}_{t-2}$. Let ν^- be the equilibrium aggregate coefficient of the other speculators. Then, the normalized expected profit of ST i is:

$$
\begin{aligned}
\tilde{\pi}_S^i &= \nu^i B_2 - \lambda\nu^i\bar{\mu}A_{12} - \lambda\nu^i\bar{\mu}_0 A_{02} - \lambda\nu^i\bar{\nu}A_{22}, \quad \text{with} \\
\bar{\nu} &= \nu^i + \nu^-, \qquad \nu^- = (N_D - 1)\nu.
\end{aligned}
\tag{IA.157}
$$

As long as A_{22} and λ are positive, which are part of the overall second order conditions, ST i maximizes his profit by setting $\nu^i = \frac{B - \lambda\nu^- A_{22}}{2\lambda A_{22}}$, where $B = B_2 - \lambda\bar{\mu}A_{12} - \lambda\bar{\mu}_0 A_{02}$. The corresponding maximum normalized profit is:

$$
\tilde{\pi}_{S,\max}^i = \frac{\left(B_2 - \lambda\bar{\mu}A_{12} - \lambda\bar{\mu}_0 A_{02} - \lambda\nu^- A_{22}\right)^2}{4\lambda A_{22}}. \tag{IA.158}
$$

Numerical results show that this profit is much smaller than $\tilde{\pi}_S$ when ST i can learn from the order flow. Thus, it makes more sense for STs to use their information to infer more recent signals using the order flow.

3 Fast and slow trading with public news

I consider the benchmark model in the paper (with FTs and STs), but assume that signals about the increments of the fundamental value $\mathrm{d}v_t = v_t - v_{t-1}$ are made public with a delay of k periods. Denote by w_t the public forecast of the fundamental value at t. Thus, the increment $\mathrm{d}w_t$ is revealed to the public just before trading at $t-k$. Recall that $t-k$ denotes $t-k\mathrm{d}t$.

3.1 Public revelation after two lags

In this simplest model, $\mathrm{d}w_t$ is observed by FTs at t (just before trading), by STs at $t-1$, and by the public at $t-2$.

I look for an equilibrium with the following properties: (i) the equilibrium is symmetric, in the sense that the FTs have identical trading strategies, and the same for the STs; and (ii) the equilibrium coefficients are constant with respect to time.

To solve for the equilibrium, in the first step the speculators' trading strategies are taken as given, and I compute the dealer's pricing functions. In the second step, the dealer's pricing functions are taken as given, and I compute the optimal trading strategies of the FTs and STs.

Dealer's pricing rules (λ, α, ρ)

According to the model timeline, before trading at t the dealer observes $\mathrm{d}w_{t-2}$ (or equivalently w_{t-2}). Then, she observes the order flow $\mathrm{d}y_t$ and sets the price p_t at which trading takes place:

$$p_t \;=\; \mathsf{E}\big(w_t \mid \mathcal{I}_t, \mathrm{d}y_t\big), \quad \text{with} \quad \mathcal{I}_t \;=\; \{\mathrm{d}y_{t-1}, \mathrm{d}y_{t-2}, \ldots, \mathrm{d}w_{t-2}, \mathrm{d}w_{t-3}, \ldots\}. \tag{IA.159}$$

The aggregate order flow satisfies:

$$\mathrm{d}y_t \;=\; \bar{\gamma}\mathrm{d}w_t + \bar{\mu}\widetilde{\mathrm{d}w}_{t-1} + \mathrm{d}u_t, \tag{IA.160}$$

where $\bar{\gamma}$ and $\bar{\mu}$ are the aggregate trading coefficients, and $\widetilde{\mathrm{d}w}_{t-1}$ is the component of $\mathrm{d}w_{t-1}$ orthogonal on the dealer's information set $\mathcal{I}_t$, i.e., the dealer computes:

$$\widetilde{\mathrm{d}w}_t \;=\; \mathrm{d}w_t - \mathsf{E}\big(\mathrm{d}w_t \mid \mathcal{I}_{t+1}\big). \tag{IA.161}$$

By definition, $\mathcal{I}_{t+1} = \mathcal{I}_t \cup \langle \mathrm{d}y_t, \mathrm{d}w_{t-1} \rangle = \mathcal{I}_t \cup \langle \mathrm{d}y_t, \widetilde{\mathrm{d}w}_{t-1} \rangle$. But $\mathrm{d}w_t$, $\mathrm{d}y_t$ and $\widetilde{\mathrm{d}w}_{t-1}$ are all orthogonal on $\mathcal{I}_t$, hence the dealer computes:

$$\begin{aligned}\widehat{\mathrm{d}w}_t &= \mathsf{E}\big(\mathrm{d}w_t \mid \mathcal{I}_{t+1}\big) = \mathsf{E}\big(\mathrm{d}w_t \mid \mathcal{I}_t, \mathrm{d}y_t, \widetilde{\mathrm{d}w}_{t-1}\big) = \mathsf{E}\big(\mathrm{d}w_t \mid \mathrm{d}y_t, \widetilde{\mathrm{d}w}_{t-1}\big) \\ &= \rho_1 \mathrm{d}y_t + \rho_2 \widetilde{\mathrm{d}w}_{t-1},\end{aligned} \tag{IA.162}$$

where:

$$\begin{bmatrix} \rho_1 \\ \rho_2 \end{bmatrix} = \mathsf{Var}\left(\begin{bmatrix} \mathrm{d}y_t \\ \widetilde{\mathrm{d}w}_{t-1} \end{bmatrix}\right)^{-1} \mathsf{Cov}\left(\mathrm{d}w_t, \begin{bmatrix} \mathrm{d}y_t \\ \widetilde{\mathrm{d}w}_{t-1} \end{bmatrix}\right). \tag{IA.163}$$

Define:

$$\begin{aligned}
&\tilde{\sigma}_u = \frac{\sigma_u}{\sigma_v}, \qquad W_t = \frac{\mathsf{E}\big[(\widetilde{\mathrm{d}w}_t)^2\big]}{\sigma_v^2 \mathrm{d}t}, \qquad Y_t = \frac{\mathsf{E}\big[(\mathrm{d}y_t)^2\big]}{\sigma_v^2 \mathrm{d}t}, \qquad X_t = \frac{\mathsf{E}\big[\mathrm{d}y_t \widetilde{\mathrm{d}w}_{t-1}\big]}{\sigma_v^2 \mathrm{d}t}, \\
&\rho = \frac{\bar{\gamma}}{\tilde{\sigma}_u^2 + \bar{\gamma}^2}, \qquad a = \rho\bar{\gamma}, \qquad b = \rho\bar{\mu}, \qquad c = \rho^2 \tilde{\sigma}_u^2 = a - a^2, \\
&\lambda = \rho \frac{a + b(1-a)}{a + b^2(1-a)}, \qquad \alpha = 1 - a.
\end{aligned} \tag{IA.164}$$

From (IA.160) it follows that $Y_t = \bar{\gamma}^2 + \bar{\mu}^2 W_{t-1} + \tilde{\sigma}_u^2$ and $X_t = \bar{\mu} W_{t-1}$, hence one obtains:

$$\begin{bmatrix} \rho_1 \\ \rho_2 \end{bmatrix} = \begin{bmatrix} \bar{\gamma}^2 + \bar{\mu}^2 W_{t-1} + \tilde{\sigma}_u^2 & \bar{\mu} W_{t-1} \\ \bar{\mu} W_{t-1} & W_{t-1} \end{bmatrix}^{-1} \begin{bmatrix} \bar{\gamma} \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{\bar{\gamma}}{\tilde{\sigma}_u^2 + \bar{\gamma}^2} \\ -\frac{\bar{\mu}\bar{\gamma}}{\tilde{\sigma}_u^2 + \bar{\gamma}^2} \end{bmatrix} = \begin{bmatrix} \rho \\ -b \end{bmatrix}, \tag{IA.165}$$

which implies (using the formula $\mathrm{d}y_t = \bar{\gamma}\mathrm{d}w_t + \bar{\mu}\widetilde{\mathrm{d}w}_{t-1} + \mathrm{d}u_t$):

$$\begin{aligned}\widehat{\mathrm{d}w}_t &= \rho \mathrm{d}y_t - b\widetilde{\mathrm{d}w}_{t-1} = a\mathrm{d}w_t + \rho \mathrm{d}u_t, \\ \widetilde{\mathrm{d}w}_t &= \alpha \mathrm{d}w_t - \rho \mathrm{d}u_t.\end{aligned} \tag{IA.166}$$

Using (IA.166) the order flow at t can be expressed as:

$$\mathrm{d}y_t = \bar{\gamma}\mathrm{d}w_t + \bar{\mu}\alpha \mathrm{d}w_{t-1} + \mathrm{d}u_t - \rho\bar{\mu}\mathrm{d}u_{t-1}. \tag{IA.167}$$

I now compute the price at t. Note that $\widetilde{\mathrm{d}w}_{t-2}$ is in $\mathcal{I}_t$: indeed, $\mathrm{d}w_{t-2}$ is in $\mathcal{I}_t$ and $\widehat{\mathrm{d}w}_{t-2}$ by definition is in $\mathcal{I}_{t-2}$ which is included in $\mathcal{I}_t$. Thus, if I define:

$$\mathrm{d}y_{t-1}^{\perp} = \mathrm{d}y_{t-1} - \bar{\mu}\widetilde{\mathrm{d}w}_{t-2} = \bar{\gamma}\mathrm{d}w_{t-1} + \mathrm{d}u_{t-1}, \tag{IA.168}$$

it follows that $\mathrm{d}y_{t-1}^{\perp}$ is the component of $\mathrm{d}y_{t-1}$ orthogonal on $\mathcal{I}_{t-1} \cup \langle \mathrm{d}w_{t-2} \rangle$. Hence, one has $\mathcal{I}_t = \mathcal{I}_{t-1} \cup \langle \mathrm{d}y_{t-1}, \mathrm{d}w_{t-2} \rangle = \mathcal{I}_{t-1} \cup \langle \mathrm{d}y_{t-1}^{\perp}, \mathrm{d}w_{t-2} \rangle$. Using the fact that $\mathrm{d}w_{t-1}$ and $\mathrm{d}w_t$ are orthogonal on $\mathcal{I}_{t-1}$ and $\mathrm{d}w_{t-2}$, the dealer sets the price at t:

$$\begin{aligned} p_t &= \mathsf{E}\big(w_t \mid \mathcal{I}_{t-1}, \mathrm{d}y_{t-1}, \mathrm{d}w_{t-2}, \mathrm{d}y_t\big) = w_{t-2} + \mathsf{E}\big(\mathrm{d}w_{t-1} + \mathrm{d}w_t \mid \mathrm{d}y_{t-1}^{\perp}, \mathrm{d}y_t\big) \\ &= w_{t-2} + \lambda_1 \mathrm{d}y_{t-1}^{\perp} + \lambda_2 \mathrm{d}y_t, \end{aligned} \tag{IA.169}$$

where the constants λ_1 and λ_2 are computed using equations (IA.168) and (IA.167):

$$\begin{aligned} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} &= \mathsf{Var}\left(\begin{bmatrix} \mathrm{d}y_{t-1}^{\perp} \\ \mathrm{d}y_t \end{bmatrix} \right)^{-1} \mathsf{Cov}\left(\mathrm{d}w_{t-1} + \mathrm{d}w_t, \begin{bmatrix} \mathrm{d}y_{t-1}^{\perp} \\ \mathrm{d}y_t \end{bmatrix} \right) \\ &= \begin{bmatrix} \bar{\gamma}^2 + \tilde{\sigma}_u^2 & \bar{\gamma}\bar{\mu}\alpha - \bar{\mu}\rho\tilde{\sigma}_u^2 \\ \bar{\gamma}\bar{\mu}\alpha - \bar{\mu}\rho\tilde{\sigma}_u^2 & \bar{\gamma}^2 + \bar{\mu}^2\alpha^2 + \bar{\mu}^2\rho^2\tilde{\sigma}_u^2 + \tilde{\sigma}_u^2 \end{bmatrix}^{-1} \begin{bmatrix} \bar{\gamma} \\ \bar{\gamma} + \alpha\bar{\mu} \end{bmatrix}. \end{aligned} \tag{IA.170}$$

One computes:

$$\lambda_1 = \rho, \qquad \lambda_2 = \lambda = \rho \frac{a + b(1-a)}{a + b^2(1-a)}. \tag{IA.171}$$

Therefore, the price at t satisfies:

$$p_t = w_{t-2} + \rho \mathrm{d}y_{t-1}^{\perp} + \lambda \mathrm{d}y_t, \tag{IA.172}$$

or, using the formula $\mathrm{d}y_{t-1}^{\perp} = \mathrm{d}y_{t-1} - \bar{\mu}\widetilde{\mathrm{d}w}_{t-2}$, it satisfies:

$$p_t = w_{t-2} + \rho \mathrm{d}y_{t-1} - b\widetilde{\mathrm{d}w}_{t-2} + \lambda \mathrm{d}y_t. \tag{IA.173}$$

Note that the last formula uses only quantities observed directly by the dealer: $\mathrm{d}y_t$, $\mathrm{d}y_{t-1}$ and $\widetilde{\mathrm{d}w}_{t-2}$. According to (IA.166), the last quantity is computed by the dealer using the recursive equation $\widetilde{\mathrm{d}w}_t = \mathrm{d}w_t - \rho \mathrm{d}y_t + b\widetilde{\mathrm{d}w}_{t-1}$, which implies $\rho \mathrm{d}y_{t-1} - b\widetilde{\mathrm{d}w}_{t-2} = \mathrm{d}w_{t-1} - \widetilde{\mathrm{d}w}_{t-1}$. The pricing formula is then equivalent to:[15]

$$p_t = w_{t-1} - \widetilde{\mathrm{d}w}_{t-1} + \lambda \mathrm{d}y_t. \tag{IA.174}$$

I am interested in computing the infinitesimal covariance of $\widetilde{\mathrm{d}w}_t$ from the perspective of the dealer, but also from the perspective of the speculators. According to (IA.166), $\widetilde{\mathrm{d}w}_t$ satisfies the recursive equation $\widetilde{\mathrm{d}w}_t = \mathrm{d}w_t - \rho \mathrm{d}y_t + b\widetilde{\mathrm{d}w}_{t-1} = (1 - \rho\bar{\gamma})\mathrm{d}w_t - \rho \mathrm{d}u_t +$

[15] Note that neither w_{t-1} nor $\widetilde{\mathrm{d}w}_{t-1}$ is observed by the dealer at t, but the difference $w_{t-1} - \widetilde{\mathrm{d}w}_{t-1}$ is observed as it is equal to $w_{t-2} + \rho \mathrm{d}y_{t-1} - b\widetilde{\mathrm{d}w}_{t-1}$.

$(b-\rho\bar{\mu})\widetilde{\mathrm{d}w}_{t-1}$. For the dealer, $b=\rho\bar{\mu}$, and therefore $\widetilde{\mathrm{d}w}_t = \alpha \mathrm{d}w_t - \rho \mathrm{d}u_t$, which implies:

$$W_t \;=\; \frac{\mathsf{E}\big[(\widetilde{\mathrm{d}w}_t)^2\big]}{\sigma_v^2\mathrm{d}t} \;=\; \frac{\mathsf{E}\Big[\big(\alpha\mathrm{d}w_t - \rho\mathrm{d}u_t\big)^2\Big]}{\sigma_v^2\mathrm{d}t} \;=\; \alpha^2+\rho^2\tilde{\sigma}_u^2 \;=\; \alpha. \tag{IA.175}$$

For a speculator, the aggregate coefficients $\bar{\gamma}$ and $\bar{\mu}$ include his own coefficients γ^i and μ^i, which can be different from the equilibrium values. Hence, the speculator computes:

$$\begin{aligned} W_t \;&=\; \frac{\mathsf{E}\Big[\big((1-\rho\bar{\gamma})\mathrm{d}w_t - \rho\mathrm{d}u_t + (b-\rho\bar{\mu})\widetilde{\mathrm{d}w}_{t-1}\big)^2\Big]}{\sigma_v^2\mathrm{d}t} \\ &=\; (1-\rho\bar{\gamma})^2 + \rho^2\tilde{\sigma}_u^2 + (b-\rho\bar{\mu})^2 W_{t-1}. \end{aligned} \tag{IA.176}$$

Using Lemma A.1 in the Appendix in the paper, if the coefficient $b-\rho\bar{\mu}\in(-1,1)$, the covariance W_t is constant and equal to:

$$W \;=\; \frac{(1-\rho\bar{\gamma})^2+\rho^2\tilde{\sigma}_u^2}{1-(b-\rho\bar{\mu})^2}. \tag{IA.177}$$

Speculators' optimal strategy (γ, μ)

Consider a FT, indexed by $i=1,\dots,N_F$. At $t=0$ he chooses a trading strategy of the form:

$$\mathrm{d}x_t^i \;=\; \gamma^i\mathrm{d}w_t + \mu^i\widetilde{\mathrm{d}w}_{t-1}, \tag{IA.178}$$

where $\widetilde{\mathrm{d}w}_t$ satisfies a recursive equation of the form:

$$\widetilde{\mathrm{d}w}_t \;=\; \mathrm{d}w_t - \rho\mathrm{d}y_t + b\widetilde{\mathrm{d}w}_{t-1}, \tag{IA.179}$$

and the coefficients ρ and b are fixed.[16] The aggregate order flow is of the form:

$$\begin{aligned} \mathrm{d}y_t \;&=\; \bar{\gamma}\mathrm{d}w_t + \bar{\mu}\widetilde{\mathrm{d}w}_{t-1} + \mathrm{d}u_t, \quad \text{with} \\ \bar{\gamma} \;&=\; \gamma^i + \gamma^{-i}, \qquad \bar{\mu} \;=\; \mu^i + \mu^{-i}, \end{aligned} \tag{IA.180}$$

where the superscript "$-i$" indicates the aggregate quantity from the other speculators. FT i takes as given the coefficients γ^{-i} and μ^{-i}, and assumes the following functional form for the price:

$$p_t \;=\; w_{t-1} - \widetilde{\mathrm{d}w}_{t-1} + \lambda\mathrm{d}y_t, \tag{IA.181}$$

[16] In equilibrium, $\widetilde{\mathrm{d}w}_{t-1}$ is the component of $\mathrm{d}w_{t-1}$ orthogonal to the information set $\mathcal{I}_t$ before trading at t.

where the coefficient λ is fixed. Using (IA.177), the FT also computes:

$$W \;=\; \frac{\mathsf{E}\big[(\widetilde{\mathrm{d}w_t})^2\big]}{\sigma_v^2 \mathrm{d}t} \;=\; \frac{(1-\rho\bar{\gamma})^2 + \rho^2\tilde{\sigma}_u^2}{1-(b-\rho\bar{\mu})^2}. \tag{IA.182}$$

The normalized expected profit of FT i at $t=0$ is $\frac{1}{\sigma_v^2}\,\mathsf{E}\int_0^T (w_t - p_t)\mathrm{d}x_t^i$:

$$\begin{aligned}
\tilde{\pi}_F^i \;&=\; \frac{1}{\sigma_v^2}\,\mathsf{E}\int_0^T \Big[w_t - w_{t-1} + \widetilde{\mathrm{d}w}_{t-1} - \lambda\big(\bar{\gamma}\mathrm{d}w_t + \bar{\mu}\widetilde{\mathrm{d}w}_{t-1} + \mathrm{d}u_t\big)\Big]\cdot\Big(\gamma^i\mathrm{d}w_t + \mu^i\widetilde{\mathrm{d}w}_{t-1}\Big)\\
&=\; \frac{1}{\sigma_v^2}\,\mathsf{E}\int_0^T \Big[\mathrm{d}w_t\big(1-\lambda\bar{\gamma}\big) + \widetilde{\mathrm{d}w}_{t-1}\big(1-\lambda\bar{\mu}\big) - \lambda\mathrm{d}u_t\Big]\cdot\Big(\gamma^i\mathrm{d}w_t + \mu^i\widetilde{\mathrm{d}w}_{t-1}\Big)\\
&=\; \gamma^i - \lambda\gamma^i\bar{\gamma} + W\big(\mu^i - \lambda\mu^i\bar{\mu}\big)\\
&=\; \gamma^i - \lambda\gamma^i\bar{\gamma} + \frac{(1-\rho\bar{\gamma})^2 + c}{1-(b-\rho\bar{\mu})^2}\,\big(\mu^i - \lambda\mu^i\bar{\mu}\big),
\end{aligned} \tag{IA.183}$$

where the last equality follows from (IA.182). This optimization problem can be solved by considering the first order conditions and then imposing the equilibrium conditions derived below.

Equilibrium conditions

Besides the equations that describe the optimization problem of the FTs and STs, the equilibrium conditions for the dealer's coefficients are already in (IA.164).

An important quantity is the "speculator participation ratio," which I define as the ratio of speculator trading variance over total trading variance:

$$SPR \;=\; \frac{\mathsf{Var}(\mathrm{d}y_t) - \mathsf{Var}(\mathrm{d}u_t)}{\mathsf{Var}(\mathrm{d}y_t)} \;=\; \frac{\Omega^{\mathrm{d}y,\mathrm{d}y} - \Omega^{\mathrm{d}u,\mathrm{d}u}}{\Omega^{\mathrm{d}y,\mathrm{d}y}}. \tag{IA.184}$$

Using the formulas in (IA.164), one computes:

$$SPR \;=\; \frac{a^2 + (1-a)b^2}{a + (1-a)b^2}. \tag{IA.185}$$

But a and b converge to one when N_F and N_L are large, hence the speculator participation rate can be arbitrarily close to one. This closely mirror the results in the model without public information.

3.2 Imperfect public information

In this subsection, I consider a market that is not strong-form efficient. I assume that just before trading at t, the FTs observe $\mathrm{d}v_t$, while the STs observe $\mathrm{d}v_{t-1}$. The difference is that now the public does not observe $\mathrm{d}v_{t-2}$ but an imprecise signal about it:

$$\mathrm{d}s_{t-2} \;=\; \mathrm{d}v_{t-2} + \mathrm{d}\eta_{t-2}, \qquad \mathrm{d}\eta_{t-2} \;\sim\; \mathcal{N}\big(0, \sigma_\eta^2 \mathrm{d}t\big). \tag{IA.186}$$

Denote by w_t the public forecast of the fundamental value. Then, the above setup is equivalent to assuming that before trading at t the public observes $\mathrm{d}w_{t-2}$, which satisfies:

$$\mathrm{d}w_{t-2} \;=\; \frac{\sigma_v^2}{\sigma_v^2 + \sigma_\eta^2}\mathrm{d}s_{t-2} \;\sim\; \mathcal{N}\big(0, \sigma_w^2 \mathrm{d}t\big), \quad \text{with} \quad \sigma_w^2 \;=\; \frac{\sigma_v^4}{\sigma_v^2 + \sigma_\eta^2}. \tag{IA.187}$$

Equation (IA.187) implies that:

$$\tilde{\sigma}_w^2 \;=\; \frac{\sigma_w^2}{\sigma_v^2} \;=\; \frac{\sigma_v^2}{\sigma_v^2 + \sigma_\eta^2} \;\in\; [0, 1]. \tag{IA.188}$$

Note that a more precise public signal leads to a ratio $\tilde{\sigma}_w$ closer to one, while a less precise signal leads to a ratio $\tilde{\sigma}_w$ closer to zero.

I look for an equilibrium with the following properties: (i) the equilibrium is symmetric, in the sense that the FTs have identical trading strategies, and the same for the STs; and (ii) the equilibrium coefficients are constant with respect to time.

To solve for the equilibrium, in the first step the speculators' trading strategies are taken as given, and I compute the dealer's pricing functions. In the second step the dealer's pricing functions are taken as given, and I solve for the optimal trading strategies for the FTs and STs.

Dealer's pricing rules (λ, θ, ρ)

According to the model timeline, before trading at t the dealer observes $\mathrm{d}w_{t-2}$ (or equivalently w_{t-2}). Then, she observes the order flow $\mathrm{d}y_t$ and sets the price p_t at which trading takes place:

$$p_t \;=\; \mathsf{E}\big(v_t \mid \mathcal{I}_t, \mathrm{d}y_t\big), \quad \text{with} \quad \mathcal{I}_t \;=\; \{\mathrm{d}y_{t-1}, \mathrm{d}y_{t-2}, \ldots, \mathrm{d}w_{t-2}, \mathrm{d}w_{t-3}, \ldots\}. \tag{IA.189}$$

Define:

$$\begin{aligned} \widehat{\mathrm{d}v}_{t-1} &= \mathsf{E}\big(\mathrm{d}v_{t-1} \mid \mathcal{I}_t\big), & \widehat{\mathrm{d}w}_{t-1} &= \mathsf{E}\big(\mathrm{d}w_{t-1} \mid \mathcal{I}_t\big), \\ \widetilde{\mathrm{d}v}_{t-1} &= \mathrm{d}v_{t-1} - \widehat{\mathrm{d}v}_{t-1}, & \widetilde{\mathrm{d}w}_{t-1} &= \mathrm{d}w_{t-1} - \widehat{\mathrm{d}v}_{t-1}. \end{aligned} \tag{IA.190}$$

By definition, $\mathcal{I}_{t+1} = \mathcal{I}_t \cup \langle \mathrm{d}y_t, \mathrm{d}w_{t-1} \rangle = \mathcal{I}_t \cup \langle \mathrm{d}y_t, \widetilde{\mathrm{d}w}_{t-1} \rangle$. But $\mathrm{d}v_t$, $\mathrm{d}y_t$ and $\widetilde{\mathrm{d}w}_{t-1}$ are all orthogonal on $\mathcal{I}_t$, hence the dealer computes:

$$\begin{aligned} \widehat{\mathrm{d}v}_t &= \mathsf{E}\big(\mathrm{d}v_t \mid \mathcal{I}_{t+1}\big) = \mathsf{E}\big(\mathrm{d}v_t \mid \mathcal{I}_t, \mathrm{d}y_t, \widetilde{\mathrm{d}w}_{t-1}\big) = \mathsf{E}\big(\mathrm{d}v_t \mid \mathrm{d}y_t, \widetilde{\mathrm{d}w}_{t-1}\big) \\ &= \theta_1 \mathrm{d}y_t + \theta_2 \widetilde{\mathrm{d}w}_{t-1}, \end{aligned} \tag{IA.191}$$

where:

$$\begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = \mathsf{Var}\left(\begin{bmatrix} \mathrm{d}y_t \\ \widetilde{\mathrm{d}w}_{t-1} \end{bmatrix} \right)^{-1} \mathsf{Cov}\left(\mathrm{d}v_t, \begin{bmatrix} \mathrm{d}y_t \\ \widetilde{\mathrm{d}w}_{t-1} \end{bmatrix} \right). \tag{IA.192}$$

Also, the dealer computes:

$$\begin{aligned} \widehat{\mathrm{d}w}_t &= \mathsf{E}\big(\mathrm{d}w_t \mid \mathcal{I}_{t+1}\big) = \mathsf{E}\big(\mathrm{d}w_t \mid \mathcal{I}_t, \mathrm{d}y_t, \widetilde{\mathrm{d}w}_{t-1}\big) = \mathsf{E}\big(\mathrm{d}w_t \mid \mathrm{d}y_t, \widetilde{\mathrm{d}w}_{t-1}\big) \\ &= \rho_1 \mathrm{d}y_t + \rho_2 \widetilde{\mathrm{d}w}_{t-1}, \end{aligned} \tag{IA.193}$$

where:

$$\begin{bmatrix} \rho_1 \\ \rho_2 \end{bmatrix} = \mathsf{Var}\left(\begin{bmatrix} \mathrm{d}y_t \\ \widetilde{\mathrm{d}w}_{t-1} \end{bmatrix} \right)^{-1} \mathsf{Cov}\left(\mathrm{d}w_t, \begin{bmatrix} \mathrm{d}y_t \\ \widetilde{\mathrm{d}w}_{t-1} \end{bmatrix} \right). \tag{IA.194}$$

The aggregate order flow satisfies:

$$\mathrm{d}y_t = \bar{\gamma} \mathrm{d}v_t + \mathrm{d}u_t + \bar{\mu} \widetilde{\mathrm{d}v}_{t-1}, \tag{IA.195}$$

where $\bar{\gamma}$ and $\bar{\mu}$ are the aggregate trading coefficients. One computes:

$$\begin{aligned} \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} &= \begin{bmatrix} \bar{\gamma}^2 + \tilde{\sigma}_u^2 + \bar{\mu}^2 V_{t-1} & \bar{\mu} Z_{t-1} \\ \bar{\mu} Z_{t-1} & W_{t-1} \end{bmatrix}^{-1} \begin{bmatrix} \bar{\gamma} \\ 0 \end{bmatrix}, \\ \begin{bmatrix} \rho_1 \\ \rho_2 \end{bmatrix} &= \begin{bmatrix} \bar{\gamma}^2 + \tilde{\sigma}_u^2 + \bar{\mu}^2 V_{t-1} & \bar{\mu} Z_{t-1} \\ \bar{\mu} Z_{t-1} & W_{t-1} \end{bmatrix}^{-1} \begin{bmatrix} \bar{\gamma} \tilde{\sigma}_w^2 \\ 0 \end{bmatrix} = \tilde{\sigma}_w^2 \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}. \end{aligned} \tag{IA.196}$$

Define:

$$\tilde{\sigma}_u = \frac{\sigma_u}{\sigma_v}, \qquad \tilde{\sigma}_w = \frac{\sigma_w}{\sigma_v}, \qquad Y_t = \frac{\mathsf{E}\big[(\mathrm{d}y_t)^2\big]}{\sigma_v^2 \mathrm{d}t},$$
$$W_t = \frac{\mathsf{E}\big[(\widetilde{\mathrm{d}w_t})^2\big]}{\sigma_v^2 \mathrm{d}t}, \qquad V_t = \frac{\mathsf{E}\big[(\widetilde{\mathrm{d}v_t})^2\big]}{\sigma_v^2 \mathrm{d}t}, \qquad Z_t = \frac{\mathsf{E}\big[\widetilde{\mathrm{d}v_t}\widetilde{\mathrm{d}w_t}\big]}{\sigma_v^2 \mathrm{d}t}. \tag{IA.197}$$

Using (IA.195), one obtains:

$$\begin{aligned}
Y_t &= \bar{\gamma}^2 + \tilde{\sigma}_u^2 + \bar{\mu}^2 V_{t-1}, \\
V_t &= \frac{\mathsf{E}\Big[\big((1-\theta_1\bar{\gamma})\mathrm{d}v_t - \theta_1 \mathrm{d}u_t - \theta_1\bar{\mu}\widetilde{\mathrm{d}v}_{t-1} - \theta_2\widetilde{\mathrm{d}w}_{t-1}\big)^2\Big]}{\sigma_v^2 \mathrm{d}t} \\
&= (1-\theta_1\bar{\gamma})^2 + \theta_1^2\tilde{\sigma}_u^2 + \theta_1^2\bar{\mu}^2 V_{t-1} + \theta_2^2 W_{t-1} + 2\theta_1\theta_2\bar{\mu} Z_{t-1}, \\
W_t &= \frac{\mathsf{E}\Big[\big(\mathrm{d}w_t - \rho_1\bar{\gamma}\mathrm{d}v_t - \rho_1 \mathrm{d}u_t - \rho_1\bar{\mu}\widetilde{\mathrm{d}v}_{t-1} - \rho_2\widetilde{\mathrm{d}w}_{t-1}\big)^2\Big]}{\sigma_v^2 \mathrm{d}t} \\
&= (1-2\rho_1\bar{\gamma})\tilde{\sigma}_w^2 + \rho_1^2\bar{\gamma}^2 + \rho_1^2\tilde{\sigma}_u^2 + \rho_1^2\bar{\mu}^2 V_{t-1} + \rho_2^2 W_{t-1} + 2\rho_1\rho_2\bar{\mu} Z_{t-1}, \\
Z_t &= (1-\theta_1\bar{\gamma})(\tilde{\sigma}_w^2 - \rho_1\bar{\gamma}) + \theta_1\rho_1\tilde{\sigma}_u^2 + \theta_1\rho_1\bar{\mu}^2 V_{t-1} + \theta_2\rho_2 W_{t-1} + (\theta_1\rho_2 + \theta_2\rho_1)\bar{\mu} Z_{t-1}.
\end{aligned} \tag{IA.198}$$

I solve this system of recursive equations in V, W, and Z. Define:

$$A = \begin{bmatrix} \theta_1^2\bar{\mu}^2 & \theta_2^2 & 2\theta_1\theta_2\bar{\mu} \\ \rho_1^2\bar{\mu}^2 & \rho_2^2 & 2\rho_1\rho_2\bar{\mu} \\ \theta_1\rho_1\bar{\mu}^2 & \theta_2\rho_2 & (\theta_1\rho_2+\theta_2\rho_1)\bar{\mu} \end{bmatrix}, \quad B = \begin{bmatrix} (1-\theta_1\bar{\gamma})^2 + \theta_1^2\tilde{\sigma}_u^2 \\ (1-2\rho_1\bar{\gamma})\tilde{\sigma}_w^2 + \rho_1^2\bar{\gamma}^2 + \rho_1^2\tilde{\sigma}_u^2 \\ (1-\theta_1\bar{\gamma})(\tilde{\sigma}_w^2 - \rho_1\bar{\gamma}) + \theta_1\rho_1\tilde{\sigma}_u^2 \end{bmatrix}. \tag{IA.199}$$

The eigenvalues of $I - A$ are: 1, 1 and $1 - (\theta_1\bar{\mu} + \theta_2\tilde{\sigma}_w^2)^2$. One needs to impose the condition that all eigenvalues are between -1 and 1, which is equivalent to:

$$(\theta_1\bar{\mu} + \theta_2\tilde{\sigma}_w^2)^2 \; < \; 2. \tag{IA.200}$$

According to Lemma A.1 in the Appendix in the paper, if the condition in (IA.200) is satisfied, the numbers V_t, W_t, and Z_t are constant and satisfy $[V, W, Z]^T = (I - A)^{-1}B$, which implies:

$$V = \frac{(1-\theta_1\bar{\gamma})^2 + \theta_1^2\tilde{\sigma}_u^2 + \theta_2^2\tilde{\sigma}_w^2(1-\tilde{\sigma}_w^2)}{1-(\theta_1\bar{\mu}+\theta_2\tilde{\sigma}_w^2)^2}, \qquad Z = \tilde{\sigma}_w^2 V, \qquad Y = \bar{\gamma}^2 + \tilde{\sigma}_u^2 + \bar{\mu}^2 V,$$
$$W = \frac{1+\theta_1^2(\tilde{\sigma}_u^2\tilde{\sigma}_w^2 + \tilde{\sigma}_w^2\bar{\gamma}^2 + \tilde{\sigma}_w^2\bar{\mu}^2 - \bar{\mu}^2) - 2\theta_1\theta_2\tilde{\sigma}_w^2(1-\tilde{\sigma}_w^2)\bar{\mu} - 2\theta_1\tilde{\sigma}_w^2\bar{\gamma}}{1-(\theta_1\bar{\mu}+\theta_2\tilde{\sigma}_w^2)^2}. \tag{IA.201}$$

Using the equations in (IA.196), some algebraic manipulation shows that θ_1 and θ_2 satisfy:

$$\theta_1^2 X\bar{\gamma} + \theta_1\big(\tilde{\sigma}_w^2\bar{\mu}^2 - \tilde{\sigma}_w^2\bar{\gamma}^2 - \tilde{\sigma}_u^2 - \bar{\gamma}^2 - \bar{\mu}^2\big) + \bar{\gamma} = 0,$$
$$\theta_2 = \frac{\theta_1^2 X\big(3\bar{\gamma} - \theta_1(\tilde{\sigma}_u^2+\bar{\gamma}^2)\big) + \theta_1\big(2\tilde{\sigma}_w^2\bar{\mu}^2 - 2\tilde{\sigma}_w^2\bar{\gamma}^2 - \tilde{\sigma}_u^2 - \bar{\gamma}^2 - 2\bar{\mu}^2\big) + \bar{\gamma}}{(1-\tilde{\sigma}_w^2)\big(1-\theta_1^2 X\big)\,\bar{\mu}}, \tag{IA.202}$$
$$\text{with} \quad X = \tilde{\sigma}_w^2(\tilde{\sigma}_u^2 + \bar{\gamma}^2 + \bar{\mu}^2) - \bar{\mu}^2.$$

I now compute the price at t, which is set by the dealer as $p_t = \mathsf{E}(v_t|\mathcal{I}_t, \mathrm{d}y_t)$. I also define the quote at t as the dealer's expectation of the fundamental value just before trading at t:

$$q_t = \mathsf{E}(v_t \mid \mathcal{I}_t). \tag{IA.203}$$

Then one has $p_t = \mathsf{E}(v_t|\mathcal{I}_{t-1}, \mathrm{d}y_{t-1}, \mathrm{d}w_{t-2}, \mathrm{d}y_t) = \mathsf{E}(v_t|\mathcal{I}_{t-1}, \mathrm{d}y_{t-1}, \widetilde{\mathrm{d}w}_{t-2}, \mathrm{d}y_t)$. The variables $v_t - q_{t-1}$, $\mathrm{d}y_{t-1}$, $\widetilde{\mathrm{d}w}_{t-2}$, and $\mathrm{d}y_t$ are orthogonal to $\mathcal{I}_{t-1}$, which includes q_{t-1}, therefore the price satisfies:

$$\begin{aligned} p_t &= q_{t-1} + \mathsf{E}\big(v_t - q_{t-1} \mid \mathcal{I}_{t-1}, \mathrm{d}y_{t-1}, \widetilde{\mathrm{d}w}_{t-2}, \mathrm{d}y_t\big) \\ &= q_{t-1} + \lambda_1 \mathrm{d}y_{t-1} + \lambda_2 \widetilde{\mathrm{d}w}_{t-2} + \lambda_3 \mathrm{d}y_t, \end{aligned} \tag{IA.204}$$

where the constants λ_1, λ_2, and λ_3 satisfy:

$$\begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \mathsf{Var}\left(\begin{bmatrix} \mathrm{d}y_{t-1} \\ \widetilde{\mathrm{d}w}_{t-2} \\ \mathrm{d}y_t \end{bmatrix}\right)^{-1} \mathsf{Cov}\left(v_t, \begin{bmatrix} \mathrm{d}y_{t-1} \\ \widetilde{\mathrm{d}w}_{t-2} \\ \mathrm{d}y_t \end{bmatrix}\right). \tag{IA.205}$$

One obtains the following formulas relating the price p_t and the quote q_t:

$$p_t = p_{t-1} + \lambda_2 \widetilde{\mathrm{d}w}_{t-2} + \lambda_3 \mathrm{d}y_t, \qquad p_t = q_t + \lambda_1 \mathrm{d}y_t. \tag{IA.206}$$

Recall the formulas:

$$\begin{aligned} \widetilde{\mathrm{d}v}_t &= \mathrm{d}v_t - \theta_1 \mathrm{d}y_t - \theta_2 \widetilde{\mathrm{d}w}_{t-1}, \qquad \widetilde{\mathrm{d}w}_t \;=\; \mathrm{d}w_t - \rho_1 \mathrm{d}y_t - \rho_2 \widetilde{\mathrm{d}w}_{t-1}, \\ \mathrm{d}y_t &= \bar{\gamma}\mathrm{d}v_t + \mathrm{d}u_t + \bar{\mu}\widetilde{\mathrm{d}v}_{t-1}. \end{aligned} \tag{IA.207}$$

Define:

$$B_t \;=\; \widetilde{\mathsf{Cov}}(v_t, \widetilde{\mathrm{d}w}_t) \qquad C_t \;=\; \widetilde{\mathsf{Cov}}(\widetilde{\mathrm{d}w}_t, \widetilde{\mathrm{d}w}_{t-1}). \tag{IA.208}$$

One computes the recursive equations for several covariances involved in (IA.205):

$$\begin{aligned} \widetilde{\mathsf{Cov}}(\mathrm{d}y_t, \widetilde{\mathrm{d}w}_{t-2}) &= \bar{\mu}\widetilde{\mathsf{Cov}}(\widetilde{\mathrm{d}v}_{t-1}, \widetilde{\mathrm{d}w}_{t-2}) \;=\; -\theta_1\bar{\mu}\widetilde{\mathsf{Cov}}(\mathrm{d}y_{t-1}, \widetilde{\mathrm{d}w}_{t-2}) - \theta_2\bar{\mu}W, \\ \widetilde{\mathsf{Cov}}(\mathrm{d}y_{t-1}, \widetilde{\mathrm{d}w}_{t-2}) &= \bar{\mu}Z, \qquad \widetilde{\mathsf{Cov}}(v_t, \mathrm{d}y_t) \;=\; \bar{\gamma} + \bar{\mu}\widetilde{\mathsf{Cov}}(v_{t-1}, \widetilde{\mathrm{d}v}_{t-1}), \\ \widetilde{\mathsf{Cov}}(\widetilde{\mathrm{d}w}_t, \widetilde{\mathrm{d}w}_{t-1}) &= -\rho_1\widetilde{\mathsf{Cov}}(\mathrm{d}y_t, \widetilde{\mathrm{d}w}_{t-1}) - \rho_2\widetilde{\mathsf{Cov}}(\widetilde{\mathrm{d}w}_{t-1}, \widetilde{\mathrm{d}w}_{t-1}) \;=\; -\rho_1\bar{\mu}Z - \rho_2 W, \\ \widetilde{\mathsf{Cov}}(v_t, \widetilde{\mathrm{d}v}_t) &= 1 - \theta_1\widetilde{\mathsf{Cov}}(v_t, \mathrm{d}y_t) - \theta_2\widetilde{\mathsf{Cov}}(v_t, \widetilde{\mathrm{d}w}_{t-1}) \\ &= 1 - \theta_1\bar{\gamma} - \theta_1\bar{\mu}\widetilde{\mathsf{Cov}}(v_{t-1}, \widetilde{\mathrm{d}v}_{t-1}) - \theta_2\widetilde{\mathsf{Cov}}(v_{t-1}, \widetilde{\mathrm{d}w}_{t-1}), \\ \widetilde{\mathsf{Cov}}(v_t, \widetilde{\mathrm{d}w}_t) &= \tilde{\sigma}_w^2 - \rho_1\widetilde{\mathsf{Cov}}(v_t, \mathrm{d}y_t) - \rho_2\widetilde{\mathsf{Cov}}(v_t, \widetilde{\mathrm{d}w}_{t-1}) \\ &= \tilde{\sigma}_w^2 - \rho_1\bar{\gamma} - \rho_1\bar{\mu}\widetilde{\mathsf{Cov}}(v_{t-1}, \widetilde{\mathrm{d}v}_{t-1}) - \rho_2\widetilde{\mathsf{Cov}}(v_{t-1}, \widetilde{\mathrm{d}w}_{t-1}) \\ &= \tilde{\sigma}_w^2\widetilde{\mathsf{Cov}}(v_t, \widetilde{\mathrm{d}v}_t), \end{aligned} \tag{IA.209}$$

with the last equality coming from $\rho_1 = \tilde{\sigma}_w^2\theta_1$ and $\rho_2 = \tilde{\sigma}_w^2\theta_2$. To apply Lemma A.1 in the Appendix in the paper, I impose the condition:

$$\theta_1\bar{\mu} \;\in\; (-1, 1). \tag{IA.210}$$

From (IA.209) it follows that all covariances involved are constant and satisfy:

$$\begin{aligned} \widetilde{\mathsf{Cov}}(\mathrm{d}y_t, \widetilde{\mathrm{d}w}_{t-2}) &= \frac{\theta_2\bar{\mu}W}{1+\theta_1\bar{\mu}}, \qquad \widetilde{\mathsf{Cov}}(\mathrm{d}y_{t-1}, \widetilde{\mathrm{d}w}_{t-2}) \;=\; \bar{\mu}Z, \\ \widetilde{\mathsf{Cov}}(v_t, \widetilde{\mathrm{d}v}_t) &= \frac{1-\theta_1\bar{\gamma}}{1+\theta_1\bar{\mu}+\theta_2\tilde{\sigma}_w^2}, \qquad B \;=\; \widetilde{\mathsf{Cov}}(v_t, \widetilde{\mathrm{d}w}_t) \;=\; \frac{(1-\theta_1\bar{\gamma})\tilde{\sigma}_w^2}{1+\theta_1\bar{\mu}+\theta_2\tilde{\sigma}_w^2}, \\ \widetilde{\mathsf{Cov}}(v_t, \mathrm{d}y_t) &= \frac{(1+\theta_2\tilde{\sigma}_w^2)\bar{\gamma}+\bar{\mu}}{1+\theta_1\bar{\mu}+\theta_2\tilde{\sigma}_w^2}, \qquad C \;=\; \widetilde{\mathsf{Cov}}(\widetilde{\mathrm{d}w}_t, \widetilde{\mathrm{d}w}_{t-1}) \;=\; -\rho_1\bar{\mu}Z - \rho_2 W. \end{aligned} \tag{IA.211}$$

From (IA.205) one obtains:

$$\begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \frac{1}{1+\theta_1\bar{\mu}+\theta_2\tilde{\sigma}_w^2} \begin{bmatrix} Y & \bar{\mu}Z & 0 \\ \bar{\mu}Z & W & \frac{\theta_2\bar{\mu}W}{1+\theta_1\bar{\mu}} \\ 0 & \frac{\theta_2\bar{\mu}W}{1+\theta_1\bar{\mu}} & Y \end{bmatrix}^{-1} \begin{bmatrix} (1+\theta_2\tilde{\sigma}_w^2)\bar{\gamma}+\bar{\mu} \\ (1-\theta_1\bar{\gamma})\tilde{\sigma}_w^2 \\ (1+\theta_2\tilde{\sigma}_w^2)\bar{\gamma}+\bar{\mu} \end{bmatrix}. \tag{IA.212}$$

Speculators' optimal strategy (γ, μ)

Consider a FT, indexed by $i = 1, \ldots, N_F$. At $t = 0$ he chooses a trading strategy of the form:

$$\mathrm{d}x_t^i \;=\; \gamma^i \mathrm{d}v_t + \mu^i \widetilde{\mathrm{d}v}_{t-1}, \tag{IA.213}$$

where $\widetilde{\mathrm{d}v}_t$ satisfies:

$$\widetilde{\mathrm{d}v}_t \;=\; \mathrm{d}v_t - \theta_1 \mathrm{d}y_t - \theta_2 \widetilde{\mathrm{d}w}_{t-1}, \tag{IA.214}$$

and the coefficients θ_1 and θ_2 are fixed.[17] The aggregate order flow is of the form:

$$\begin{aligned} \mathrm{d}y_t \;&=\; \bar{\gamma}\mathrm{d}w_t + \bar{\mu}\widetilde{\mathrm{d}w}_{t-1} + \mathrm{d}u_t, \quad \text{with} \\ \bar{\gamma} \;&=\; \gamma^i + \gamma^{-i}, \qquad \bar{\mu} \;=\; \mu^i + \mu^{-i}, \end{aligned} \tag{IA.215}$$

where the superscript "$-i$" indicates the aggregate quantity from the other speculators. FT i takes as given the coefficients γ^{-i} and μ^{-i}, and assumes the following functional form for the price:

$$p_t \;=\; p_{t-1} + \lambda_2 \widetilde{\mathrm{d}w}_{t-2} + \lambda_3 \mathrm{d}y_t, \tag{IA.216}$$

where the coefficients λ_1, λ_2 and λ_3 are fixed. The normalized expected profit of FT i at $t = 0$ is $\frac{1}{\sigma_v^2}\,\mathsf{E}\int_0^T (v_t - p_t)\mathrm{d}x_t^i$:

$$\begin{aligned} \tilde{\pi}_F^i \;&=\; \frac{1}{\sigma_v^2}\,\mathsf{E}\int_0^T \Big(v_t - p_{t-1} - \lambda_2\widetilde{\mathrm{d}w}_{t-2} - \lambda_3 \mathrm{d}y_t\Big)\Big(\gamma^i \mathrm{d}v_t + \mu^i \widetilde{\mathrm{d}v}_{t-1}\Big) \\ &=\; \frac{1}{\sigma_v^2}\,\mathsf{E}\int_0^T \Big(v_t - \lambda_3\bar{\gamma}\mathrm{d}v_t - \lambda_2\widetilde{\mathrm{d}w}_{t-2} - \lambda_3\bar{\mu}\widetilde{\mathrm{d}v}_{t-1}\Big)\Big(\gamma^i \mathrm{d}w_t + \mu^i \widetilde{\mathrm{d}w}_{t-1}\Big) \\ &=\; \big(\gamma^i - \lambda_3\gamma^i\bar{\gamma}\big)\tilde{\sigma}_w^2 + \mu^i\big(B - \lambda_2 C\big) - \lambda_3\mu^i\bar{\mu}Z. \end{aligned} \tag{IA.217}$$

This is the same problem as in the benchmark model (without public information) where B is replaced by $B - \lambda_2 C$ and A is replaced by Z. The second order condition for this

[17] In equilibrium, $\widetilde{\mathrm{d}v}_{t-1}$ is the component of $\mathrm{d}v_{t-1}$ orthogonal to the information set $\mathcal{I}_t$ before trading at t.

maximization is:

$$\lambda_3 > 0, \qquad Z > 0. \tag{IA.218}$$

One has thus a unique symmetric equilibrium, with the following coefficients:

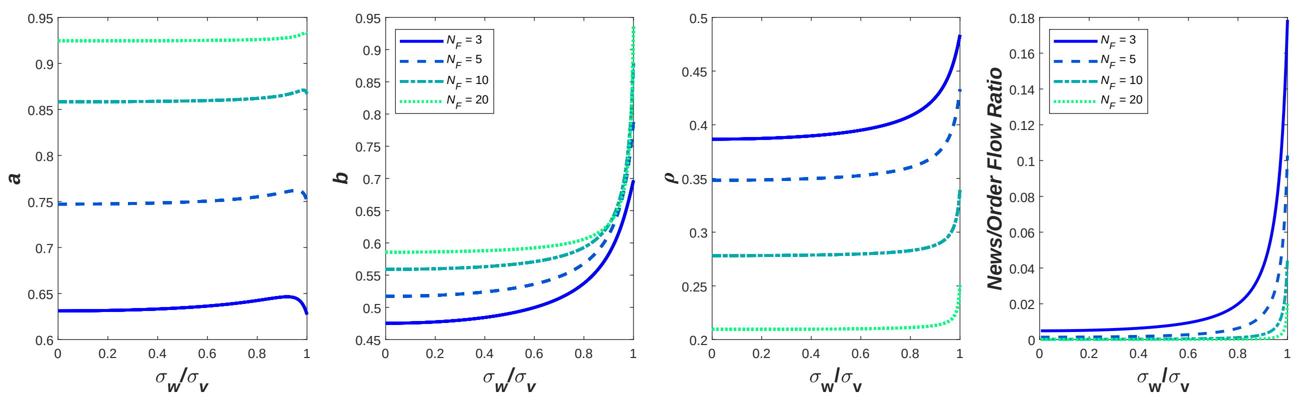
$$\gamma = \frac{1}{\lambda_3} \frac{1}{N_F + 1}, \qquad \mu = \frac{(B - \lambda_2 C)/Z}{\lambda_3} \frac{1}{N_L + 1}. \tag{IA.219}$$

The aggregate coefficients are:

$$\bar{\gamma} = \frac{1}{\lambda_3} \frac{N_F}{N_F + 1}, \qquad \bar{\mu} = \frac{(B - \lambda_2 C)/Z}{\lambda_3} \frac{N_L}{N_L + 1}. \tag{IA.220}$$

Figure IA.3: Optimal inventory mean reversion

This figure shows various equilibrium quantities against the precision of public information, $\tilde{\sigma}_w = \sigma_w/\sigma_v$. The equilibrium values in the four plots are: (i) a, (ii) b, (iii) ρ, and (iv) news-to-order-flow ratio, measured as the variance ratio of the public information compnent $\lambda_2 \widetilde{\mathrm{d}w}_{t-2}$ to the order flow component $\lambda_3 \mathrm{d}y_t$. The parameter values are: $\sigma_u = 1$, $N_S = 2$, and $N_F \in \{3, 5, 10, 20\}$, and $N_L = N_F + N_S$.



Numerical results

Figure IA.3 illustrates the results. When the public precision $\tilde{\sigma}_w = \frac{\sigma_w}{\sigma_v}$ is close to zero, the model approaches the benchmark model in the paper (without public information). At the other end, when the public precision $\tilde{\sigma}_w$ is close to one, the model resembles the model with perfect public information described in Subsection 3.1.

In general, unless the public precision is very high, the equilibrium is closer to the benchmark model, which does not involve public information. Furthermore, the fourth plot in Figure IA.3 shows that the price variance caused by public news is much smaller than (usually less than 1% of) the price variance caused by order flow.

These results suggest than under plausible conditions (that the public precision is not very high) the behavior of the "correct" model is much closer to the benchmark model discussed in the paper. Moreover, under these conditions the effect of public information on prices is very small and can be ignored.

4 Robust trading strategies

The purpose of this section is twofold. First, I verify that the intuition of the benchmark model in the paper extends to a setup in which the fundamental value has more than one component. For simplicity, I focus on extending the model $\mathcal{M}_0$ in which all speculators trade only on their current signal (with no lags). Thus, if $\mathrm{d}w_t$ is the current signal about only one of two orthogonal components of the fundamental value, I verify that trading strategies of the form:

$$\mathrm{d}x_t \;=\; \gamma_t \mathrm{d}w_t \tag{IA.221}$$

remain profitable.

Second, when the fundamental value has two components, I study the decision of speculators to use "smooth" strategies of the Kyle (1985) type:[18]

$$\mathrm{d}x_t \;=\; \beta_t(w_t - p_t)\mathrm{d}t. \tag{IA.222}$$

These strategies are not allowed in the paper, because by using the forecast w_t, the speculator would use an infinite number of lags: $w_t = \mathrm{d}w_t + \mathrm{d}w_{t-1} + \mathrm{d}w_{t-2} + \cdots$ (recall that, by notation, $X_{t-1} = X_{t-\mathrm{d}t}$). In this section, I show that using any smooth strategy as in (IA.222) would produce an expected loss for certain parameter values. In this sense, smooth strategies are not robust to the alternate model in which the asset value is multidimensional.

4.1 Motivation

In most trading models with asymmetric information, speculators learn only about one component of the asset's fundamental value. For instance, in Kyle (1985), the unique informed trader (the "insider") uses private information to generate profits smoothly over time, using a strategy as in equation (IA.222). Thus, the insider compares his forecast with the price, and then buys slowly if his forecast is above the price, and sells otherwise. The implicit assumption in Kyle (1985) is that the price only contains information about his signal, and thus the insider has no inference problem: he knows his information to be superior to that of the public's.

I now introduce a second component of the fundamental value, as in Subrahmanyam and Titman (1999), and allow a different group of speculators to learn about this sec-

[18]In Kyle (1985) w_t is in fact constant. Back and Pedersen (1998), however, show that the same type of strategies are optimal even if the fundamental value changes over time.

ond component.[19] I show that in this case the smooth strategy in (IA.222) starts losing money if the parameters related to the other component of the asset value are large enough. In other words, smooth strategies are not robust. By contrast, quick strategies as in equation (IA.221) are robust. Indeed, Proposition IA.5 below shows that the expected profit from this strategy is positive, and stays constant under all these specifications (taking the price impact coefficient λ as given).

Intuitively, when the fundamental value has multiple components, a speculator who specializes in only one component is potentially adversely selected when using the price to decide his strategy. For instance, suppose the value of IBM has both a domestic and an international component. Then, suppose that a hedge fund that specializes only in the IBM's domestic component uses a smooth strategy as in (IA.222). Then, by buying and selling at the public price, the hedge fund essentially behaves as a noise trader with respect to the international component, and can therefore make losses on average. If instead, the hedge fund uses a quick strategy and buys if its signal about the domestic component is positive, its average profit is not affected by what happens in the international component.

4.2 Multidimensional asset value

I now describe formally the model with two components of the fundamental value. Suppose the liquidation value of the risk asset v_T (at $T = 1$) can be decomposed as a sum:

$$v_T \;=\; w_T + e_T, \qquad T \;=\; 1. \tag{IA.223}$$

I consider a model similar to $\mathcal{M}_0$ from Section 2 in the paper, in which speculators only use their current signal (see also Proposition 3). There are $N_w \geq 1$ speculators, called the "w-speculators," who learn about w_T by observing at each t the increment $\mathrm{d}w_t$ of a diffusion process with terminal value w_T. Also, there are $N_e \geq 1$ speculators, called the "e-speculators," who learn about e_T by observing at each t the increment $\mathrm{d}e_t$ of a diffusion process with terminal value e_T.[20] Recall that in the benchmark model, the speculators receive signals of the form $\mathrm{d}s_t = \mathrm{d}v_t + \mathrm{d}\eta_t$, such that the increment of

[19]Subrahmanyam and Titman (1999) have a one-period model with information acquisition. They find that multidimensional asset values generate liquidity complementarity, in the sense that informed traders in one component of the asset value behave as noise traders in the second component, and thus encourage information production in that component.

[20]I do not model the information acquisition explicitly. Subrahmanyam and Titman (1999) solve a one-period model with endogenous information acquisition, and analyze the liquidity externalities that result from this choice.

their forecast w_t is equal to $\mathrm{d}w_t = a\mathrm{d}s_t$, with $a = \frac{\sigma_v^2}{\sigma_v^2+\sigma_\eta^2}$. In this section, however, I do not need these explicit formulas, and assume instead that the w-speculators directly observe $\mathrm{d}w_t$. Thus, the fundamental value increment $\mathrm{d}v_t$ has the following orthogonal decomposition:

$$\mathrm{d}v_t \;=\; \mathrm{d}w_t + \mathrm{d}e_t, \quad \text{with} \quad \sigma_e^2 \;=\; \sigma_v^2 - \sigma_w^2 \;=\; \frac{\sigma_v^2 \sigma_\eta^2}{\sigma_v^2 + \sigma_\eta^2}. \tag{IA.224}$$

Proposition IA.5 describes the equilibrium of the model.

Proposition IA.5. *Consider $N_w + N_e$ speculators, of which N_w speculators learn about w_t, and N_e speculators learn about e_t. Denote the position in the risky asset of an w- or e-speculator, respectively, by $x_{w,t}$ and $x_{e,t}$. Assume that the speculators can only trade on their most recent signal, $\mathrm{d}w_t$ or $\mathrm{d}e_t$, respectively. Then, there exists a unique linear equilibrium, in which the speculators' trading strategies, and the dealer's pricing functions are of the form:*

$$\mathrm{d}x_{w,t} \;=\; \gamma_w \mathrm{d}w_t, \qquad \mathrm{d}x_{e,t} \;=\; \gamma_e \mathrm{d}e_t, \qquad \mathrm{d}p_t \;=\; \lambda \mathrm{d}y_t, \tag{IA.225}$$

with equilibrium coefficients:

$$\gamma_w \;=\; \frac{1}{\lambda}\frac{1}{N_w+1}, \quad \gamma_e \;=\; \frac{1}{\lambda}\frac{1}{N_e+1}, \quad \lambda \;=\; \left(\frac{N_w}{(N_w+1)^2}\frac{\sigma_w^2}{\sigma_u^2} + \frac{N_e}{(N_e+1)^2}\frac{\sigma_e^2}{\sigma_u^2}\right)^{1/2}. \tag{IA.226}$$

The expected profits at $t = 0$ of the w- and e-speculators, are, respectively,

$$\pi_w \;=\; \frac{\sigma_w^2}{\lambda(N_w+1)^2}, \quad \pi_e \;=\; \frac{\sigma_w^2}{\lambda(N_e+1)^2}. \tag{IA.227}$$

Proof. See Subsection 4.4 below. ■

Proposition IA.5 shows that, taking λ as given, the w-speculators have indeed the same strategy, and the same expected profits regardless of the structure of the e-component. That is to say, the strategy and profits of the w speculators do not depend on N_e or σ_e. The magnitude of the price impact coefficient λ, however, does change with the specification, because of the increase in adverse selection in the other component.

4.3 Smooth trading with multidimensional value

I next analyze the expected profit of a speculator that combines smooth strategies as in (IA.222) with quick strategies as in (IA.221). Proposition IA.6 provides a formula for the expected profit of this type of speculator.

Proposition IA.6. *Suppose now that one of the w-speculators, now called the "β-speculator," adds a smooth component to his equilibrium trading strategy:*

$$\mathrm{d}x_t^\beta \;=\; \beta_t(w_{t-1} - p_{t-1})\mathrm{d}t + \gamma_w \mathrm{d}w_t, \tag{IA.228}$$

while the other traders and the dealer maintain their equilibrium strategies. Define:

$$\varepsilon_t \;=\; \mathrm{e}^{-\lambda\left(\int_t^1 \beta_\tau \mathrm{d}\tau\right)}. \tag{IA.229}$$

Then, the expected profit of the β-speculator at $t = 0$ equals:

$$\begin{aligned} \pi \;&=\; \pi^0 + \pi^\beta, \quad \text{with} \quad \pi^0 \;=\; \frac{\sigma_w^2}{\lambda(N_w+1)^2}, \quad \text{and} \\ \pi^\beta \;&=\; \frac{1}{2\lambda}\int_0^T \big(1-\varepsilon_t\big)\left((1+\varepsilon_t)\frac{1}{N_w+1}\sigma_w^2 - (1-\varepsilon_t)\frac{N_e}{N_e+1}\sigma_e^2\right)\mathrm{d}t. \end{aligned} \tag{IA.230}$$

Proof. See Subsection 4.4 below. ■

An important implication of Proposition IA.6 is that the profit π^β depends on the specification of the model. Consider the following cases:

- If $\sigma_e = 0$, the fundamental value has only one component. Then, the β-speculator increases his profit by β-trading: $\frac{\sigma_w^2}{2\lambda(N_w+1)}\int_0^T(1-\varepsilon_t^2)\mathrm{d}t > 0$. Moreover, if as in Kyle (1985) the β-speculator sets $\beta_t = \frac{\beta_0}{1-t} > 0$, then $\varepsilon_t = 0$, and the profit is maximized.

- If $\sigma_e > 0$, there is more than one component of the fundamental value. Then, I show that $\beta_t > 0$ produces a loss for certain values of σ_e (and N_e). Indeed, the condition $\beta_t > 0$ translates into ε_t not being identically equal to 1, or equivalently $\int_0^T(1-\varepsilon_t)^2\mathrm{d}t > 0$. Choose a value σ_e such that:

  $$\sigma_e^2 \;>\; \frac{N_e+1}{N_e}\,\frac{\frac{2}{(N_w+1)^2} + \frac{\int_0^T(1-\varepsilon_t^2)\mathrm{d}t}{N_w+1}}{\int_0^T(1-\varepsilon_t)^2\mathrm{d}t}\,\sigma_w^2. \tag{IA.231}$$

  Then, one can easily verify that $\pi^\beta < 0$.

In other words, the β-strategy in equation (IA.228) is not robust to the fundamental value having more than one component.

Corollary IA.1. *The maximum profit of the β-speculator is $\pi_{\max} = \pi^0 + \pi^{\beta}_{\max}$, where:*

$$\frac{\pi^{\beta}_{\max}}{\pi^0} \;=\; \frac{1}{2}\,\frac{1}{\frac{1}{N_w+1} + \frac{N_e}{N_+1}\frac{\sigma_e^2}{\sigma_w^2}} \;<\; \frac{\sigma_w^2}{\sigma_e^2}. \tag{IA.232}$$

Thus, the profit of the β-speculator that comes from the β component is at most equal to the variance ratio σ_w^2/σ_e^2. In general, one can think of the e-component as the orthogonal component to the w-component, in which case it is plausible that its instantaneous variance σ_e^2 is much larger than σ_w^2. Then, Corollary IA.1 implies that the profit of the β-speculator that comes from the β-component is very small. This justifies why I ignore the β-component in the paper.

4.4 Proofs

Proof of Proposition IA.5. I first determine the optimal strategies of the speculators, taking the dealer's pricing rule as given. The expected profit at $t = 0$ of the i'th w-speculator is:

$$\begin{aligned}\pi^i_{w,0} \;&=\; \mathsf{E}\int_0^T \left(w_t + e_t - p_{t-1} - \lambda_t\Big((\gamma^i_{w,t} + \gamma^{-i}_{w,t})\mathrm{d}w_t + \sum_{j=1}^{N_e}\gamma^j_{e,t}\mathrm{d}e_t + \mathrm{d}u_t\Big)\right)\gamma^i_{w,t}\mathrm{d}w_t \\ &=\; \int_0^T \gamma^i_{w,t}\sigma_w^2\mathrm{d}t - \lambda_t\gamma^i_{w,t}\big(\gamma^i_{w,t} + \gamma^{-i}_{w,t}\big)\sigma_w^2\mathrm{d}t,\end{aligned} \tag{IA.233}$$

where $\gamma^{-i}_{w,t}$ denotes the aggregate coefficient of the other $N_w - 1$ w-speculators. This is a pointwise quadratic optimization problem, with solution $\lambda_t\gamma^i_{w,t} = \frac{1-\lambda_t\gamma^{-i}_{w,t}}{2}$. Since this is true for each w-speculator, one obtains that all $\gamma^i_{w,t}$ are equal to $\gamma_{w,t} = \frac{1}{\lambda_t(N_w+1)}$. Similarly, all $\gamma^j_{e,t}$ are equal to $\gamma_{e,t} = \frac{1}{\lambda_t(N_e+1)}$. Combining these two equations, one obtains:

$$\gamma^i_{w,t} \;=\; \gamma_{w,t} \;=\; \frac{1}{\lambda_t(N_w+1)}, \qquad \gamma^j_{e,t} \;=\; \gamma_{e,t} \;=\; \frac{1}{\lambda_t(N_e+1)}. \tag{IA.234}$$

I now determine the dealer's pricing rule, taking the behavior of the speculators as given. The dealer assumes that the aggregate order flow is $\mathrm{d}y_t = N_w\gamma_t\mathrm{d}w_t + N_e\phi_t\mathrm{d}e_t + \mathrm{d}u_t$.

To set λ_t, the dealer sets p_t such that the market is efficient, which implies:

$$\mathrm{d}p_t \;=\; \lambda_t \mathrm{d}y_t, \quad \text{with} \quad \lambda_t \;=\; \frac{\mathsf{Cov}(w_t + e_t, \mathrm{d}y_t)}{\mathsf{Var}(\mathrm{d}y_t)} \;=\; \frac{N_w \gamma_{w,t} \sigma_w^2 + N_e \gamma_{e,t} \sigma_e^2}{N_w^2 \gamma_{w,t}^2 \sigma_w^2 + N_e^2 \gamma_{t,e}^2 \sigma_e^2 + \sigma_u^2}. \tag{IA.235}$$

This implies $(N_w \lambda_t \gamma_{w,t})^2 \sigma_w^2 + (N_e \lambda_t \gamma_{e,t})^2 \sigma_e^2 + \lambda_t^2 \sigma_u^2 = N_w \lambda_t \gamma_{w,t} \sigma_w^2 + N_e \lambda_t \gamma_{e,t} \sigma_e^2$. But $N_w \lambda_t \gamma_{w,t} = \frac{N_w}{N_w+1}$ and $N_e \lambda_t \gamma_{e,t} = \frac{N_e}{N_e+1}$. Hence, $\lambda_t^2 \sigma_u^2 = \frac{N_w}{(N_w+1)^2} \sigma_w^2 + \frac{N_e}{(N_e+1)^2} \sigma_e^2$, which implies:

$$\lambda_t \;=\; \lambda \;=\; \left(\frac{N_w}{(N_w+1)^2} \frac{\sigma_w^2}{\sigma_u^2} + \frac{N_e}{(N_e+1)^2} \frac{\sigma_e^2}{\sigma_u^2} \right)^{1/2}. \tag{IA.236}$$

This proves the stated formulas. ■

Proof of Proposition IA.6. The trading strategy of the β-speculator is of the form:

$$\mathrm{d}x_t \;=\; \beta_t (w_{t-1} - p_{t-1}) \mathrm{d}t + \gamma_w \mathrm{d}w_t \tag{IA.237}$$

where γ_w is the equilibrium (constant) value. Denote the aggregate coefficients:

$$\bar{\gamma}_w \;=\; N_w \gamma_w, \qquad \bar{\gamma}_e \;=\; N_e \gamma_e. \tag{IA.238}$$

Define the following (normalized) covariances:

$$\Sigma_t \;=\; \frac{\mathsf{E}\big((w_t - p_t)^2\big)}{\sigma_w^2}, \qquad \Omega_t \;=\; \frac{\mathsf{E}\big(e_t p_t\big)}{\sigma_w^2}, \qquad \tilde{\sigma}_e^2 \;=\; \frac{\sigma_e^2}{\sigma_w^2}. \tag{IA.239}$$

Since $w_0 = e_0 = p_0$, one obtains:

$$\Sigma_0 \;=\; \Omega_0 \;=\; 0. \tag{IA.240}$$

The normalized expected profit of the β-speculator at $t = 0$ is:

$$\begin{aligned}
\tilde{\pi} \;&=\; \frac{1}{\sigma_w^2} \mathsf{E} \int_0^T \Big(w_t + e_t - p_{t-1} - \lambda \Big(\beta_t (w_{t-1} - p_{t-1}) \mathrm{d}t + \bar{\gamma}_w \mathrm{d}w_t + \bar{\gamma}_e \mathrm{d}e_t + \mathrm{d}u_t \Big) \Big) \times \\
&\qquad \times \Big(\beta_t (w_{t-1} - p_{t-1}) \mathrm{d}t + \gamma_w \mathrm{d}w_t \Big) \\
&=\; \int_0^T \Big(\gamma_w - \lambda \gamma_w \bar{\gamma}_w + \beta_t \Sigma_{t-1} - \beta_t \Omega_{t-1} \Big) \mathrm{d}t \\
&=\; \tilde{\pi}^0 \;+\; \int_0^T \Big(\beta_t \Sigma_{t-1} - \beta_t \Omega_{t-1} \Big) \mathrm{d}t,
\end{aligned} \tag{IA.241}$$

where $\tilde{\pi}^0$ is the normalized profit of the β-speculator when $\beta = 0$, that is:

$$\tilde{\pi}^0 \;=\; \gamma_w - \lambda\gamma_w\bar{\gamma}_w \;=\; \frac{\gamma_w}{N_w+1} \;=\; \frac{1}{\lambda(N_w+1)^2}. \tag{IA.242}$$

Since $\mathrm{d}w_t - \mathrm{d}p_t = -\lambda\beta_t(w_{t-1} - p_{t-1})\mathrm{d}t + (1-\lambda\bar{\gamma}_w)\mathrm{d}w_t - \lambda\bar{\gamma}_e\mathrm{d}e_t - \lambda\mathrm{d}u_t$, Σ_t satisfies:

$$\begin{aligned}\frac{\mathrm{d}\Sigma_t}{\mathrm{d}t} \;&=\; \frac{1}{\sigma_w^2\mathrm{d}t}\,\mathsf{E}\big(2(w_{t-1}-p_{t-1})(\mathrm{d}w_t-\mathrm{d}p_t) + (\mathrm{d}w_t-\mathrm{d}p_t)^2\big)\\ &=\; -2\lambda\beta_t\Sigma_{t-1} + (1-\lambda\bar{\gamma}_w)^2 + (\lambda\bar{\gamma}_e)^2\tilde{\sigma}_e^2 + \lambda^2\tilde{\sigma}_u^2.\end{aligned} \tag{IA.243}$$

This is a first order ODE with solution:

$$\begin{aligned}\Sigma_t \;&=\; D_\Sigma\,\mathrm{e}^{-2\lambda B_t}\int_0^t \mathrm{e}^{2\lambda B_\tau}\,\mathrm{d}\tau, \quad \text{with}\\ B_t \;&=\; \int_0^t \beta_\tau \mathrm{d}\tau, \qquad D_\Sigma \;=\; (1-\lambda\bar{\gamma}_w)^2 + (\lambda\bar{\gamma}_e)^2\tilde{\sigma}_e^2 + \lambda^2\tilde{\sigma}_u^2.\end{aligned} \tag{IA.244}$$

Recall a formula derived in the computation of λ:

$$(\lambda\bar{\gamma}_w)^2 + (\lambda\bar{\gamma}_e)^2\tilde{\sigma}_e^2 + \lambda^2\tilde{\sigma}_u^2 \;=\; \lambda\bar{\gamma}_w + \lambda\bar{\gamma}_e\tilde{\sigma}_e^2. \tag{IA.245}$$

Then, one computes:

$$D_\Sigma \;=\; 1 - \lambda\bar{\gamma}_w + \lambda\bar{\gamma}_e\tilde{\sigma}_e^2 \;=\; \frac{1}{N_w+1} + \frac{N_e}{N_e+1}\tilde{\sigma}_e^2. \tag{IA.246}$$

By integrating (IA.243) over $[0,T]$ (and using $\Sigma_0 = 0$), one also computes:

$$\int_0^T \beta_t\Sigma_{t-1}\mathrm{d}t \;=\; \frac{D_\Sigma - \Sigma_1}{2\lambda}. \tag{IA.247}$$

Since $\mathrm{d}p_t = \lambda\beta_t(w_{t-1}-p_{t-1})\mathrm{d}t + \lambda\bar{\gamma}_w\mathrm{d}w_t + \lambda\bar{\gamma}_e\mathrm{d}e_t + \lambda\mathrm{d}u_t$, Ω_t satisfies:

$$\begin{aligned}\frac{\mathrm{d}\Omega_t}{\mathrm{d}t} \;&=\; \frac{1}{\sigma_w^2\mathrm{d}t}\,\mathsf{E}\big(p_{t-1}\mathrm{d}e_t + e_{t-1}\mathrm{d}p_t + \mathrm{d}e_t\mathrm{d}p_t\big)\\ &=\; -\lambda\beta_t\Omega_{t-1} + \lambda\bar{\gamma}_e\tilde{\sigma}_e^2.\end{aligned} \tag{IA.248}$$

This is a first order ODE with solution:

$$\begin{aligned}\Omega_t \;&=\; D_\Omega\,\mathrm{e}^{-\lambda B_t}\int_0^t \mathrm{e}^{\lambda B_\tau}\,\mathrm{d}\tau, \quad \text{with}\\ D_\Omega \;&=\; \lambda\bar{\gamma}_e\tilde{\sigma}_e^2 \;=\; \frac{N_e}{N_e+1}\tilde{\sigma}_e^2.\end{aligned} \tag{IA.249}$$

By integrating (IA.248) over $[0,T]$ (and using $\Omega_0 = 0$), one also computes:

$$\int_0^T \beta_t \Omega_{t-1} \mathrm{d}t \;=\; \frac{D_\Omega - \Omega_1}{\lambda}. \tag{IA.250}$$

One computes:

$$\Sigma_1 \;=\; D_\Sigma \int_0^T \mathrm{e}^{-2\lambda(B_1 - B_t)}\,\mathrm{d}t, \qquad \Omega_1 \;=\; D_\Omega \int_0^T \mathrm{e}^{-\lambda(B_1 - B_t)}\,\mathrm{d}t. \tag{IA.251}$$

Combining the formulas above, one computes:

$$\begin{aligned} \tilde{\pi} \;&=\; \tilde{\pi}^0 + \frac{D_\Sigma}{2\lambda}\left(1 - \int_0^T \mathrm{e}^{-2\lambda(B_1 - B_t)}\,\mathrm{d}t\right) - \frac{D_\Omega}{\lambda}\left(1 - \int_0^T \mathrm{e}^{-\lambda(B_1 - B_t)}\,\mathrm{d}t\right), \\ B_t \;&=\; \int_0^t \beta_\tau \mathrm{d}\tau, \qquad D_\Sigma \;=\; \frac{1}{N_w + 1} + \frac{N_e}{N_e + 1}\tilde{\sigma}_e^2, \qquad D_\Omega \;=\; \frac{N_e}{N_e + 1}\tilde{\sigma}_e^2. \end{aligned} \tag{IA.252}$$

If I define:

$$\varepsilon_t \;=\; \mathrm{e}^{-\lambda(B_1 - B_t)} \;=\; \mathrm{e}^{-\lambda(\int_t^1 \beta_\tau \mathrm{d}\tau)} \;\in\; [0,T], \tag{IA.253}$$

One computes:

$$\begin{aligned} \tilde{\pi} \;&=\; \tilde{\pi}^0 + \frac{1}{\lambda}\int_0^T (1 - \varepsilon_t)\left(D_\Sigma \frac{1 + \varepsilon_t}{2} - D_\Omega\right)\mathrm{d}t \\ &=\; \tilde{\pi}^0 + \frac{1}{2\lambda}\int_0^T (1 - \varepsilon_t)\left(\frac{1 + \varepsilon_t}{N_w + 1} - \frac{(1 - \varepsilon_t) N_e \tilde{\sigma}_e^2}{N_e + 1}\right)\mathrm{d}t. \end{aligned} \tag{IA.254}$$

This completes the proof. ■

Proof of Corollary IA.1. For two constants $a, b > 0$, define the following function:

$$F \;:\; (0,1) \;\to\; \mathbb{R}, \quad F(\varepsilon) \;=\; (1 - \varepsilon^2)a - (1 - \varepsilon)^2 b. \tag{IA.255}$$

The first order condition for a maximum is $F'(\varepsilon) \;=\; -2\varepsilon a + 2(1 - \varepsilon)b \;=\; 0$, and the second order condition is $F''(\varepsilon) \;=\; -2(a + b) \;<\; 0$, which is satisfied at all ε. The optimum corresponds to:

$$\varepsilon^* \;=\; \frac{a}{a + b}, \qquad F(\varepsilon^*) \;=\; \frac{a^2}{a + b}. \tag{IA.256}$$

Then, if I define:

$$a \;=\; \frac{\sigma_w^2}{N_w + 1}, \qquad b \;=\; \frac{N_e \sigma_e^2}{N_e + 1}, \tag{IA.257}$$

one has $\pi^\beta = \frac{1}{2\lambda}\int_0^1 F(\varepsilon_t)\mathrm{d}t$. Thus, according to (IA.256), the maximum value of π^β is $\frac{a^2}{a+b}$, where a and b are as in (IA.257). Using $\pi^0 = \frac{\sigma_w^2}{\lambda(N_w+1)^2}$, one computes:

$$\pi^\beta_{\max} = \frac{1}{2\lambda}\frac{\frac{\sigma_w^4}{(N_w+1)^2}}{\frac{\sigma_w^2}{N_w+1}+\frac{N_e\sigma_e^2}{N_e+1}}, \qquad \frac{\pi^\beta_{\max}}{\pi^0} = \frac{1}{2}\frac{1}{\frac{1}{N_w+1}+\frac{N_e}{N_e+1}\frac{\sigma_e^2}{\sigma_w^2}} < \frac{N_e+1}{2N_e}\frac{\sigma_e^2}{\sigma_w^2}. \quad \text{(IA.258)}$$

Since $N_e \geq 1$, one gets $\frac{\pi^\beta_{\max}}{\pi^0} < \frac{\sigma_w^2}{\sigma_e^2}$, which proves (IA.232).

Note that the maximum profit of the β-speculator is attained when ε_t is constant and equal to $a/(a+b)$, with a and b as in (IA.257). But ε_t depends on β_t via equation (IA.253): $\varepsilon_t = \mathrm{e}^{-\lambda(\int_t^1 \beta_\tau \mathrm{d}\tau)}$. This implies that the maximum is attained when $\int_t^1 \beta_\tau \mathrm{d}\tau$ is constant for all t, and is equal to $\frac{\ln(a+b)-\ln(a)}{\lambda}$. This occurs when β_t is equal to zero for all $t < 1$, and approaches infinity at $t = 1$.[21] Interestingly, at the optimum, β_t is zero for all values of $t < 1$. ■

[21] This a multiple of the Dirac delta function at $t = 1$. In fact, there is no actual function for which $B_t = \int_t^1 \beta_\tau \mathrm{d}\tau$ is a positive constant for all t, but there are functions which are arbitrarily close to the Dirac delta function, such that B_t is arbitrarily close to the given constant.

5 Quick inventory management

5.1 Inventory management with one IFT

In this subsection, I discuss the general equilibrium of the model with inventory management from Section 4 in the paper, and provide the proofs that have been left out of the paper. Recall that in this model there are N_F FTs (fast traders), N_L STs (slow traders), and one IFT (inventory-averse fast trader) who maximizes his expected profit subject to a quadratic penalty in his inventory in the risky asset. (More details are given below.) As in Section 4 in the paper, I assume that the IFT's trading strategy is in the "quick regime," meaning that it has the form:

$$\mathrm{d}x_t \;=\; -\Theta x_{t-1} + G\mathrm{d}w_t, \quad \text{with} \quad \Theta \in [0,2), \quad G \in \mathbb{R}. \tag{IA.259}$$

In Section 6 below, I also discuss the "smooth regime," in which the IFT's trading strategy is of the form $\mathrm{d}x_t = -\theta x_{t-1}\mathrm{d}t + G\mathrm{d}w_t$, with $\theta \in (0,\infty)$. This case corresponds to a strategy of the form (IA.259) for which $\Theta = \theta\mathrm{d}t$ is infinitesimal. However, in the next section I show numerically that the smooth regime is never optimal for the IFT, and thus it can be ignored in this section.

More specifically, the agents are:

- One IFT, who chooses a trading strategy of the form (IA.259) to maximize his expected utility:

$$U \;=\; \mathsf{E}\left(\int_0^T (v_T - p_t)\mathrm{d}x_t\right) \;-\; C_I\,\mathsf{E}\left(\int_0^T x_t^2\mathrm{d}t\right), \tag{IA.260}$$

  where $T = 1$, and $C_I > 0$ is the IFT's inventory aversion coefficient;

- N_F risk-neutral FTs, who choose a trading strategy $\mathrm{d}x_{i,t} = \gamma_i\mathrm{d}w_t$, with $\gamma_i \in \mathbb{R}$;

- N_L risk-neutral STs, who choose a trading strategy $\mathrm{d}x_{j,t} = \mu_j\widetilde{\mathrm{d}w}_{t-1}$, with $\mu_j \in \mathbb{R}$; the term $\widetilde{\mathrm{d}w}_{t-1}$ is of the form $\widetilde{\mathrm{d}w}_{t-1} = \mathrm{d}w_{t-1} - z_{t-1,t}$, where $z_{t-1,t}$ is the dealer's expectation of $\mathrm{d}w_{t-1}$ given past order flow;

- A dealer who sets a linear pricing rule $\mathrm{d}p_t = \lambda\mathrm{d}y_t$, such that the dealer's expected profit is zero; she also computes $z_{t-1,t} = \mathsf{E}_t(\mathrm{d}w_{t-1}) = \rho\mathrm{d}y_{t-1}$;

- Exogenous noise traders, who on aggregate submit at each t a market order $\mathrm{d}u_t$.

I introduce the following coefficients:

$$
\begin{aligned}
R &= \frac{\lambda}{\rho}, \qquad \gamma^- = \sum_{i=1}^{N_F} \gamma_i, \qquad \bar{\gamma} = \gamma^- + G, \qquad \bar{\mu} = \sum_{j=1}^{N_L} \mu_j, \\
a^- &= \rho\gamma^-, \qquad a = \rho\bar{\gamma}, \qquad b = \rho\bar{\mu}.
\end{aligned}
\tag{IA.261}
$$

Below I describe the equilibrium of the model, by considering one agent at a time and taking the behavior of the other agents as given. Then, I put together all the equilibrium conditions and derive a single system of equations that the coefficients should satisfy. In doing so, I also prove Theorem 3.

Optimal inventory management

I describe the optimal choice of the IFT, while taking the behavior of the FTs, the STs, and the dealer as given. Since I want to prove a more general result than Theorem 2 in the paper, I also analyze the case when the slow trading coefficient b is below the threshold $\leq \frac{\sqrt{17}-1}{8}$. Proposition IA.7 below shows when b is below the threshold, a sufficiently averse IFT optimally chooses Θ positive but as small as possible. I denote this case by:

$$
\Theta = 0_+. \tag{IA.262}
$$

This case is different from $\Theta = 0$. Indeed, at $\Theta = 0$, the IFT's inventory follows a random walk, while at $\Theta > 0$, the IFT's inventory is negligible, as one can see for instance in equation (69). This shows that inventory management has a discontinuity at $\Theta = 0$.

Corollary IA.3 in Section 6 in this Internet Appendix shows that the cases $\Theta = 0$ and $\Theta = 0_+$ are joined continuously by a smooth regime, in which the IFT has a strategy of the form $\mathrm{d}x_t = -\theta x_{t-1}\mathrm{d}t + G\mathrm{d}w_t$, with $\theta \in [0, \infty]$. (Continuity here means that the IFT's expected utility varies continuously across the regimes.) When $\theta = \infty$, the IFT has the same expected utility as in the case $\Theta = 0_+$.

Proposition IA.7. *Consider the behavior of the other speculators and the dealer as given, and fix the coefficients $\gamma \geq 0$, $\mu > 0$, $\lambda > 0$, $\rho > 0$. Define $\bar{\gamma}$, $\bar{\mu}$, a^-, b and R as in (IA.261). Moreover, suppose that $b = \rho\bar{\mu} < 1$. Then, if $\bar{C}_I$ and $\bar{C}_I^0$ as in (IA.293), the optimal strategy of the IFT is as follows:*

(i) If $b \le \frac{\sqrt{17}-1}{8} = 0.3904$*, and* $C_I \le \bar{C}_I^0$*, the IFT sets:*

$$\Theta \;=\; 0, \qquad G \;=\; \frac{1 - Ra^-}{2\lambda + C_I}. \tag{IA.263}$$

If $b \le \frac{\sqrt{17}-1}{8}$*, and* $C_I > \bar{C}_I^0$*, the IFT sets:*

$$\Theta \;=\; 0_+, \qquad G \;=\; \frac{b(1 - a^-)}{2\rho\big(b + \frac{1}{2}\big)}. \tag{IA.264}$$

In the latter case, the maximum expected utility of the IFT is:

$$U^{\max}_{\Theta=0_+} \;=\; \frac{\big(Rb(1 - a^-)\big)^2}{4\lambda(1+b)\big(b + \frac{1}{2}\big)}\sigma_w^2. \tag{IA.265}$$

(ii) If $b > \frac{\sqrt{17}-1}{8}$*, and* $C_I \le \bar{C}_I$*, the IFT sets:*

$$\Theta \;=\; 0, \qquad G \;=\; \frac{1 - Ra^-}{2\lambda + C_I}. \tag{IA.266}$$

If $b > \frac{\sqrt{17}-1}{8}$*, and* $C_I > \bar{C}_I$*, the IFT sets:*

$$\Theta \;=\; 2 - \frac{\sqrt{1-b}}{b}, \qquad G \;=\; \frac{1 - a^-}{2\rho\left(1 + \frac{1}{\sqrt{1-b}}\right)}. \tag{IA.267}$$

In the latter case, the maximum expected utility of the IFT is:

$$U^{\max}_{\Theta>0} \;=\; \frac{\big(Rb(1 - a^-)\big)^2}{4\lambda b(1 + \sqrt{1-b})^2}\sigma_w^2. \tag{IA.268}$$

Proof. See Subsection 5.2. ■

Optimal strategies of fast and slow traders

I describe the optimal choice of the FTs and STs, while taking the behavior of the IFT and the dealer as given.

Proposition IA.8. *Consider the behavior of the IFT and the dealer as given, and fix the coefficients* $G \in \mathbb{R}$*,* $\Theta \in (0, 2)$*,* $\rho \ge 0$*,* $\lambda > 0$*. Suppose there exists a solution to the* *following system of equations ($\phi = 1 - \Theta$):*

$$
\begin{aligned}
\bar{\gamma} &= \frac{N_F}{\lambda(N_F+1)} + \frac{G}{N_F+1}, \qquad \bar{\mu} = \frac{E+\lambda\Theta X}{\lambda W}\,\frac{N_L}{N_L+1}, \\
X &= \frac{\frac{\rho\phi G^2}{1+\phi} + G(1-\rho\bar{\gamma})}{1+\phi\rho\bar{\mu}}, \qquad Z = \frac{\bar{\mu}G(1-\rho\bar{\gamma})}{1+\phi\rho\bar{\mu}} - \frac{G^2}{(1+\phi)(1+\phi\rho\bar{\mu})}, \\
Y &= \frac{-\frac{\Theta G^2}{1+\phi} - 2\Theta Z + \bar{\gamma}^2 + \bar{\mu}^2(1-2\rho\bar{\gamma}) + \tilde{\sigma}_u^2}{1-\rho^2\bar{\mu}^2}, \\
W &= 1 - 2\rho\bar{\gamma} + \rho^2 Y = \frac{-\frac{\Theta\rho^2 G^2}{1+\phi} - 2\rho^2\Theta Z + (1-\rho\bar{\gamma})^2 + \rho^2\tilde{\sigma}_u^2}{1-\rho^2\bar{\mu}^2}, \\
E &= \frac{1-\lambda\bar{\gamma}-\rho\bar{\gamma}+\rho G(1-\lambda\bar{\gamma}) - \lambda\rho\phi Z + \rho\lambda Y}{1+\rho\bar{\mu}},
\end{aligned}
\tag{IA.269}
$$

and that this solution satisfies:

$$
0 \le \rho\bar{\mu} < 1, \qquad W > 0. \tag{IA.270}
$$

Define the coefficients γ and μ:

$$
\gamma = \frac{1-\lambda G}{\lambda(N_F+1)}, \qquad \mu = \frac{E+\lambda\Theta X}{\lambda W(N_L+1)}. \tag{IA.271}
$$

Then, the optimal trading strategies of the FTs and the STs satisfy $\gamma_i = \gamma$ for all $i = 1,\ldots,N_F$, and all $\mu_j = \mu$ for $j = 1,\ldots,N_L$.

Proof. See Subsection 5.2. ■

For future reference, Corollary IA.2 describes the profit function of the FTs and STs if their trading strategy is symmetric (the same for the FTs and the same for the STs), but not necessarily the optimal one.

Corollary IA.2. *Consider the behavior of the IFT and the dealer as given, and suppose the trading strategies of the FTs and the STs satisfy, respectively,*

$$
\mathrm{d}x_t^F = \gamma \mathrm{d}w_t + \mu\widetilde{\mathrm{d}w}_{t-1}, \qquad \mathrm{d}x_t^S = \mu\widetilde{\mathrm{d}w}_{t-1}, \tag{IA.272}
$$

with $\widetilde{\mathrm{d}w}_{t-1} = \mathrm{d}w_{t-1} - \rho\mathrm{d}y_{t-1}$. Denote by $\gamma^- = N_F\gamma$, $\bar{\gamma} = \gamma^- + G$, $\bar{\mu} = N_L\mu$. Then, the expected profits of the FTs and STs satisfy, respectively,

$$
\pi^F = \gamma(1-\lambda\bar{\gamma})\sigma_w^2, \qquad \pi^S = \mu\big(E + \lambda\Theta X - \lambda W\bar{\mu}\big)\sigma_w^2, \tag{IA.273}
$$

where E, X, and W are defined as in (IA.269).

Proof. See equation (IA.312) from the proof of Proposition IA.8. ■

Dealer's pricing rules with inventory management

I describe the dealer's pricing functions, while taking the behavior of the IFT and of the FTs and STs as given.

Proposition IA.9. *Consider the behavior of the speculators as given, and fix the coefficients $G \in \mathbb{R}$, $\Theta \in (0,2)$, $\gamma \geq 0$, $\mu > 0$. Denote by $\phi = 1 - \Theta$, and by $\gamma^- = N_F\gamma$, $\bar{\gamma} = G + \gamma^-$, $\bar{\mu} = N_L\mu$. Suppose the following third degree equation in ρ has a solution $\rho > 0$:*

$$\begin{aligned}\bar{\gamma}(1-\rho^2\bar{\mu}^2)(1+\phi\rho\bar{\mu}) \;=\; & \Theta\,\frac{G^2(1-\phi\rho\bar{\mu}) - 2\bar{\mu}G(1+\phi)(1-\rho\bar{\gamma})}{(1+\phi)} \\ & + \big(\bar{\gamma}^2 + \bar{\mu}^2(1-2\rho\bar{\gamma}) + \tilde{\sigma}_u^2\big)(1+\phi\rho\bar{\mu}).\end{aligned} \tag{IA.274}$$

Define Z and Y by the formulas:

$$Z \;=\; \frac{\bar{\mu}G(1-\rho\bar{\gamma})}{1+\phi\rho\bar{\mu}} - \frac{G^2}{(1+\phi)(1+\phi\rho\bar{\mu})}, \qquad Y \;=\; \frac{\bar{\gamma}}{\rho}. \tag{IA.275}$$

Then, the dealer sets ρ equal to the solution of (IA.274), and sets λ as follows:

$$\lambda \;=\; \frac{\bar{\mu} + (\bar{\gamma} - G)}{Y + \bar{\gamma}\bar{\mu} - \bar{\gamma}G - \phi Z}. \tag{IA.276}$$

Proof. See Subsection 5.2. ■

Equilibrium conditions

I solve for the equilibrium of the inventory management model with one IFT, N_F FTs, and N_L STs. Define:

$$n_F \;=\; \frac{N_F}{N_F+1}, \qquad n_L \;=\; \frac{N_L}{N_L+1}. \tag{IA.277}$$

I now collect all the partial equilibrium conditions obtained thus far, and generate the full equilibrium conditions. Theorem IA.3 generalizes Theorem 3 in the paper, and provides necessary and sufficient conditions for an equilibrium of the model.

Theorem IA.3. *Suppose there is an equilibrium in which the speculators's strategies are: $\mathrm{d}x_t = -\Theta x_{t-1} + G\mathrm{d}w_t$ (the IFT), $\mathrm{d}x_t^F = \gamma\mathrm{d}w_t$ (the FTs), $\mathrm{d}x_t^S = \mu\widetilde{\mathrm{d}w}_{t-1}$ (the*

STs); and the dealer's pricing rules are: $\mathrm{d}p_t = \lambda \mathrm{d}y_t$*, and* $\widetilde{\mathrm{d}w}_t = \mathrm{d}w_t - \rho \mathrm{d}y_t$*. Denote the coefficients* R*,* a^-*, and* b *as follows:*

$$R \;=\; \frac{\lambda}{\rho}, \qquad \gamma^- \;=\; N_F \gamma, \qquad \bar{\mu} \;=\; N_L \mu, \qquad a^- \;=\; \rho \gamma^-, \qquad b \;=\; \rho \bar{\mu}. \tag{IA.278}$$

Suppose $b > \frac{\sqrt{17}-1}{8}$*. Then, the equilibrium coefficients satisfy the following equations:*[22]

$$\begin{aligned}
\frac{2b(1+b)(2B+1)}{n_L} \;&=\; \frac{Q}{B^2(a^-+b)} \;+\; \frac{3bB + 2b^2B - 1 - b}{b}(1-a^-) \;-\; 2, \\
B \;&=\; \frac{1}{\sqrt{1-b}}, \qquad q \;=\; (B+1)\Big(2(B^2-1) - n_F(3B^2-2)\Big), \\
a^- \;&=\; \frac{-q \pm \sqrt{q^2 + n_F B^5\big((4-n_F)B + 2(2-n_F)\big)}}{B^2\big((4-n_F)B + 2(2-n_F)\big)}, \\
Q \;&=\; B^3(a^-)^2 + 2(3B^3 + 3B^2 - 2B - 1)a^- + (B^3 + 2B^2 - 2), \\
R \;&=\; \frac{4(B+1)B^2\big(a^-+b\big)}{Q}, \qquad a \;=\; \frac{(2B+1)a^- + 1}{2(B+1)} \\
\rho^2 \;&=\; \left((a-b^2) + \frac{2bB-1}{2B+1}\,(1-a)\right)\,(1-a)\,\frac{\sigma_w^2}{\sigma_u^2}, \qquad \lambda \;=\; R\,\rho \\
\Theta \;&=\; 2 - \frac{\sqrt{1-b}}{b}, \qquad G \;=\; \frac{1-a}{\rho(2B+1)}, \qquad \gamma \;=\; \frac{a^-}{\rho N_F}, \qquad \mu \;=\; \frac{b}{\rho N_L}.
\end{aligned} \tag{IA.279}$$

Conversely, suppose the equations (IA.279) have a real solution such that $\frac{\sqrt{17}-1}{8} < b < 1$*,* $a < 1$*,* $\lambda > 0$*. Then, the speculators' strategies and the dealer's pricing rules with these coefficients provide an equilibrium of the model.*

In equilibrium, the expected profits of the IFT, FTs, and STs are respectively,

$$\pi \;=\; \frac{Rb}{\rho}\,\frac{(1-a^-)^2}{(1+\sqrt{1-b})^2}\,\sigma_w^2, \qquad \pi^F \;=\; \lambda\gamma^2\sigma_w^2, \qquad \pi^S \;=\; \lambda\mu^2(1-a)\sigma_w^2. \tag{IA.280}$$

The results in Theorem IA.3 suggest a procedure to search numerically for an equilibrium, once the parameters N_F, N_L, σ_w, and σ_u are given. Indeed, if one substitutes the formulas for a^-, q, and B in the first equation of (IA.279), this becomes a non-linear equation in one variable, b. This equation can be solved numerically very efficiently. Then, one needs to verify that the conditions $\frac{\sqrt{17}-1}{8} < b < 1$, $a < 1$, $\lambda > 0$ are satisfied. Then, the equations in (IA.279) provide formulas for all the equilibrium coefficients of

[22] To be rigorous, I include the case when a^- is negative. Numerically, however, this case never occurs in equilibrium, because it leads to $\lambda < 0$, which contradicts the FT's second order condition (IA.315) in Section 5 in this Internet Appendix.

the model.

5.2 Proofs

Proof of Proposition IA.7. I follow the proof of Theorem 2 in the paper, with a few modifications.

When $\Theta = 0$, the trading strategy of the IFT is $\mathrm{d}x_t = G\,\mathrm{d}w_t$. As in the proof of Theorem 2, the normalized expected utility of the IFT is $\tilde{U}_{\Theta=0} = G(1-\lambda\bar{\gamma}) - \frac{C_I}{2}G^2$. Since $\bar{\gamma} = \gamma^- + G$, one has $\tilde{U}_{\Theta=0} = G(1-\lambda\gamma^-) - \lambda G^2 - \frac{C_I}{2}G^2$. Since $\lambda\gamma^- = \frac{\lambda}{\rho}\rho\gamma^- = Ra^-$, one obtains:

$$\tilde{U}_{\Theta=0} \;=\; G\big(1-Ra^-\big) - G^2\left(\lambda + \frac{C_I}{2}\right). \tag{IA.281}$$

The function $\tilde{U}_{\Theta=0}$ attains its maximum at:

$$G \;=\; \frac{1-Ra^-}{2\lambda + C_I}, \tag{IA.282}$$

as stated in Proposition IA.7. The maximum value is:

$$\tilde{U}^{\max}_{\Theta=0} \;=\; \frac{\big(1-Ra^-\big)^2}{2(2\lambda + C_I)}. \tag{IA.283}$$

When $\Theta > 0$, the trading strategy of the IFT is $\mathrm{d}x_t = -\Theta\, x_{t-1} + G\,\mathrm{d}w_t$. As in the proof of Theorem 2, the IFT's inventory costs are zero, hence the IFT's expected utility is the same as his expected profit. Then, equation (74) shows that the IFT's normalized expected utility/profit is:

$$\tilde{U}_{\Theta>0} \;=\; G\,\frac{Rb(1-a^-)}{1+\phi b} - G^2\,\frac{\lambda\big(b+\frac{1}{1+\phi}\big)}{1+\phi b}. \tag{IA.284}$$

Fix ϕ. Then, the first order condition with respect to G implies that the optimum G satisfies:

$$G \;=\; \frac{Rb(1-a^-)}{2\lambda\big(b+\frac{1}{1+\phi}\big)} \;=\; \frac{b(1-a^-)}{2\rho\big(b+\frac{1}{1+\phi}\big)}, \tag{IA.285}$$

as stated in Proposition IA.7. For this G, the normalized expected utility (profit) of the IFT is:

$$\tilde{U}_{\Theta>0} \;=\; \frac{\big(Rb(1-a^-)\big)^2}{4\lambda(1+\phi b)\big(b+\frac{1}{1+\phi}\big)}. \tag{IA.286}$$

Consider the function:

$$f(\phi) \;=\; (1+\phi b)\left(b+\frac{1}{1+\phi}\right) \qquad \Longrightarrow \qquad f'(\phi) \;=\; \frac{b^2(1+\phi)^2+b-1}{(1+\phi)^2}. \tag{IA.287}$$

The polynomial in the numerator of $f'(\phi)$ has two roots:

$$\phi_1 \;=\; -1+\frac{\sqrt{1-b}}{b} \qquad \phi_2 \;=\; -1-\frac{\sqrt{1-b}}{b}. \tag{IA.288}$$

Note that $b = \rho\bar{\mu} > 0$, and since one has assumed that $b < 1$, the two roots are real and distinct. Clearly, $\phi_2 < -1$ and $\phi_1 > -1$. Since the numerator of $f'(\phi)$ is a quadratic function of ϕ, it follows that $f'(\phi) < 0$ for $\phi \in (\phi_2, \phi_1)$, and positive everywhere else. As the ϕ must belong to the interval $(-1, 1]$, there are two cases:

(i) If $\phi_1 \geq 1$, f is strictly decreasing on $(-1, 1]$, hence it attains its minimum at $\phi = 1$. Thus, the maximum normalized expected utility $\tilde{U}_{\Theta>0}$ from (IA.286) attains its maximum at $\phi = 1$, or equivalently at $\Theta = 0_+$ (recall that there is a discontinuity at $\Theta = 0$). This maximum value is:

$$\tilde{U}^{\max}_{\Theta=0_+} \;=\; \frac{\big(Rb(1-a^-)\big)^2}{4\lambda(1+b)\big(b+\frac{1}{2}\big)}. \tag{IA.289}$$

To determine the cutoff value for C_I, set $\tilde{U}^{\max}_{\Theta=0_+} = \tilde{U}^{\max}_{\Theta=0}$. One obtains:

$$\bar{C}^0_I \;=\; 2\lambda\left(\frac{(1-Ra^-)^2(1+b)(b+\frac{1}{2})}{R^2b^2(1-a^-)^2}-1\right). \tag{IA.290}$$

(ii) If $\phi_1 \in (-1, 1)$, f is strictly decreasing on $(-1, \phi_1)$ and strictly increasing on $(\phi_1, 1)$, hence it attains its minimum at $\phi = \phi_1$. Thus, the maximum normalized expected utility $\tilde{U}_{\Theta>0}$ from (IA.286) attains its maximum at $\phi = \phi_1$, or equivalently at $\Theta = 2 - \frac{\sqrt{1-b}}{b} \in (0, 2)$. This maximum value is:

$$\tilde{U}^{\max}_{\Theta>0} \;=\; \frac{\big(Rb(1-a^-)\big)^2}{4\lambda b(1+\sqrt{1-b})^2}. \tag{IA.291}$$

To determine the cutoff value for C_I, I set $\tilde{U}^{\max}_{\Theta>0} = \tilde{U}^{\max}_{\Theta=0}$. One obtains:

$$\bar{C}_I \;=\; 2\lambda\left(\frac{(1-Ra^-)^2(1+\sqrt{1-b})^2}{R^2b(1-a^-)^2}-1\right). \tag{IA.292}$$

But $\phi_1 \geq 1$ is equivalent to $b \leq \frac{\sqrt{17}-1}{8}$, hence the two cases (i) and (ii) described here are the same as the cases described in the statement of Proposition IA.7.

Finally, I collect the equations for $\bar{C}_I$ and $\bar{C}_I^0$:

$$\begin{aligned} \bar{C}_I &= 2\lambda\left(\frac{(1-Ra^-)^2(1+\sqrt{1-b})^2}{R^2 b(1-a^-)^2} - 1\right), \\ \bar{C}_I^0 &= 2\lambda\left(\frac{(1-Ra^-)^2(1+b)(b+\frac{1}{2})}{R^2 b^2(1-a^-)^2} - 1\right), \end{aligned} \tag{IA.293}$$

For future reference, I also compute the normalized expected utility at $\Theta = 0_+$. By using $\bar{\gamma} = \gamma^- + G$, some algebraic manipulation of (IA.284) shows that:

$$\tilde{U}_{\Theta>0} = \frac{\lambda\bar{\mu}G(1-\rho\bar{\gamma})}{1+\phi\rho\bar{\mu}} - \frac{\lambda G^2}{(1+\phi)(1+\phi\rho\bar{\mu})}. \tag{IA.294}$$

Taking the limit when $\Theta \to 0$ (or equivalently when $\phi \to 1$), one obtains:

$$\tilde{U}_{\Theta=0_+} = \frac{\lambda\bar{\mu}G(1-\rho\bar{\gamma})}{1+\rho\bar{\mu}} - \frac{\lambda G^2}{2(1+\rho\bar{\mu})}. \tag{IA.295}$$

■

Proof of Proposition IA.8. As in Theorem 2, I define some normalized covariances that are used throughout this proof. If x_t is the IFT's inventory in the risky asset, denote:

$$\begin{aligned} \Omega_t^{xx} &= \frac{\mathsf{E}\big(x_t^2\big)}{\sigma_w^2 \mathrm{d}t}, \quad \Omega_t^{xw} = \frac{\mathsf{E}\big(x_t w_t\big)}{\sigma_w^2 \mathrm{d}t}, \quad \Omega_t^{xp} = \frac{\mathsf{E}\big(x_t p_t\big)}{\sigma_w^2 \mathrm{d}t}, \quad \Omega_t^{xe} = \frac{\mathsf{E}\big(x_t(w_t-p_t)\big)}{\sigma_w^2 \mathrm{d}t}, \\ E_t &= \frac{\mathsf{E}\big((w_t-p_t)\widetilde{\mathrm{d}w_t}\big)}{\sigma_w^2 \mathrm{d}t}, \quad X_t = \frac{\mathsf{E}\big(x_t\widetilde{\mathrm{d}w_t}\big)}{\sigma_w^2 \mathrm{d}t}, \quad Y_t = \frac{\mathsf{E}\big((\mathrm{d}y_t)^2\big)}{\sigma_w^2 \mathrm{d}t}, \quad Z_t = \frac{\mathsf{E}\big(x_{t-1}\mathrm{d}y_t\big)}{\sigma_w^2 \mathrm{d}t}, \\ W_t &= \frac{\mathsf{E}\big((\widetilde{\mathrm{d}w_t})^2\big)}{\sigma_w^2 \mathrm{d}t}, \quad H_t = \frac{\mathsf{E}\big((w_t-p_t)\mathrm{d}y_t\big)}{\sigma_w^2 \mathrm{d}t}, \quad H_t^w = \frac{\mathsf{E}\big(w_t \mathrm{d}y_t\big)}{\sigma_w^2 \mathrm{d}t}, \quad H_t^p = \frac{\mathsf{E}\big(p_t \mathrm{d}y_t\big)}{\sigma_w^2 \mathrm{d}t}. \end{aligned} \tag{IA.296}$$

Recall that in Theorem 2 the following formulas were proved:

$$\begin{aligned} \Theta\Omega^{xx} &= \frac{G^2}{1+\phi}, \qquad \Theta\Omega^{xw} = G, \qquad \Theta\Omega^{xp} = \lambda G\bar{\gamma} + \lambda\phi Z, \\ \Theta\Omega^{xe} &= G(1-\lambda\bar{\gamma}) - \lambda\phi Z, \qquad X = \frac{\frac{\rho\phi G^2}{1+\phi} + G(1-\rho\bar{\gamma})}{1+\phi\rho\bar{\mu}}, \\ Z &= -\Theta\Omega^{xx} + \bar{\mu}X = \frac{\bar{\mu}G(1-\rho\bar{\gamma})}{1+\phi\rho\bar{\mu}} - \frac{G^2}{(1+\phi)(1+\phi\rho\bar{\mu})}. \end{aligned} \tag{IA.297}$$

The formula for W_t is (recall that $\widetilde{\mathrm{d}w}_t = \mathrm{d}w_t - \rho \mathrm{d}y_t$):

$$W_t \;=\; \frac{\mathsf{E}\big((\widetilde{\mathrm{d}w}_t)^2\big)}{\sigma_w^2 \mathrm{d}t} \;=\; 1 - 2\rho\bar{\gamma} + \rho^2 Y_t. \tag{IA.298}$$

The formula for Y_t is (recall that $\tilde{\sigma}_u = \frac{\sigma_u}{\sigma_w}$):

$$\begin{aligned} Y_t \;&=\; \frac{\mathsf{E}\big((\mathrm{d}y_t)^2\big)}{\sigma_w^2 \mathrm{d}t} \;=\; \Theta^2 \Omega^{xx}_{t-1} + \bar{\gamma}^2 + \bar{\mu}^2 W_{t-1} - 2\Theta\bar{\mu} X_{t-1} + \tilde{\sigma}_u^2 \\ &=\; \Theta^2 \Omega^{xx}_{t-1} + \bar{\gamma}^2 + \bar{\mu}^2 \big(1 - 2\rho\bar{\gamma} + \rho^2 Y_{t-1}\big) - 2\Theta\bar{\mu} X_{t-1} + \tilde{\sigma}_u^2. \end{aligned} \tag{IA.299}$$

Because $\rho\bar{\mu} \in [0,1)$, I apply Lemma A.1 in the Appendix in the paper to deduce that Y_t is constant, and equal to:

$$Y \;=\; \frac{\Theta^2 \Omega^{xx} + \bar{\gamma}^2 + \bar{\mu}^2 (1 - 2\rho\bar{\gamma}) - 2\Theta\bar{\mu} X + \tilde{\sigma}_u^2}{1 - \rho^2 \bar{\mu}^2}. \tag{IA.300}$$

From (IA.297), one computes $\Theta^2 \Omega^{xx} - 2\Theta\bar{\mu}X = -\Theta^2\Omega^{xx} - 2\Theta(-\Theta\Omega^{xx} + \bar{\mu}X) = -\frac{\Theta G^2}{1+\phi} - 2\Theta Z$, hence:

$$Y \;=\; \frac{-\frac{\Theta G^2}{1+\phi} - 2\Theta Z + \bar{\gamma}^2 + \bar{\mu}^2(1 - 2\rho\bar{\gamma}) + \tilde{\sigma}_u^2}{1 - \rho^2 \bar{\mu}^2}, \tag{IA.301}$$

as desired. Therefore, W_t is constant and equal to:

$$W \;=\; 1 - 2\rho\bar{\gamma} + \rho^2 Y \;=\; \frac{-\frac{\Theta\rho^2 G^2}{1+\phi} - 2\Theta\rho^2 Z + (1 - \rho\bar{\gamma})^2 + \rho^2 \tilde{\sigma}_u^2}{1 - \rho^2\bar{\mu}^2}, \tag{IA.302}$$

as desired. Since $H^w_t = \frac{\mathsf{E}(w_t \mathrm{d}y_t)}{\sigma_w^2 \mathrm{d}t}$ and $\Theta\Omega^{xw} = G$, one computes:

$$\begin{aligned} H^w_t \;&=\; \frac{\mathsf{E}\big((w_{t-1} + \mathrm{d}w_t)(-\Theta x_{t-1} + \bar{\gamma}\mathrm{d}w_t + \bar{\mu}\widetilde{\mathrm{d}w}_{t-1})\big)}{\sigma_w^2 \mathrm{d}t} \\ &=\; \bar{\gamma} + \frac{\mathsf{E}\big(w_{t-1}(-\Theta x_{t-1} + \bar{\gamma}\mathrm{d}w_t + \bar{\mu}\widetilde{\mathrm{d}w}_{t-1})\big)}{\sigma_w^2 \mathrm{d}t} \\ &=\; \bar{\gamma} - \Theta\Omega^{xw} + \bar{\mu} E^w_{t-1} \\ &=\; \bar{\gamma} - G + \bar{\mu} E^w_{t-1}. \end{aligned} \tag{IA.303}$$

One obtains the following recursive equation for E^w:

$$\begin{aligned} E^w_t \;&=\; \frac{\mathsf{E}\big(w_t \widetilde{\mathrm{d}w}_t\big)}{\sigma_w^2 \mathrm{d}t} \;=\; \frac{\mathsf{E}\big(w_t(\mathrm{d}w_t - \rho\mathrm{d}y_t)\big)}{\sigma_w^2 \mathrm{d}t} \;=\; 1 - \rho H^w_t \\ &=\; 1 - \rho(\bar{\gamma} - G) - \rho\bar{\mu} E^w_{t-1}. \end{aligned} \tag{IA.304}$$

As long as $b = \rho\bar{\mu} < 1$ (and $b \geq 0$), I apply Lemma A.1 in the Appendix in the paper to deduce that E_t^w is constant, and equal to:

$$E^w = \frac{1 - \rho(\bar{\gamma} - G)}{1 + \rho\bar{\mu}}. \tag{IA.305}$$

Equation (IA.304) implies $H^w = \frac{1-E^w}{\rho}$, from which one computes:

$$H^w = \frac{\bar{\mu} + (\bar{\gamma} - G)}{1 + \rho\bar{\mu}}. \tag{IA.306}$$

Since $\Theta\Omega^{xp} = \lambda G\bar{\gamma} + \lambda\phi Z$ and $H_t^p = \frac{\mathsf{E}(p_t \mathrm{d}y_t)}{\sigma_w^2 \mathrm{d}t} = \frac{\mathsf{E}((p_{t-1} + \lambda \mathrm{d}y_t)\mathrm{d}y_t)}{\sigma_w^2 \mathrm{d}t} = \frac{\mathsf{E}(p_{t-1} \mathrm{d}y_t)}{\sigma_w^2 \mathrm{d}t} + \lambda Y$, one computes:

$$\begin{aligned} H_t^p &= \lambda Y + \frac{\mathsf{E}\big(p_{t-1}(-\Theta x_{t-1} + \bar{\gamma}\mathrm{d}w_t + \bar{\mu}\widetilde{\mathrm{d}w}_{t-1})\big)}{\sigma_w^2 \mathrm{d}t} \\ &= \lambda Y - \Theta\Omega^{xp} + \bar{\mu}E_{t-1}^p \\ &= \lambda\big(Y - G\bar{\gamma} - \phi Z\big) + \bar{\mu}E_{t-1}^p. \end{aligned} \tag{IA.307}$$

One obtains the following recursive equation for E^p:

$$\begin{aligned} E_t^p &= \frac{\mathsf{E}\big(p_t\widetilde{\mathrm{d}w}_t\big)}{\sigma_w^2 \mathrm{d}t} = \frac{\mathsf{E}\big(p_t(\mathrm{d}w_t - \rho \mathrm{d}y_t)\big)}{\sigma_w^2 \mathrm{d}t} = \lambda\bar{\gamma} - \rho H_t^p \\ &= \lambda\big(\bar{\gamma} - \rho Y + \rho G\bar{\gamma} + \rho\phi Z\big) - \rho\bar{\mu}E_{t-1}^p. \end{aligned} \tag{IA.308}$$

As long as $b = \rho\bar{\mu} < 1$ (and $b \geq 0$), I apply Lemma A.1 in the Appendix in the paper to deduce that E_t^w is constant, and equal to:

$$E^p = \lambda\,\frac{\bar{\gamma} - \rho Y + \rho G\bar{\gamma} + \rho\phi Z}{1 + \rho\bar{\mu}}. \tag{IA.309}$$

Equation (IA.308) implies $H^p = \frac{\lambda\bar{\gamma} - E^p}{\rho}$, from which one computes:

$$H^p = \lambda\,\frac{\bar{\gamma}\bar{\mu} + Y - \bar{\gamma}G - \phi Z}{1 + \rho\bar{\mu}}. \tag{IA.310}$$

Putting together (IA.305) and (IA.309), one obtains:

$$E = E^w - E^p = \frac{1 - \rho(\bar{\gamma} - G) - \lambda(\bar{\gamma} - \rho Y + \rho G\bar{\gamma} + \rho\phi Z)}{1 + \rho\bar{\mu}}, \tag{IA.311}$$

which is equivalent to the desired equation for E.

To simplify presentation, I combine the FTs and STs by considering a speculator with trading strategy of the form $\mathrm{d}x_{i,t} = \gamma_i \mathrm{d}w_t + \mu_i \widetilde{\mathrm{d}w}_{t-1}$. If the speculator is a FT, I set $\mu_i = 0$; and if the speculator is a ST, I set$\gamma_i = 0$. The normalized expected profit of this speculator is:

$$\begin{aligned}
\tilde{\pi}_i &= \frac{1}{\sigma_w^2} \mathsf{E} \int_0^T (w_t - p_t)\big(\gamma_i \mathrm{d}w_t + \mu_i \widetilde{\mathrm{d}w}_{t-1}\big) \\
&= \frac{1}{\sigma_w^2} \mathsf{E} \int_0^T \Big(w_{t-1} - p_{t-1} + \mathrm{d}w_t - \lambda\big(-\Theta x_{t-1} + \bar{\gamma}\mathrm{d}w_t + \bar{\mu}\widetilde{\mathrm{d}w}_{t-1}\big)\Big)\big(\gamma_i \mathrm{d}w_t + \mu_i \widetilde{\mathrm{d}w}_{t-1}\big) \\
&= \frac{1}{\sigma_w^2} \mathsf{E} \int_0^T \Big(w_{t-1} - p_{t-1} + \mathrm{d}w_t(1 - \lambda\bar{\gamma}) + \lambda\Theta x_{t-1} - \lambda\bar{\mu}\widetilde{\mathrm{d}w}_{t-1}\big)\Big)\big(\gamma_i \mathrm{d}w_t + \mu_i \widetilde{\mathrm{d}w}_{t-1}\big) \\
&= \gamma_i(1 - \lambda\bar{\gamma}) + \mu_i\big(E + \lambda\Theta X - \lambda\bar{\mu}W\big).
\end{aligned} \tag{IA.312}$$

Recall that by assumption (see equation (13)) the covariances E, X, and W do not depend on speculators' strategies. That is, the speculator regards them as constant and not as functions of γ_i, μ_i.

I compute the optimal weight of a FT indexed by $i = 1, \ldots, N_F$. From (IA.312) with $\mu_i = 0$, his normalized expected profit is:

$$\tilde{\pi}_i^F = \gamma_i(1 - \lambda G) - \gamma_i(\gamma_i + \gamma_{-i})\lambda, \tag{IA.313}$$

where γ_{-i} is the aggregate weight on $\mathrm{d}w_t$ of the other FTs. The first order condition with respect to γ_i implies:

$$1 - \lambda G = \lambda(2\gamma_i + \gamma_{-i}), \tag{IA.314}$$

and the second order condition for a maximum is:

$$\lambda > 0. \tag{IA.315}$$

Note that this second order condition is satisfied by assumption. The first order condition is true for all FTs, hence all γ_i are equal to γ, where:

$$\gamma = \frac{1 - \lambda G}{\lambda(N_F + 1)}. \tag{IA.316}$$

From this, one computes $\gamma^- = N_F \gamma = \left(\frac{1}{\lambda} - G\right) \frac{N_F}{N_F+1}$, which implies $\bar{\gamma} = \gamma^- + G = \frac{N_F}{\lambda(N_F+1)} + \frac{G}{N_F+1}$. This proves the desired formula for $\bar{\gamma}$.

I compute the optimal weight of a ST indexed by $i = 1, \ldots, N_L$. From (IA.312) with

$\gamma_i = 0$, his normalized expected profit is:

$$\tilde{\pi}_i^S \;=\; \mu_i(E + \lambda\Theta X) - \mu_i(\mu_i + \mu_{-i})\lambda W, \tag{IA.317}$$

where μ_{-i} is the aggregate weight on $\widetilde{\mathrm{d}w}_{t-1}$ of the other STs. The first order condition with respect to μ_i implies:

$$E + \lambda\Theta X - (2\mu_i + \mu_{-i})\lambda W \;=\; 0, \tag{IA.318}$$

and the second order condition for a maximum is:

$$\lambda W \;>\; 0. \tag{IA.319}$$

From (IA.315), this condition is equivalent to $W > 0$, which is assumed true. The first order condition is true for all STs, hence all μ_i are equal to μ, where:

$$\mu \;=\; \frac{E + \lambda\Theta X}{\lambda W(N_L + 1)}. \tag{IA.320}$$

From this, $\bar{\mu} = N_L\mu = \frac{E+\lambda\Theta X}{\lambda W}\frac{N_L}{N_L+1}$. This proves the desired formula for $\bar{\mu}$. ■

Proof of Proposition IA.9. I compute the pricing functions set by the dealer. As in the proof of Theorem 1, the definition $\widetilde{\mathrm{d}w}_t = \mathrm{d}w_t - \mathsf{E}_{t+1}(\mathrm{d}w_t)$ implies $\rho = \frac{\widetilde{\mathsf{Cov}}(\mathrm{d}w_t, \mathrm{d}y_t)}{\widetilde{\mathsf{Var}}(\mathrm{d}y_t)} = \frac{\bar{\gamma}}{Y}$. Hence, one obtains:

$$\rho \;=\; \frac{\bar{\gamma}}{Y} \qquad \Longrightarrow \qquad \rho Y \;=\; \bar{\gamma}. \tag{IA.321}$$

Note that this is equivalent to the second equation in (IA.275).

To compute λ, I impose the zero expected profit condition for the dealer. Recall the notations from equation (IA.296):

$$\begin{aligned} \Omega_t^{xe} &= \frac{\mathsf{E}\big(x_t(w_t - p_t)\big)}{\sigma_w^2 \mathrm{d}t}, \qquad Y_t = \frac{\mathsf{E}\big((\mathrm{d}y_t)^2\big)}{\sigma_w^2 \mathrm{d}t}, \qquad Z_t = \frac{\mathsf{E}\big(x_{t-1}\mathrm{d}y_t\big)}{\sigma_w^2 \mathrm{d}t} \\ H_t^w &= \frac{\mathsf{E}\big(w_t \mathrm{d}y_t\big)}{\sigma_w^2 \mathrm{d}t}, \qquad H_t^p = \frac{\mathsf{E}\big(p_t \mathrm{d}y_t\big)}{\sigma_w^2 \mathrm{d}t}. \end{aligned} \tag{IA.322}$$

In the proof of Proposition IA.8, all these numbers are constant. Equations (IA.306) and (IA.310) imply that:

$$H^w \;=\; \frac{\bar{\mu} + (\bar{\gamma} - G)}{1 + \rho\bar{\mu}}, \qquad H^p \;=\; \lambda\,\frac{Y + \bar{\gamma}\bar{\mu} - \bar{\gamma}G - \phi Z}{1 + \rho\bar{\mu}}. \tag{IA.323}$$

The dealer's normalized expected profit at $t = 0$ is:

$$\begin{aligned}\tilde{\pi}_d &= \frac{1}{\sigma_w^2}\,\mathsf{E}\int_0^T (p_t - w_t)\mathrm{d}y_t \;=\; H^p - H^w \\ &= \lambda\,\frac{Y + \bar{\gamma}\bar{\mu} - \bar{\gamma}G - \phi Z}{1+\rho\bar{\mu}} \;-\; \frac{\bar{\mu} + (\bar{\gamma} - G)}{1+\rho\bar{\mu}}.\end{aligned} \tag{IA.324}$$

Setting the dealer's expected profit to zero is then equivalent to:

$$\lambda \;=\; \frac{\bar{\mu} + (\bar{\gamma} - G)}{Y + \bar{\gamma}\bar{\mu} - \bar{\gamma}G - \phi Z}, \tag{IA.325}$$

which proves equation (IA.276).

I now compute Y. From Proposition IA.8, one obtains:

$$Z \;=\; \frac{\bar{\mu}G(1-\rho\bar{\gamma})}{1+\phi\rho\bar{\mu}} - \frac{G^2}{(1+\phi)(1+\phi\rho\bar{\mu})}, \qquad Y \;=\; \frac{-\frac{\Theta G^2}{1+\phi} - 2\Theta Z + \bar{\gamma}^2 + \bar{\mu}^2(1-2\rho\bar{\gamma}) + \tilde{\sigma}_u^2}{1-\rho^2\bar{\mu}^2}. \tag{IA.326}$$

Note that the equation for Z is identical to the first equation in (IA.275). By substituting Z in the equation for Y, one obtains:

$$Y \;=\; \frac{\Theta\,\frac{G^2(1-\phi\rho\bar{\mu}) - 2\bar{\mu}G(1+\phi)(1-\rho\bar{\gamma})}{(1+\phi)(1+\phi\rho\bar{\mu})} + \bar{\gamma}^2 + \bar{\mu}^2(1-2\rho\bar{\gamma}) + \tilde{\sigma}_u^2}{1-\rho^2\bar{\mu}^2}. \tag{IA.327}$$

Multiply this equation by $\rho(1-\rho^2\bar{\mu}^2)(1+\phi\rho\bar{\mu})$. Because (IA.321) implies $\rho Y = \bar{\gamma}$, one obtains:

$$\begin{aligned}\bar{\gamma}(1-\rho^2\bar{\mu}^2)(1+\phi\rho\bar{\mu}) \;=\; &\Theta\,\frac{G^2(1-\phi\rho\bar{\mu}) - 2\bar{\mu}G(1+\phi)(1-\rho\bar{\gamma})}{(1+\phi)} \\ &+ \big(\bar{\gamma}^2 + \bar{\mu}^2(1-2\rho\bar{\gamma}) + \tilde{\sigma}_u^2\big)(1+\phi\rho\bar{\mu}).\end{aligned} \tag{IA.328}$$

This is the third degree equation in ρ stated in (IA.274). ■

Proof of Theorem IA.3 (Theorem 3). To find necessary conditions, suppose there is an equilibrium. Since the IFT is sufficiently averse and $b > \frac{\sqrt{17}-1}{8}$, according to Proposition IA.7, he chooses optimally $\Theta > 0$. Then, one can put together all the equations from Propositions IA.7–IA.9. To simplify the equations, note that (see equation (IA.321)):

$$\rho Y \;=\; \bar{\gamma}. \tag{IA.329}$$

Then, equation (IA.298) becomes:

$$W \;=\; 1 - \rho\bar{\gamma} \;=\; 1 - a. \tag{IA.330}$$

I now assume that the IFT is sufficiently inventory averse, so that his inventory mean reversion is strictly positive ($\Theta > 0$). The relevant equations from Propositions IA.7–IA.9 are:

$$\begin{aligned}
\gamma^{-} &= N_F\gamma \;=\; \frac{1-\lambda G}{\lambda}\,\frac{N_F}{N_F+1}, \qquad \bar{\mu} \;=\; N_L\mu \;=\; \frac{E+\lambda\Theta X}{\lambda W}\,\frac{N_L}{N_L+1},\\
G &= \frac{1-a^{-}}{2\rho\left(1+\frac{1}{\sqrt{1-b}}\right)}, \qquad \Theta \;=\; 2-\frac{\sqrt{1-b}}{b} \;\in\; (0,1), \qquad \phi \;=\; 1-\Theta,\\
W &= 1-a, \qquad \rho Y \;=\; \bar{\gamma}, \qquad \lambda \;=\; \frac{\bar{\mu}+\bar{\gamma}-G}{Y+\bar{\gamma}\bar{\mu}-\bar{\gamma}G-\phi Z},\\
X &= \frac{\frac{\rho\phi G^2}{1+\phi}+G(1-\rho\bar{\gamma})}{1+\phi\rho\bar{\mu}}, \qquad Y \;=\; \frac{-\frac{\Theta G^2}{1+\phi}-2\Theta Z+\bar{\gamma}^2+\bar{\mu}^2(1-2\rho\bar{\gamma})+\tilde{\sigma}_u^2}{1-\rho^2\bar{\mu}^2},\\
Z &= \frac{\bar{\mu}G(1-\rho\bar{\gamma})}{1+\phi\rho\bar{\mu}}-\frac{G^2}{(1+\phi)(1+\phi\rho\bar{\mu})},\\
E &= \frac{1-\rho\bar{\gamma}+\rho G}{1+\rho\bar{\mu}}-\lambda\,\frac{\bar{\gamma}-\rho Y+\rho G\bar{\gamma}+\rho\phi Z}{1+\rho\bar{\mu}} \;=\; \frac{1-\rho\gamma^{-}}{1+\rho\bar{\mu}}-\lambda\,\frac{\rho G\bar{\gamma}+\rho\phi Z}{1+\rho\bar{\mu}}.
\end{aligned} \tag{IA.331}$$

The goal is to express all the equations in (IA.331) as functions of a^{-} and b, and then use the first two equations to solve numerically for a^{-} and b. Denote:

$$R \;=\; \frac{\lambda}{\rho}, \qquad B \;=\; \frac{1}{\sqrt{1-b}}, \qquad a^{-} \;=\; \rho\gamma^{-}, \qquad a \;=\; \rho\bar{\gamma}, \qquad b \;=\; \rho\bar{\mu}. \tag{IA.332}$$

Rewrite the equations in (IA.331), except for the first two, as functions of a^{-}, b (or B), and R:

$$\begin{aligned}
\rho G &= \frac{1-a^-}{2(B+1)}, \qquad \Theta = 2-\frac{1}{bB}, \qquad \phi = \frac{1}{bB}-1,\\
W &= 1-a = (1-a^-)\frac{2B+1}{2(B+1)},\\
\rho^2 Y &= a = a^- + \rho G = \frac{1+a^-(2B+1)}{2(B+1)},\\
R &= \frac{a^-+b}{\rho^2 Y + ab - a\rho G - \phi\rho^2 Z} = \frac{a^-+b}{a+ab-a\rho G-\phi\rho^2 Z},\\
\rho X &= \frac{(1-a^-)^2 B}{4(B+1)}, \qquad \rho^2 Z = \frac{(1-a^-)^2}{4(B+1)^2}\, bB^2,\\
(1-b^2)\rho^2 Y &= (1-b^2)a = -\frac{\Theta\rho^2 G^2}{1+\phi} - 2\Theta\rho^2 Z + a^2 + b^2(1-2a) + \rho^2\tilde{\sigma}_u^2,\\
E &= \frac{1-a^-}{1+b} - R\,\frac{2(1-a^-)-(1-a^-)^2 B}{4(B+1)(1+b)},
\end{aligned} \tag{IA.333}$$

From the corresponding equation for R in (IA.333), one computes (after some algebraic manipulation):

$$R = \frac{4(B+1)B^2\big(a^-+b\big)}{B^3(a^-)^2 + 2(3B^3+3B^2-2B-1)a^- + (B^3+2B^2-2)}. \tag{IA.334}$$

By setting:

$$Q = B^3(a^-)^2 + 2(3B^3+3B^2-2B-1)a^- + (B^3+2B^2-2), \tag{IA.335}$$

I have proved the equation for R in (IA.279).

Now (IA.331) implies $a = \frac{1+a^-(2B+1)}{2(B+1)}$, which proves the corresponding equation in (IA.279). Recall that:

$$n_F = \frac{N_F}{N_F+1}, \qquad n_L = \frac{N_L}{N_L+1}. \tag{IA.336}$$

Equation $\gamma^- = \frac{1-\lambda G}{\lambda}\,\frac{N_F}{N_F+1}$ from (IA.331) can be written as $Ra^- = (1-R\rho G)n_F$, or equivalently $R\big(a^-+\rho G n_F\big) = n_F$. Using the formula for R from (IA.334), (after some algebraic manipulation) one obtains the following second degree equation in a^-:

$$B^2\Big(4(B+1)-n_F(B+2)\Big)(a^-)^2 + 2(B+1)\Big(2(B^2-1)-n_F(3B^2-2)\Big)a^- - B^3 n_F = 0. \tag{IA.337}$$

This second degree polynomial has two real roots:

$$\begin{aligned} a^- &= \frac{-q \pm \sqrt{q^2 + n_F B^5\big((4-n_F)B + 2(2-n_F)\big)}}{B^2\big((4-n_F)B + 2(2-n_F)\big)}, \quad \text{with} \\ q &= (B+1)\Big(2(B^2-1) - n_F(3B^2-2)\Big). \end{aligned} \tag{IA.338}$$

This proves the equation for a^- in (IA.279).

From (IA.333), one computes:

$$\frac{E}{1-a^-} = \frac{1}{1+b} - R\,\frac{2-(1-a^-)B}{4(B+1)(1+b)}, \quad \frac{\rho X}{1-a^-} = \frac{(1-a^-)B}{4(B+1)}, \tag{IA.339}$$

which implies

$$\begin{aligned} \frac{E + R\Theta\rho X}{R(1-a^-)} &= \frac{1}{R(1+b)} + \frac{\big(\Theta(1+b)+1\big)(1-a^-)B - 2}{4(B+1)(1+b)} \\ &= \frac{Q}{4(B+1)(1+b)B^2(a^-+b)} + \frac{\big(\Theta(1+b)+1\big)(1-a^-)B - 2}{4(B+1)(1+b)}, \end{aligned} \tag{IA.340}$$

where the last equation comes from (IA.334).

Now, multiply equation $\bar{\mu} = \frac{E+\lambda\Theta X}{\lambda W}\,\frac{N_L}{N_L+1}$ from (IA.331) by ρ to obtain:

$$b = \frac{E + R\Theta\rho X}{R(1-a^-)}\,\frac{2(B+1)}{2B+1}\,n_L. \tag{IA.341}$$

Multiplying this equation by $\frac{2(1+b)(2B+1)}{n_L}$ and using (IA.340), one obtains:

$$\frac{2b(1+b)(2B+1)}{n_L} = \frac{Q}{B^2(a^-+b)} + \big(\Theta(1+b)+1\big)(1-a^-)B - 2. \tag{IA.342}$$

Since $\Theta(1+b)+1 = \frac{3bB+2b^2B-1-b}{b}$, I have proved the first equation in (IA.279).

It remains just to prove the equation for ρ. The penultimate equation in (IA.333) implies $\rho^2\tilde{\sigma}_u^2 = (1-a)(a-b^2) + \frac{\Theta\rho^2G^2}{1+\phi} + 2\Theta b\rho^2 Z$, hence:

$$\begin{aligned} \rho^2 &= \frac{(1-a)(a-b^2) + \frac{\Theta\rho^2G^2}{1+\phi} + 2\Theta\rho^2 Z}{\tilde{\sigma}_u^2} \\ &= \frac{(1-a)(a-b^2) + \frac{\Theta}{1+\phi}\,\frac{(1-a^-)^2}{4(B+1)^2}\big(1 + 2bB^2(1+\phi)\big)}{\tilde{\sigma}_u^2}. \end{aligned} \tag{IA.343}$$

Using $1 - a = (1 - a^-)\frac{2B+1}{2(B+1)}$, one computes (after some algebraic manipulation):

$$\frac{\Theta}{1+\phi}\,\frac{(1-a^-)^2}{4(B+1)^2}\big(1 + 2bB^2(1+\phi)\big) \;=\; \frac{2bB-1}{2B+1}(1-a)^2, \tag{IA.344}$$

which proves the corresponding formula for ρ in (IA.279).

I have just finished the proof that the equations in (IA.279) are necessary for the existence of an equilibrium. I now show that they are sufficient if I assume that the solution to (IA.279) also satisfies $\frac{\sqrt{17}-1}{8} < b < 1$, $a < 1$, $\lambda > 0$. I now follow the proofs of Propositions IA.7–IA.9 to show that the strategies defined by using these coefficients provide an equilibrium. The condition $b < 1$ is used to perform the computations in Proposition IA.7. The condition $b > \frac{\sqrt{17}-1}{8}$ is used in showing that the IFT chooses $\Theta > 0$. The condition $\lambda > 0$ is used as the second order condition for maximization for all three types of speculators (see in particular the second order condition (IA.315) for the FT). The condition $a < 1$ or equivalently $W = 1 - a > 0$ is used as a second order condition for the ST (see equation (IA.319)).

Finally, I compute the equilibrium expected profits of the IFT, FTs and STs, denoted respectively by π, π^F, π^S. From equation (IA.291), the normalized profit of the IFT is:

$$\tilde{\pi} \;=\; \frac{Rb(1-a^-)^2}{4\rho(1+\sqrt{1-b})^2}, \tag{IA.345}$$

as stated in the Thorem. From (IA.273), the normalized expected profits of the FTs and STs are respectively:

$$\tilde{\pi}^F \;=\; \gamma(1 - \lambda G - \lambda\gamma^-), \qquad \tilde{\pi}^S \;=\; \mu\big(E + \lambda\Theta X - \lambda W\bar{\mu}\big). \tag{IA.346}$$

From (IA.271), one obtains:

$$\gamma \;=\; \frac{1-\lambda G}{\lambda(N_F+1)}, \qquad \mu \;=\; \frac{E+\lambda\Theta X}{\lambda W(N_L+1)}. \tag{IA.347}$$

One computes $\lambda\gamma^- = N_F\lambda\gamma = \frac{N_F}{N_F+1}(1-\lambda G)$. Therefore, $\tilde{\pi}^F = \gamma\,\frac{1-\lambda G}{N_F+1} = \lambda\gamma^2$. Similarly, $\tilde{\pi}^S = \lambda W\mu^2$. But in equilibrium $W = 1 - a$, hence $\tilde{\pi}^S = \lambda\mu^2(1-a)$. One obtains:

$$\tilde{\pi}^F \;=\; \lambda\gamma^2, \qquad \tilde{\pi}^S \;=\; \lambda\mu^2(1-a), \tag{IA.348}$$

as stated in Theorem IA.3. The proof is now complete. ■

Proof of Proposition 7. The asymptotic notation in this proof is:

$$X \approx X_\infty \quad \Longleftrightarrow \quad \lim_{N_F, N_L \to \infty} \frac{X}{X_\infty} = 1. \tag{IA.349}$$

(Note that $N_L \to \infty$ is also included as part of the definition.) Denote:

$$b_\infty = \frac{\sqrt{5}-1}{2}, \qquad B_\infty = 1 + b_\infty = \frac{\sqrt{5}+1}{2}, \qquad a_\infty = 1. \tag{IA.350}$$

First, I prove that:

$$b \approx b_\infty, \qquad a \approx a_\infty, \qquad 1 - a \approx \frac{1+b_\infty}{N_F+1}, \qquad 1 - a^- \approx \frac{2}{N_F+1}. \tag{IA.351}$$

Define the function of two variables f:

$$f(B,\varepsilon) = \frac{-q + \sqrt{q^2 + (1-\varepsilon)B^5\big((3+\varepsilon)B + 2(1+\varepsilon)\big)}}{B^2\big((3+\varepsilon)B + 2(1+\varepsilon)\big)}, \quad \text{with} \quad q = (B+1)\big(-B^2 + \varepsilon(3B^2-2)\big). \tag{IA.352}$$

Also, define the function of two variables g by:

$$g(B,\varepsilon) = -\frac{2b(1+b)(2B+1)}{n_L} + \frac{Q}{B^2(a^- + b)} + \frac{3bB + 2b^2B - 1 - b}{b}(1-a^-) - 2,$$
$$\text{with} \quad n_L = 1, \qquad b = 1 - \frac{1}{B^2}, \qquad a^- = f(B,\varepsilon),$$
$$\text{and} \quad Q = B^3(a^-)^2 + 2(3B^3 + 3B^2 - 2B - 1)a^- + (B^3 + 2B^2 - 2). \tag{IA.353}$$

I now use the formulas $B_\infty = 1 + b_\infty$ and $b_\infty(1+b_\infty) = 1$ to compute the values of f and g and of their partial derivatives at $B = B_\infty$ and $\varepsilon = 0$. After some algebraic manipulation, one computes:

$$g(B_\infty, 0) = 0, \quad f(B_\infty, 0) = 1, \quad \frac{\partial f}{\partial B}(B_\infty, 0) = 0, \quad \frac{\partial f}{\partial \varepsilon}(B_\infty, 0) = -2. \tag{IA.354}$$

Denote by $B(\varepsilon)$ the solution of $g(B,\varepsilon) = 0$:

$$B(\varepsilon) \quad \Longleftrightarrow \quad g(B,\varepsilon) = 0. \tag{IA.355}$$

From (IA.354), $g(B_\infty, 0) = 0$, therefore:

$$B(0) \;=\; B_\infty. \tag{IA.356}$$

Denote by $a^-(\varepsilon)$ the function:

$$a^-(\varepsilon) \;=\; f(B(\varepsilon), \varepsilon). \tag{IA.357}$$

Using (IA.354), one computes the derivative of a^- at $\varepsilon = 0$:

$$\begin{aligned} \frac{\mathrm{d}a^-}{\mathrm{d}\varepsilon}(0) \;&=\; \frac{\partial f}{\partial B}(B_\infty, 0) B'(\varepsilon) \;+\; \frac{\partial f}{\partial \varepsilon}(B_\infty, 0) \\ &=\; 0 \times B'(\varepsilon) \;+\; (-2) \\ &=\; -2. \end{aligned} \tag{IA.358}$$

Fix $N_F \geq 0$. Let a^-_* and B_* be, respectively, the equilibrium values of a^- and $B = \frac{1}{\sqrt{1-b}}$ when N_L approaches infinity:

$$a^-_* \;=\; \lim_{N_L \to \infty} a^-, \qquad B_* \;=\; \lim_{N_L \to \infty} B. \tag{IA.359}$$

Theorem 3 shows that the equations (IA.279) are necessary conditions for an equilibrium, hence a^- and B satisfy (IA.279). Taking the limit when $N_L \to \infty$ ($n_L \to 1$), it follows that a^-_* and B_* satisfy equations (IA.279) with $n_L = 1$. But, by definition, the numbers $a^-(\varepsilon)$ and $B(\varepsilon)$ satisfy the same equations when ε is:

$$\varepsilon \;=\; \frac{1}{N_F + 1}. \tag{IA.360}$$

Therefore, one obtains:

$$a^-_* \;=\; a^-(\varepsilon), \qquad B_* \;=\; B(\varepsilon). \tag{IA.361}$$

From (IA.354) and (IA.358), one has $B(0) = B_\infty$, $a^-(0) = 1$ and $\frac{\mathrm{d}a^-}{\mathrm{d}\varepsilon}(0) = -2$. Therefore, $B(\varepsilon) \approx B_\infty$, $a^-(\varepsilon) \approx 1$, and $1 - a^-(\varepsilon) \approx 2\varepsilon$. From (IA.361), this translates into $B_* \approx B_\infty$, $a^-_* \approx 1$, and $1 - a^-_* \approx \frac{2}{N_F+1}$. But B_* and a^-_* are limits when $N_L \to \infty$, therefore one obtains the following asymptotic formulas:

$$B \;\approx\; B_\infty, \qquad a^- \;\approx\; 1, \qquad 1 - a^- \;\approx\; 2\varepsilon. \tag{IA.362}$$

From (IA.279), one has $a = a^- + \frac{1-a^-}{2(B+1)}$, which implies $1 - a = (1 - a^-)\frac{2B+1}{2(B+1)}$. But

$\frac{2B_\infty+1}{2(B_\infty+1)} = \frac{1+b_\infty}{2}$. Therefore, one obtains:

$$a \;\approx\; 1, \qquad 1-a \;\approx\; (1+b_\infty)\varepsilon. \tag{IA.363}$$

One has $a = a^- + \rho G$, hence $\rho G = a - a^- = (1-a^-) - (1-a) \approx 2\varepsilon - (1+b_\infty)\varepsilon = (1-b_\infty)\varepsilon$. One obtains:

$$\rho G \;\approx\; (1-b_\infty)\varepsilon. \tag{IA.364}$$

I now analyze $R = \frac{\lambda}{\rho}$. From (IA.331), one has $\gamma^- = \frac{1-\lambda G}{\lambda}\frac{N_F}{N_F+1}$, which multiplying by ρ becomes $a^- = \big(\frac{1}{R} - \rho G\big)(1-\varepsilon)$. From this, $\frac{1}{R} = \frac{a^-}{1-\varepsilon} + \rho G$, which implies $1 - \frac{1}{R} = \frac{1-a^- -\varepsilon}{1-\varepsilon} - \rho G$. Using $1-a^- \approx 2\varepsilon$ and $\rho G \approx (1-b_\infty)\varepsilon$, one gets $1 - \frac{1}{R} \approx b_\infty \varepsilon$. From this, one gets $R \approx 1$ and $\frac{R-1}{R} \approx b_\infty\varepsilon$, hence:

$$R \;\approx\; 1, \qquad R-1 \;\approx\; b_\infty\varepsilon. \tag{IA.365}$$

One computes $1 - Ra^- \approx 1 - (1+b_\infty\varepsilon)(1-2\varepsilon) \approx (2-b_\infty)\varepsilon$. Similarly, $1 - Ra \approx 1 - (1+b_\infty\varepsilon)\big(1-(1+b_\infty)\varepsilon\big) \approx \varepsilon$. One obtains:

$$1-Ra^- \;\approx\; (2-b_\infty)\varepsilon, \qquad 1-Ra \;\approx\; \varepsilon. \tag{IA.366}$$

Since $b = 1 - \frac{1}{B^2}$, $b_\infty = 1 - \frac{1}{B_\infty^2}$, $\Theta = 2 - \frac{\sqrt{1-b}}{b}$, one obtains:

$$b \;\approx\; b_\infty, \qquad \Theta \;\approx\; 2 - \frac{\sqrt{1-b_\infty}}{b_\infty} \;=\; 1, \qquad \phi \;\approx\; 0. \tag{IA.367}$$

From (IA.279), one has $\rho^2\tilde{\sigma}_u^2 = (1-a)(a-b^2) + \frac{\Theta}{1+\phi}\frac{(1-a^-)^2}{4(B+1)^2}\big(1+2bB^2(1+\phi)\big)$, which implies $\frac{\rho^2\tilde{\sigma}_u^2}{1-a} \approx a_\infty - b_\infty^2$. Using $1-a \approx \frac{1+b_\infty}{N_F+1}$ one gets $\rho^2\tilde{\sigma}_u^2 \approx \frac{(1-b_\infty^2)(1+b_\infty)}{N_F+1}$, hence:

$$\rho^2\tilde{\sigma}_u^2 \;\approx\; \frac{1}{N_F+1}, \qquad \text{or} \quad \rho \;\approx\; \frac{\sigma_w}{\sigma_u}\frac{1}{\sqrt{N_F+1}}. \tag{IA.368}$$

Since $R \approx 1$, one has $\lambda \approx \rho$. Therefore, one obtains:

$$\lambda \;\approx\; \rho \;\approx\; \frac{\sigma_w}{\sigma_u}\frac{1}{\sqrt{N_F+1}}. \tag{IA.369}$$

I now compare the asymptotic results with the corresponding results in the benchmark model. Denote by γ_0, μ_0, λ_0, ρ_0, a_0, b_0 the equilibrium coefficients from Theorem 1, and by γ_∞, μ_∞, λ_∞, ρ_∞, a_∞, b_∞, respectively, their asymptotic limits. I have already shown that $\lambda \approx \rho \approx \lambda_\infty = \rho_\infty = \frac{\sigma_w}{\sigma_u}\frac{1}{\sqrt{N_F+1}}$; also $a \approx a_\infty = 1$, and $b \approx b_\infty = \frac{\sqrt{5}-1}{2}$. Moreover, in the inventory management equilibrium, one has $\gamma = \frac{\gamma^-}{N_F} = \frac{a^-}{\rho N_F} \approx \frac{1}{\rho_0(N_F+1)} \approx \frac{a_0}{\rho_0(N_F+1)} = \frac{\bar{\gamma}_0}{N_F+1} = \gamma_0$; and $\mu = \frac{\bar{\mu}}{N_L} = \frac{b}{\rho N_L} \approx \frac{b_\infty}{\rho_0 N_L} \approx \frac{b_0}{\rho_0 N_L} = \frac{\bar{\mu}_0}{N_L} = \mu_0$. Thus, $\gamma \approx \gamma_0$ and $\mu \approx \mu_0$. I have just proved that:

$$\gamma \;\approx\; \gamma_0, \qquad \mu \;\approx\; \mu_0, \qquad \lambda \;\approx\; \lambda_0, \qquad \rho \;\approx\; \rho_0, \qquad a \;\approx\; a_0, \qquad b \;\approx\; b_0. \quad \text{(IA.370)}$$

I also report the asymptotic results for $1-a_0$, $1-a_0^-$, $1-R_0 a_0$, $1-R_0 a_0^-$. From Theorem 1 (with N_F+1 fast traders), $a_0 = \frac{(N_F+1)-b_0}{(N_F+1)+1}$, hence $1-a_0 = \frac{1+b_0}{(N_F+1)+1} \approx (1+b_\infty)\varepsilon$. Also, $a_0^- = \frac{N_F}{N_F+1} a_0 = (1-\varepsilon)a_0$, hence $1-a_0^- \approx 1-(1-\varepsilon)\big(1-(1+b_\infty)\varepsilon\big) \approx (2+b_\infty)\varepsilon$. From Corollary 1, $\lambda_0 \bar{\gamma}_0 = \frac{N_F+1}{N_F+2}$, hence $1 - R_0 a_0 = 1 - \lambda_0 \bar{\gamma}_0 = \frac{1}{N_F+2} \approx \varepsilon$. Also, $R_0 a_0^- = R_0 \frac{N_F}{N_F+1} a_0 = \frac{N_F}{N_F+1} \lambda_0 \bar{\gamma}_0 = \frac{N_F}{N_F+2}$, hence $1 - R_0 a_0^- = \frac{2}{N_F+2} \approx 2\varepsilon$. Putting together these formulas, it follows that in the benchmark model:

$$1-a_0 \;\approx\; (1+b_\infty)\varepsilon, \quad 1-a_0^- \;\approx\; (2+b_\infty)\varepsilon, \quad 1-R_0 a_0 \;\approx\; \varepsilon, \quad 1-R_0 a_0^- \;\approx\; 2\varepsilon. \quad \text{(IA.371)}$$

By contrast, in the inventory management model:

$$1-a \;\approx\; (1+b_\infty)\varepsilon, \quad 1-a^- \;\approx\; 2\varepsilon, \quad 1-Ra \;\approx\; \varepsilon, \quad 1-Ra^- \;\approx\; (2-b_\infty)\varepsilon. \quad \text{(IA.372)}$$

The difference comes from the fact that the IFT's equilibrium weight G is not equal asymptotically to the FT's weight $\gamma \approx \gamma_0$ in either the inventory management model or the benchmark model. To see this, note that in the inventory management model one has $1-a = 1-a^- - \rho G$, while in the benchmark model $1-a_0 = 1-a_0^- - \rho_0\gamma_0$. But $\rho G \approx (1-b_\infty)\varepsilon$ (from equation (IA.364)), while $\rho_0\gamma_0 \approx \lambda_0\gamma_0 \approx \varepsilon$, where the last approximation follows from $\lambda_0\bar{\gamma}_0 = \frac{N_F+1}{N_F+2}$, which implies $\lambda_0\gamma_0 = \frac{1}{N_F+2} \approx \varepsilon$. I record this result for future reference:

$$\lambda_0\gamma_0 \;\approx\; \varepsilon. \quad \text{(IA.373)}$$

If I now use $\rho G \approx (1-b_\infty)\varepsilon$ and $\rho_0\gamma_0 \approx \varepsilon$, by taking their ratio one obtains:

$$\frac{G}{\gamma} \;\approx\; \frac{G}{\gamma_0} \;\approx\; 1-b_\infty \;=\; 0.3820. \quad \text{(IA.374)}$$

From Corollary IA.2, the normalized expected profit of a FT in the inventory management model is $\tilde{\pi}^F = \gamma(1-\lambda\bar{\gamma}) = \gamma(1-Ra)$. From (IA.372), $1-Ra \approx \varepsilon$. Since $\gamma \approx \gamma_0$, one obtains:

$$\tilde{\pi}^F \;=\; \gamma_0\,\varepsilon. \quad \text{(IA.375)}$$

From Proposition 1, the normalized expected profit of a FT in the benchmark model is

$\tilde{\pi}_0^F = \frac{\gamma_0}{N_F+2}$, which implies:

$$\tilde{\pi}_0^F \;\approx\; \gamma_0\,\varepsilon. \tag{IA.376}$$

Therefore, the FTs in the inventory management model make asymptotically the same profits as the FTs in the benchmark model:

$$\frac{\tilde{\pi}^F}{\tilde{\pi}_0^F} \;\approx\; 1. \tag{IA.377}$$

For the IFT, equation (IA.291) implies that the normalized expected utility (or profit) is $\tilde{\pi} = \frac{b\left(R(1-a^-)\right)^2}{4\lambda(1+\sqrt{1-b})^2}$. Since $R \approx 1$ and $1 - a^- \approx 2\varepsilon$, one has $\tilde{\pi} \approx \frac{b_\infty \varepsilon^2}{\lambda_0(1+b_\infty)^2}$. From (IA.373), $\frac{\varepsilon}{\lambda_0} = \gamma_0$, which implies $\tilde{\pi} \approx \frac{b_\infty}{(1+b_\infty)^2}\,\gamma_0\,\varepsilon$, or:

$$\tilde{\pi} \;\approx\; (2b_\infty - 1)\,\gamma_0\,\varepsilon. \tag{IA.378}$$

Asymptotically, the ratio of the IFT's profit to the FT's profit is given by:

$$\frac{\tilde{\pi}}{\tilde{\pi}^F} \;\approx\; 2b_\infty - 1 \;=\; 0.2361. \tag{IA.379}$$

Denote by $\tilde{\pi}_{C_I=0}$ the IFT's maximum normalized expected profit when $C_I = 0$. Equation (78) implies that:

$$\tilde{\pi}_{C_I=0} \;=\; \frac{\left(1 - Ra^-\right)^2}{4\lambda}. \tag{IA.380}$$

Since $1 - Ra^- \approx (2 - b_\infty)\varepsilon$ and $\lambda \approx \lambda_0 \approx \frac{\varepsilon}{\gamma_0}$ (equation (IA.373)), one gets $\tilde{\pi}_{C_I=0} \approx \frac{(2-b_\infty)^2}{4}\gamma_0\varepsilon$, or:

$$\tilde{\pi}_{C_I=0} \;\approx\; \frac{5}{4}(1 - b_\infty)\,\gamma_0\,\varepsilon \;=\; 0.4775\,\gamma_0\,\varepsilon. \tag{IA.381}$$

Asymptotically, the ratio of $\tilde{\pi}$ to $\tilde{\pi}_{C_I=0}$ is $\frac{2b_\infty-1}{\frac{5}{4}(1-b_\infty)} = \frac{4}{5}b_\infty$, hence:

$$\frac{\tilde{\pi}}{\tilde{\pi}_{C_I=0}} \;\approx\; \frac{4}{5}\,b_\infty \;=\; 0.4944. \tag{IA.382}$$

Thus, inventory management generates a profit loss of about 50% for the IFT.

Equation (IA.293) implies that the threshold inventory aversion for the IFT is given by $1 + \frac{\bar{C}_I}{2\lambda} = \frac{(1-Ra^-)^2(1+\sqrt{1-b})^2}{R^2 b(1-a^-)^2}$. One has $R \approx 1$, $1 - Ra^- \approx (2 - b_\infty)\varepsilon$, $1 - a^- \approx 2\varepsilon$, hence $1 + \frac{\bar{C}_I}{2\lambda} \approx \frac{(2-b_\infty)^2(1+b_\infty)^2}{4b_\infty} = \frac{5}{4}(1 + b_\infty)$, which implies $\frac{\bar{C}_I}{2\lambda} \approx \frac{1+5b_\infty}{4}$. Since $\lambda \approx \frac{\sigma_w}{\sigma_u}\,\frac{1}{\sqrt{N_F+1}}$, one obtains:

$$\bar{C}_I \;\approx\; \frac{1 + 5b_\infty}{2}\,\lambda \;\approx\; 2.0451\,\frac{\sigma_w}{\sigma_u}\,\frac{1}{\sqrt{N_F+1}}. \tag{IA.383}$$

■

Proof of Proposition 8. As in Subsection 3.2 in the paper, I equate trading volume with (instantaneous) order flow variance. From (68), $\Theta\Omega^{xx} = \frac{G^2}{1+\phi}$, where $\phi = 1 - \Theta$. Since $\mathrm{d}x_t = -\Theta x_{t-1} + G\mathrm{d}w_t$, the IFT's normalized order flow variance (or normalized trading volume) satisfies:

$$\frac{\mathsf{Var}(\mathrm{d}x_t)}{\sigma_w^2 \mathrm{d}t} \;=\; \frac{TV_x}{\sigma_w^2} \;=\; \Theta^2\Omega^{xx} + G^2 \;=\; \frac{1-\phi}{1+\phi}\, G^2 + G^2 \;=\; \frac{2G^2}{1+\phi}. \tag{IA.384}$$

Since $x_t = \phi x_{t-1} + G\mathrm{d}w_t$, the IFT's order flow autocovariance satisfies $\mathsf{Cov}\big(\mathrm{d}x_t, \mathrm{d}x_{t+1}\big) = \mathsf{Cov}\big(-\Theta x_{t-1} + G\mathrm{d}w_t, -\Theta\phi x_{t-1} - \Theta G\mathrm{d}w_t + G\mathrm{d}w_{t+1}\big)$, hence:

$$\frac{\mathsf{Cov}\big(\mathrm{d}x_t, \mathrm{d}x_{t+1}\big)}{\sigma_w^2 \mathrm{d}t} \;=\; \Theta^2\phi\Omega^{xx} - \Theta G^2 \;=\; -\Theta\left(-\frac{\phi}{1+\phi}\, G^2 + G^2\right) \;=\; -\frac{\Theta G^2}{1+\phi}. \tag{IA.385}$$

Therefore, the IFT's order flow autocorrelation is:

$$\rho_x \;=\; \mathsf{Corr}\big(\mathrm{d}x_t, \mathrm{d}x_{t+1}\big) \;=\; \frac{\mathsf{Cov}\big(\mathrm{d}x_t, \mathrm{d}x_{t+1}\big)}{\mathsf{Var}(\mathrm{d}x_t)} \;=\; -\frac{\Theta}{2}, \tag{IA.386}$$

which proves the corresponding formula in (44). Asymptotically, since $\Theta \approx 1$, $\rho_x \approx -\frac{1}{2}$.

The individual and the aggregate trading volume of FTs satisfies, respectively:

$$\frac{TV_{x^F}}{\sigma_w^2} \;=\; \gamma^2, \qquad \frac{TV_{\bar{x}^F}}{\sigma_w^2} \;=\; (\gamma^-)^2. \tag{IA.387}$$

From (IA.384) and (IA.387), one gets $\frac{TV_x}{TV_{x^F}} = \frac{2G^2}{(1+\phi)\gamma^2}$, which proves the corresponding formula in (44). Asymptotically, Proposition 7 shows that $\frac{G}{\gamma} \approx 1 - b_\infty$ and $\phi \approx 0$, hence $\frac{TV_x}{TV_{x^F}} \approx 2(b_\infty - 1)^2 = 2(2 - 3b_\infty) = 0.2918$. The aggregate trading volume of STs satisfies:

$$\frac{TV_{\bar{x}^S}}{\sigma_w^2} \;=\; \bar{\mu}^2\, \frac{\mathsf{Var}\big(\widetilde{\mathrm{d}w_t}\big)}{\sigma_w^2 \mathrm{d}t} \;=\; \bar{\mu}^2\, W \;=\; \bar{\mu}^2(1-a), \tag{IA.388}$$

where I use the equilibrium formula $W = 1 - a$ in (IA.333). From (IA.387) and (IA.388), one compute:

$$\frac{TV_{\bar{x}^S}}{TV_{\bar{x}^F}} \;=\; \frac{\bar{\mu}^2(1-a)}{(\gamma^-)^2} \;=\; \frac{b^2(1-a)}{(a^-)^2}, \tag{IA.389}$$

as stated in Proposition 8. Asymptotically, from (IA.372) one has $1 - a \approx \frac{1+b_\infty}{N_F+1}$, and I use $b \approx b_\infty$, $a \approx 1$, $b_\infty^2(1+b_\infty) = b_\infty$ to get $\frac{TV_{\bar{x}^S}}{TV_{\bar{x}^F}} \approx \frac{b_\infty}{N_F+1}$, as stated.

Recall that $X = \frac{\mathsf{Cov}(\widetilde{\mathrm{d}w_t} x_t)}{\sigma_w^2 \mathrm{d}t}$ and $W = \frac{\mathsf{Var}(\widetilde{\mathrm{d}w_t})}{\sigma_w^2 \mathrm{d}t}$. From (IA.333) and $1 - a = (1 -$

$a^-)\frac{2B+1}{2(B+1)}$, one computes ($B = \frac{1}{\sqrt{1-b}}$):

$$\rho X \;=\; (1-a)^2\,\frac{B(B+1)}{(2B+1)^2}. \tag{IA.390}$$

The regression coefficient of the IFT's strategy ($\mathrm{d}x_t$) on the slow trading component ($\mathrm{d}\bar{x}_t^S$) satisfies:

$$\beta_{x,\bar{x}^S} \;=\; \frac{\mathsf{Cov}(\mathrm{d}x_t, \mathrm{d}\bar{x}_t^S)}{\mathsf{Var}(\mathrm{d}\bar{x}_t^S)} \;=\; \frac{-\Theta\bar{\mu}\,\mathsf{Cov}\big(x_{t-1}, \widetilde{\mathrm{d}w}_{t-1}\big)}{\bar{\mu}^2\,\mathsf{Var}\big(\widetilde{\mathrm{d}w}_{t-1}\big)} \;=\; \frac{-\Theta X}{\bar{\mu}W} \;=\; \frac{-\Theta\,(\rho X)}{b(1-a)}. \tag{IA.391}$$

Using (IA.390) and $W = 1-a$, one computes:

$$\beta_{x,\bar{x}^S} \;=\; -\frac{\Theta B(B+1)}{b(2B+1)^2}\,(1-a) \;=\; -\frac{\Theta B}{2b(2B+1)}\,(1-a^-). \tag{IA.392}$$

Asymptotically, $b \approx b_\infty$, $B \approx \frac{1}{b_\infty}$, $\Theta \approx 1$, and from (IA.372) $1-a^- \approx \frac{2}{N_F+1}$. Hence:

$$\beta_{x,\bar{x}^S} \;\approx\; -\frac{1}{b_\infty(1+2b_\infty)}\,\frac{1}{N_F+1} \;=\; -\frac{3+b_\infty}{5(N_F+1)} \;=\; -\frac{0.7236}{N_F+1}, \tag{IA.393}$$

which proves the stated formula.

Since the trading strategy of a FT is $\gamma\mathrm{d}w_t$, the order flow autocorrelation of the FTs is $\rho_{\bar{x}^F} = 0$. For the STs, since $\widetilde{\mathrm{d}w}_t = \mathrm{d}w_t - \rho\mathrm{d}y_t$ and $\mathrm{d}y_t = -\Theta x_{t-1} + \bar{\gamma}\mathrm{d}w_t + \bar{\mu}\widetilde{\mathrm{d}w}_{t-1} + \mathrm{d}u_t$, one computes the normalized order flow autocoviance:

$$\frac{\mathsf{Cov}\big(\mathrm{d}\bar{x}_{t-1}^S, \mathrm{d}\bar{x}_t^S\big)}{\sigma_w^2\mathrm{d}t} \;=\; -\rho\bar{\mu}^2\frac{\mathsf{Cov}\big(\widetilde{\mathrm{d}w}_{t-1}, \mathrm{d}y_t\big)}{\sigma_w^2\mathrm{d}t} \;=\; -\rho\bar{\mu}^2\Big(-\Theta X + \bar{\mu}W\Big). \tag{IA.394}$$

From (IA.388) and (IA.394), one computes the autocorrelation of slow trading $\mathrm{d}\bar{x}_t^S$, as follows:

$$\begin{aligned}\rho_{\bar{x}^S} \;&=\; \frac{\mathsf{Cov}\big(\mathrm{d}\bar{x}_{t-1}^S, \mathrm{d}\bar{x}_t^S\big)}{\mathsf{Var}(\mathrm{d}\bar{x}_t^S)} \;=\; \frac{-\rho\bar{\mu}^2\big(-\Theta X + \bar{\mu}(1-a)\big)}{\bar{\mu}^2(1-a)} \;=\; -b + \frac{\Theta\rho X}{(1-a)} \\ &=\; -b - b\beta_{x,\bar{x}^S},\end{aligned} \tag{IA.395}$$

where for the last equality I use (IA.392). Asymptotically, the term $b + b\beta_{x,\bar{x}^S} \approx b$, since $\beta_{x,\bar{x}^S}$ is of the order of $\frac{1}{N_F+1}$. Hence, $\rho_{\bar{x}^S} \approx -b_\infty$. ■

5.3 One IFT, general strategy

In this subsection, I assume that the IFT has a more general strategy which includes trading on the lagged signal:

$$\mathrm{d}x_t \;=\; -\Theta x_{t-1} + G\mathrm{d}w_t + M\widetilde{\mathrm{d}w}_{t-1}. \tag{IA.396}$$

I introduce some useful notation. If x_t is the IFT's inventory in the risky asset, denote by:

$$\Omega_t^{xx} \;=\; \frac{\mathsf{E}\big(x_t^2\big)}{\sigma_w^2\mathrm{d}t}, \quad X_t \;=\; \frac{\mathsf{E}\big(x_t\widetilde{\mathrm{d}w}_t\big)}{\sigma_w^2\mathrm{d}t}, \quad Z_t \;=\; \frac{\mathsf{E}\big(x_{t-1}\mathrm{d}y_t\big)}{\sigma_w^2\mathrm{d}t}, \quad W_t \;=\; \frac{\mathsf{E}\big((\widetilde{\mathrm{d}w}_t)^2\big)}{\sigma_w^2\mathrm{d}t}. \tag{IA.397}$$

Define the following aggregate trading coefficients:

$$\bar{\gamma} \;=\; \gamma^- + G, \quad \bar{\mu} \;=\; \mu^- + M. \tag{IA.398}$$

Since $\mathrm{d}x_t = -\Theta x_{t-1} + G\mathrm{d}w_t + M\widetilde{\mathrm{d}w}_{t-1}$, the aggregate order flow satisfies:

$$\mathrm{d}y_t \;=\; -\Theta x_{t-1} + \bar{\gamma}\mathrm{d}w_t + \bar{\mu}\widetilde{\mathrm{d}w}_{t-1} + \mathrm{d}u_t. \tag{IA.399}$$

Hence, the lagged signal $\widetilde{\mathrm{d}w}_t$ satisfies:

$$\widetilde{\mathrm{d}w}_t \;=\; \mathrm{d}w_t - \rho\mathrm{d}y_t \;=\; \rho\Theta x_{t-1} + (1-\rho\bar{\gamma})\mathrm{d}w_t - \rho\bar{\mu}\widetilde{\mathrm{d}w}_{t-1} - \rho\mathrm{d}u_t. \tag{IA.400}$$

I now compute the IFT's expected utility. As in Proposition 6 in the paper, the IFT holds all his profits in cash, and his expected utility is the same as the expected profit. Also, from equation (35) in the paper, the IFT's normalized profit is $\tilde{\pi}_{\Theta>0} = \mathsf{E}\int_0^T x_{t-1}\mathrm{d}p_t = \lambda\,\mathsf{E}\int_0^T x_{t-1}\mathrm{d}y_t$. Using the notation in (IA.397), one obtains:[23]

$$\tilde{\pi}_{\Theta>0} \;=\; \lambda\int_0^T Z_t\mathrm{d}t. \tag{IA.401}$$

Equations (IA.399) and (IA.397) imply that:

$$Z_t \;=\; \frac{\mathsf{E}\big(x_{t-1}\mathrm{d}y_t\big)}{\sigma_w^2\mathrm{d}t} \;=\; -\Theta\Omega_{t-1}^{xx} + \bar{\mu}X_{t-1}. \tag{IA.402}$$

[23]Below I show that Z_t is constant, which implies that $\tilde{\pi}_{\Theta>0} = \lambda Z$. (Recall that $T = 1$.)

To compute Ω_t^{xx} and X_t, I follow the strategy described in the proof of Proposition 6 in the paper, and analyze the recursive equations that these variables satisfy. To that end, I begin by noticing that the IFT's inventory itself follows a recursive equation:

$$x_t \;=\; \phi x_{t-1} + G\mathrm{d}w_t + M\widetilde{\mathrm{d}w}_{t-1}, \tag{IA.403}$$

where $\phi = 1 - \Theta \in (-1, 1)$. (Recall that by assumption $\Theta \in (0, 2)$.) The recursive formula for Ω_t^{xx} is then:

$$\begin{aligned} \Omega_t^{xx} \;&=\; \frac{\mathsf{E}((x_t)^2)}{\sigma_w^2 \mathrm{d}t} \;=\; \frac{\mathsf{E}\big((\phi x_{t-1} + G\mathrm{d}w_t + M\widetilde{\mathrm{d}w}_{t-1})^2\big)}{\sigma_w^2 \mathrm{d}t} \\ &=\; \phi^2 \Omega_{t-1}^{xx} + G^2 + M^2 W_{t-1} + 2\phi M X_{t-1}. \end{aligned} \tag{IA.404}$$

Using equation (IA.400), the recursive formula for X_t is:

$$\begin{aligned} X_t \;&=\; \frac{\mathsf{E}(x_t \widetilde{\mathrm{d}w}_t)}{\sigma_w^2 \mathrm{d}t} \;=\; \frac{\mathsf{E}\big((\phi x_{t-1} + G\mathrm{d}w_t + M\widetilde{\mathrm{d}w}_{t-1})(\rho\Theta x_{t-1} + (1-\rho\bar{\gamma})\mathrm{d}w_t - \rho\bar{\mu}\widetilde{\mathrm{d}w}_{t-1})\big)}{\sigma_w^2 \mathrm{d}t} \\ &=\; \phi\rho\Theta\Omega_{t-1}^{xx} + G(1-\rho\bar{\gamma}) - M\rho\bar{\mu}W_{t-1} + \rho\big(M\Theta - \phi\bar{\mu}\big)X_{t-1}. \end{aligned} \tag{IA.405}$$

Using again equation (IA.400), the recursive formula for W_t is (recall that $\tilde{\sigma}_u = \frac{\sigma_u}{\sigma_w}$):

$$\begin{aligned} W_t \;&=\; \frac{\mathsf{E}\big((\widetilde{\mathrm{d}w}_t)^2\big)}{\sigma_w^2 \mathrm{d}t} \;=\; \frac{\mathsf{E}\big((\rho\Theta x_{t-1} + (1-\rho\bar{\gamma})\mathrm{d}w_t - \rho\bar{\mu}\widetilde{\mathrm{d}w}_{t-1} - \rho\mathrm{d}u_t)^2\big)}{\sigma_w^2 \mathrm{d}t} \\ &=\; \rho^2\Theta^2\Omega_{t-1}^{xx} + (1-\rho\bar{\gamma})^2 + \rho^2\bar{\mu}^2 W_{t-1} - 2\rho^2\Theta\bar{\mu}X_{t-1} + \rho^2\tilde{\sigma}_u^2. \end{aligned} \tag{IA.406}$$

Assume now that the following conditions are satisfied:

$$-1 \;<\; \phi^2, \rho(M\Theta - \phi\bar{\mu}), \rho^2\bar{\mu}^2 \;<\; 1. \tag{IA.407}$$

I apply Lemma A.1 in the Appendix in the paper to the recursive formulas for Ω_t^{xx}, X_t, and W_t. Then, these numbers are constant and satisfy:

$$\begin{aligned} (1-\phi^2)\Omega^{xx} \;&=\; G^2 + M^2 W + 2\phi M X, \\ (1 - \rho M\Theta + \rho\phi\bar{\mu})X \;&=\; G(1-\rho\bar{\gamma}) + \phi\rho\Theta\Omega^{xx} - M\rho\bar{\mu}W, \\ (1-\rho^2\bar{\mu}^2)W \;&=\; (1-\rho\bar{\gamma})^2 + \rho^2\tilde{\sigma}_u^2 + \rho^2\Theta^2\Omega^{xx} - 2\rho^2\Theta\bar{\mu}X. \end{aligned} \tag{IA.408}$$

Equation (IA.408) provides a 3×3-system of equations with three unknown constants:

Ω^{xx}, X, and W. Thus, I can solve explicitly for these numbers, given the model parameters and the choice variables G, M, and Θ.

Equation (IA.402) now implies that Z_t is constant, hence the IFT's expected profit can be explicitly computed as a function of the constants Ω^{xx} and X:

$$\tilde{\pi}_{\Theta>0} \;=\; \lambda Z \;=\; \lambda\left(-\Theta\Omega^{xx} + \bar{\mu}X\right). \tag{IA.409}$$

Numerical results

Numerically, for all the parameters verified, the optimal coefficients occur when $\phi = 1$, or equivalently $\Theta = 0$. In that case, $\Theta\Omega^{xx}$ has a finite limit, and one computes:

$$\tilde{\pi}_{\Theta>0} \;=\; \frac{\lambda}{\rho(1+b^-)}\left(-\frac{\rho G^2}{2} + b^-(1-a)G - \frac{M(2bb^- + \rho M)}{2(1-b^2)}\big((1-a)^2 + \rho^2\tilde{\sigma}_u^2\big)\right). \tag{IA.410}$$

With the constraints imposed by (IA.407), I find that the optimal G is positive, and the optimal M is negative. Specifically, one computes:

$$\begin{aligned} G \;&=\; \frac{\big(1+3b^- - (1+2b^-)\Delta\big)(1-a^-)(1-b^-)}{2(b^-)^2\Delta}, \\ M \;&=\; -\frac{\Delta - (1-b^-)(1+2b^-)}{2b^-}, \\ \text{with}\quad \Delta \;&=\; \sqrt{(1-b^-)(1+3b^-)}. \end{aligned} \tag{IA.411}$$

To understand why the optimal M is negative, consider again equation (35), which translates into:

$$\tilde{\pi}_{\Theta>0} \;=\; \lambda\,\mathsf{E}\int_0^T x_t \mathrm{d}y_{t+1}. \tag{IA.412}$$

This implies that in the quick regime the IFT only makes profit from his correlation between his inventory this period (x_t) and the aggregate order flow next period ($\mathrm{d}y_{t+1}$). But equations (IA.399) and (IA.400) imply that the order flow next period satisfies:

$$\begin{aligned} \mathrm{d}y_{t+1} \;&=\; -\Theta x_t + \bar{\gamma}\mathrm{d}w_{t+1} + \bar{\mu}(\mathrm{d}w_t - \rho\mathrm{d}y_t) + \mathrm{d}u_{t+1} \\ &=\; -\Theta x_t + \bar{\gamma}\mathrm{d}w_{t+1} + \bar{\mu}\Big(-\rho\Theta x_{t-1} + (1-\rho\bar{\gamma})\mathrm{d}w_t - \rho\bar{\mu}\widetilde{\mathrm{d}w}_{t-1} - \rho\mathrm{d}u_t\Big) + \mathrm{d}u_{t+1} \end{aligned} \tag{IA.413}$$

Note that in the above formula the coefficient of $\mathrm{d}w_t$ is $\bar{\mu}(1-\rho\bar{\gamma}) > 0$, while the coefficient of $\widetilde{\mathrm{d}w}_{t-1}$ is $-\rho\bar{\mu}^2 < 0$. This observation corresponds to the fact that the optimal G is positive and the optimal M is negative. To explain why $M < 0$, note that the other traders (the FTs and STs) trade on signals with lag at most one. If they also traded on signals with lag larger than one, the aggregate order flow ($\mathrm{d}y_{t+1}$) would then have a positive correlation with the signal with lag two ($\mathrm{d}w_{t-1}$), and therefore in that case the optimal M would be positive. Another way to understand why $M < 0$ is to note that this component replaces partially the mean-reverting component $-\Theta x_{t-1}$ when $M = 0$. Indeed, in that case the IFT's strategy is of the form $\mathrm{d}x_t = G\mathrm{d}w_t - \Theta x_{t-1}$, and thus the mean-reverting component $-\Theta x_{t-1}$ contains the term $-\Theta G\mathrm{d}w_{t-1}$ which is similar to the term $M\widetilde{\mathrm{d}w}_{t-1}$ when M is negative.

5.4 One IFT, predictable order flow

In this subsection, I assume that the dealer knows that the order flow is predictable and sets the price to account for this predictability. As before, I assume that the IFT has a strategy of the form:

$$\mathrm{d}x_t \;=\; -\Theta x_{t-1} + G\mathrm{d}w_t. \tag{IA.414}$$

The aggregate order flow $\mathrm{d}y_t$ then satisfies:

$$\mathrm{d}y_t \;=\; -\Theta x_{t-1} + \bar{\gamma}\mathrm{d}w_t + \bar{\mu}\widetilde{\mathrm{d}w}_{t-1} + \mathrm{d}u_t, \tag{IA.415}$$

where:

$$\bar{\gamma} \;=\; \gamma^{-} + G, \qquad \bar{\mu} \;=\; \mu^{-}. \tag{IA.416}$$

Define:

$$\begin{aligned} \widehat{\mathrm{d}y}_t &\;=\; \mathsf{E}_t(\mathrm{d}y_t), \qquad & \widetilde{\mathrm{d}y}_t &\;=\; \mathrm{d}y_t - \widehat{\mathrm{d}y}_t, \\ \hat{x}_t &\;=\; \mathsf{E}_{t+1}(x_t), \qquad & \tilde{x}_t &\;=\; x_t - \hat{x}_t, \end{aligned} \tag{IA.417}$$

where as before $\mathcal{I}_t$ is the dealer's information set before trading at t, and E_t is the expectation operator conditional on $\mathcal{I}_t$. Note that E_{t+1} is the expectation operator conditional on $\mathcal{I}_t$ and $\widetilde{\mathrm{d}y}_t$. Because part of the aggregate order flow is predictable to the dealer, one has the following modified equations: By taking expectation at t in equation (IA.415), one gets $\widehat{\mathrm{d}y}_t = -\Theta\hat{x}_{t-1}$. One computes:

$$\widetilde{\mathrm{d}y}_t \;=\; -\Theta\tilde{x}_{t-1} + \bar{\gamma}\mathrm{d}w_t + \bar{\mu}\widetilde{\mathrm{d}w}_{t-1} + \mathrm{d}u_t. \tag{IA.418}$$

This equation implies that $\mathsf{E}_{t+1}(\mathrm{d}w_t) = \rho \widetilde{\mathrm{d}y}_t$ for an appropriate constant ρ. One obtains the following equations:

$$\begin{aligned} \widetilde{\mathrm{d}w}_t &= \mathrm{d}w_t - \mathsf{E}_{t+1}(\mathrm{d}w_t) = \mathrm{d}w_t - \rho\widetilde{\mathrm{d}y}_t, \\ \mathrm{d}p_t &= \lambda\widetilde{\mathrm{d}y}_t. \end{aligned} \tag{IA.419}$$

One also computes:

$$\widetilde{\mathrm{d}w}_t = \rho\Theta\tilde{x}_{t-1} + (1-\rho\bar{\gamma})\mathrm{d}w_t - \rho\bar{\mu}\widetilde{\mathrm{d}w}_{t-1} - \rho\mathrm{d}u_t. \tag{IA.420}$$

After taking expectations at $t+1$ of equation (IA.414), one obtains:

$$\mathrm{d}\hat{x}_t = -\Theta\hat{x}_{t-1} + G\,\mathsf{E}_{t+1}(\mathrm{d}w_t) = -\Theta\hat{x}_{t-1} + \rho G\widetilde{\mathrm{d}y}_t. \tag{IA.421}$$

Subtracting (IA.421) from (IA.414), one gets $\mathrm{d}\tilde{x}_t = -\Theta\tilde{x}_{t-1} + G(\mathrm{d}w_t - \rho\widetilde{\mathrm{d}y}_t)$, which implies:

$$\mathrm{d}\tilde{x}_t = -(1-\rho G)\Theta\tilde{x}_{t-1} + (1-\rho\bar{\gamma})G\mathrm{d}w_t - \rho G\bar{\mu}\widetilde{\mathrm{d}w}_{t-1} - \rho G\mathrm{d}u_t. \tag{IA.422}$$

This in turn implies the recursive formula for $\tilde{x}_t = \tilde{x}_{t-1} + \mathrm{d}\tilde{x}_t$:

$$\tilde{x}_t = (\phi + \rho G\Theta)\tilde{x}_{t-1} + (1-\rho\bar{\gamma})G\mathrm{d}w_t - \rho G\bar{\mu}\widetilde{\mathrm{d}w}_{t-1} - \rho G\mathrm{d}u_t. \tag{IA.423}$$

I now compute the IFT's expected utility. Define:

$$\Omega_t^{x\tilde{x}} = \frac{\mathsf{E}\big(x_t\tilde{x}_t\big)}{\sigma_w^2\mathrm{d}t}, \quad X_t = \frac{\mathsf{E}\big(x_t\widetilde{\mathrm{d}w}_t\big)}{\sigma_w^2\mathrm{d}t}, \quad Z_t = \frac{\mathsf{E}\big(x_{t-1}\widetilde{\mathrm{d}y}_t\big)}{\sigma_w^2\mathrm{d}t}. \tag{IA.424}$$

As in Proposition 6 in the paper, the IFT holds all his profits in cash, and his expected utility is the same as the expected profit. Also, from equation (35) in the paper, the IFT's normalized profit is $\tilde{\pi}_{\Theta>0} = \mathsf{E}\int_0^T x_{t-1}\mathrm{d}p_t = \lambda\,\mathsf{E}\int_0^T x_{t-1}\widetilde{\mathrm{d}y}_t$. Using the notation in (IA.424), one obtains:

$$\tilde{\pi}_{\Theta>0} = \lambda\int_0^T Z_t\mathrm{d}t. \tag{IA.425}$$

Equations (IA.424) and (IA.418) imply that:

$$Z_t = \frac{\mathsf{E}\big(x_{t-1}\widetilde{\mathrm{d}y}_t\big)}{\sigma_w^2\mathrm{d}t} = -\Theta\Omega_{t-1}^{x\tilde{x}} + \bar{\mu}X_{t-1}. \tag{IA.426}$$

To compute $\Omega_t^{x\tilde{x}}$ and X_t, I follow the strategy in the proof of Proposition 6 in the paper, and analyze the recursive equations that these variables satisfy. Equation (IA.423) says that $\tilde{x}_t = (\phi + \rho G \Theta)\tilde{x}_{t-1} + (1 - \rho\bar{\gamma})G\mathrm{d}w_t - \rho G\bar{\mu}\widetilde{\mathrm{d}w}_{t-1} - \rho G\mathrm{d}u_t$, hence:

$$\begin{aligned} \Omega_t^{x\tilde{x}} &= \frac{\mathsf{E}(x_t\tilde{x}_t)}{\sigma_w^2\mathrm{d}t} = \frac{\mathsf{E}\big((\phi x_{t-1} + G\mathrm{d}w_t)((\phi + \rho G\Theta)\tilde{x}_{t-1} + (1-\rho\bar{\gamma})G\mathrm{d}w_t - \rho G\bar{\mu}\widetilde{\mathrm{d}w}_{t-1})\big)}{\sigma_w^2\mathrm{d}t} \\ &= G^2(1-\rho\bar{\gamma}) + \phi(\phi + \rho G\Theta)\Omega_t^{x\tilde{x}} - \phi\rho G\bar{\mu}X_{t-1}. \end{aligned} \tag{IA.427}$$

Equation (IA.420) says that $\widetilde{\mathrm{d}w}_t = \rho\Theta\tilde{x}_{t-1} + (1-\rho\bar{\gamma})\mathrm{d}w_t - \rho\bar{\mu}\widetilde{\mathrm{d}w}_{t-1} - \rho\mathrm{d}u_t$, hence:

$$\begin{aligned} X_t &= \frac{\mathsf{E}(x_t\widetilde{\mathrm{d}w}_t)}{\sigma_w^2\mathrm{d}t} = \frac{\mathsf{E}\big((\phi x_{t-1} + G\mathrm{d}w_t)(\rho\Theta\tilde{x}_{t-1} + (1-\rho\bar{\gamma})\mathrm{d}w_t - \rho\bar{\mu}\widetilde{\mathrm{d}w}_{t-1})\big)}{\sigma_w^2\mathrm{d}t} \\ &= G(1-\rho\bar{\gamma}) + \phi\rho\Theta\Omega_{t-1}^{x\tilde{x}} - \phi\rho\bar{\mu}X_{t-1}. \end{aligned} \tag{IA.428}$$

Assume now that the following conditions are satisfied:

$$-1 \ < \ \phi(\phi + \rho G\Theta), \phi\rho\bar{\mu} \ < \ 1. \tag{IA.429}$$

I apply Lemma A.1 in the Appendix in the paper to the recursive formulas for $\Omega_t^{x\tilde{x}}$ and X_t. Then, these numbers are constant and satisfy:

$$\begin{aligned} (1 - \phi^2 - \phi\rho G\Theta)\Omega^{x\tilde{x}} + \phi\rho G\bar{\mu}X &= G^2(1-\rho\bar{\gamma}), \\ -\phi\rho\Theta\Omega^{x\tilde{x}} + (1 + \phi\rho\bar{\mu})X &= G(1-\rho\bar{\gamma}). \end{aligned} \tag{IA.430}$$

Equation (IA.430) provides a 2×2-system of equations in $\Omega^{x\tilde{x}}$ and X. This can be solved explicitly, and from (IA.426) it follows that $Z_t = -\Theta\Omega^{x\tilde{x}} + \bar{\mu}X$ is constant and equal to:

$$Z \ = \ G(1-\rho\bar{\gamma})\,\frac{-G + (1+\phi)\bar{\mu}}{-\phi\rho G + (1+\phi) + \phi(1+\phi)\rho\bar{\mu}}. \tag{IA.431}$$

Hence the IFT's expected profit $\tilde{\pi}_{\Theta>0} \ = \ \lambda Z$ can be explicitly computed:

$$\tilde{\pi}_{\Theta>0} \ = \ \lambda G(1-\rho\bar{\gamma})\,\frac{-G + (1+\phi)\bar{\mu}}{-\phi\rho G + (1+\phi) + \phi(1+\phi)\rho\bar{\mu}}. \tag{IA.432}$$

Denote:

$$g \ = \ \rho G, \tag{IA.433}$$

The first order condition in g for a maximum in (IA.432) yields a cubic equation:

$$\begin{aligned} P(g) \;=\; & -2\phi g^3 + \big(3(1+\phi) + \phi(1-a^-) + 4\phi(1+\phi)b\big) g^2 \\ & - 2(1+\phi)(1+\phi b)(1-a^- + (1+\phi)b) g + (1+\phi)^2(1-a^-)b(1+\phi b) \;=\; 0, \end{aligned} \tag{IA.434}$$

where $a^- = \rho\gamma^-$ and $b = \rho\bar{\mu}$. Since $\tilde{\pi}_{\Theta>0} = 0$ for $g = 0$ and $g = (1+\phi)b$, the first order condition is satisfied for the unique root of P in the interval $\big(0, (1+\phi)b\big)$.

The first condition in ϕ yields the equation:

$$\big(b(1+\phi) - g\big)^2 - g \;=\; 0. \tag{IA.435}$$

This equation has two real solutions:

$$\phi \;=\; \frac{z^2 + z - b}{b}, \qquad z \;=\; \pm\sqrt{g}. \tag{IA.436}$$

Substituting (IA.436) in (IA.434), one obtains the following quartic equation in z:

$$Q(z) \;=\; z^4 + 3z^3 + 2z^2(1-b) - z(1-a^-) - (1-a^-)(1-b) \;=\; 0. \tag{IA.437}$$

Since $b < 1$, the second derivative of $Q(z)$ is positive, hence Q is convex. But $Q(0) < 0$ and $Q(-\infty) > 0$, $Q(\infty) > 0$, hence Q has only two real roots $z_1 > 0$ and $z_2 < 0$. Note that equation (IA.435) implies that $(1+\phi)b = z^2 + z$. But as shown above, g must belong to the interval $(0, (1+\phi)b) = (0, z^2 + z)$. Since $g = z^2$, it follows that z must be positive. Hence, define:

$$z \;=\; \text{unique positive root of } Q. \tag{IA.438}$$

Then, the corresponding solution (ϕ, G), or equivalently (Θ, G), is given by:

$$\phi \;=\; \frac{z^2 + z - b}{b}, \qquad \Theta \;=\; \frac{2b - (z^2 + z)}{b}, \qquad G \;=\; \frac{z^2}{\rho}. \tag{IA.439}$$

This finishes the proof.

5.5 Multiple IFTs

Suppose beside the $N_F \geq 0$ FTs there are also $I \geq 2$ IFTs with trading strategy:

$$\mathrm{d}x_{i,t} \;=\; G_i \mathrm{d}w_t - \Theta_i x_{i,t}, \quad \text{with} \quad \Theta_i \;\in\; (0,2), \quad i \;=\; 1,\ldots,I. \tag{IA.440}$$

If $x_{i,t}$ is the inventory of IFT i, denote:

$$\begin{aligned} \Omega^{xx}_{ij,t} \;&=\; \frac{\mathsf{E}\big(x_{i,t}x_{j,t}\big)}{\sigma_w^2 \mathrm{d}t}, \qquad \Omega^{xe}_{i,t} \;=\; \frac{\mathsf{E}\big(x_{i,t}(w_t-p_t)\big)}{\sigma_w^2 \mathrm{d}t}, \qquad X_{i,t} \;=\; \frac{\mathsf{E}\big(x_{i,t}\widetilde{\mathrm{d}w}_t\big)}{\sigma_w^2 \mathrm{d}t} \\ \Omega^{xw}_{i,t} \;&=\; \frac{\mathsf{E}\big(x_{i,t}w_t\big)}{\sigma_w^2 \mathrm{d}t}, \qquad \Omega^{xp}_{i,t} \;=\; \frac{\mathsf{E}\big(x_{i,t}p_t\big)}{\sigma_w^2 \mathrm{d}t}, \qquad Z_{i,t} \;=\; \frac{\mathsf{E}\big(x_{i,t-1}\mathrm{d}y_t\big)}{\sigma_w^2 \mathrm{d}t}. \end{aligned} \tag{IA.441}$$

Denote by $\phi_i = 1 - \Theta_i \in (-1,1)$. Note that $x_{i,t}$ satisfies the recursive equation $x_{i,t} = \phi_i x_{i,t-1} + G_i \mathrm{d}w_t$. One computes $\Omega^{xx}_{ij,t} = \frac{\mathsf{E}(x_{i,t}x_{j,t})}{\sigma_w^2 \mathrm{d}t} = \frac{\mathsf{E}((\phi_i x_{i,t-1}+G_i\mathrm{d}w_t)(\phi_j x_{i,t-1}+G_j\mathrm{d}w_t))}{\sigma_w^2 \mathrm{d}t} = \phi_i\phi_j\Omega^{xx}_{ij,t-1} + G_iG_j$. Since $\phi_i\phi_j \in (-1,1)$, I apply Lemma A.1 in the Appendix in the paper to the recursive formula $\Omega^{xx}_{ij,t} = \phi_i\phi_j\Omega^{xx}_{ij,t-1} + G_iG_j$. Then, $\Omega^{xx}_{ij,t}$ is constant and equal to:

$$\Omega^{xx}_{ij} \;=\; \frac{G_iG_j}{1-\phi_i\phi_j}. \tag{IA.442}$$

The aggregate order flow at t is:

$$\mathrm{d}y_t \;=\; -\sum_{j=1}^{I}\Theta_j x_{j,t-1} + \bar{\gamma}\mathrm{d}w_t + \bar{\mu}\widetilde{\mathrm{d}w}_{t-1} + \mathrm{d}u_t, \quad \text{with} \quad \bar{\gamma} \;=\; \gamma^- + \sum_{j=1}^{I} G_j, \quad \gamma^- \;=\; N_F\gamma. \tag{IA.443}$$

I express $Z_{i,t}$ as a function of $X_{i,t-1}$:

$$Z_{i,t} \;=\; \frac{\mathsf{E}\big(x_{i,t-1}\mathrm{d}y_t\big)}{\sigma_w^2 \mathrm{d}t} \;=\; -\sum_{j=1}^{I}\Theta_j\Omega^{xx}_{ij,t-1} + \bar{\mu}X_{i,t-1} \;=\; -\sum_{j=1}^{I}\frac{(1-\phi_j)G_iG_j}{1-\phi_i\phi_j} + \bar{\mu}X_{i,t-1}. \tag{IA.444}$$

One has the recursive formula $X_{i,t} = \frac{\mathsf{E}(x_{i,t}\widetilde{\mathrm{d}w}_t)}{\sigma_w^2 \mathrm{d}t} = \frac{\mathsf{E}((\phi_i x_{i,t-1}+G_i\mathrm{d}w_t)(\mathrm{d}w_t-\rho\mathrm{d}y_t))}{\sigma_w^2 \mathrm{d}t} = -\phi_i\rho Z_{i,t} + G_i - G_i\rho\bar{\gamma} = -\phi_i\rho\bar{\mu}X_{i,t-1} + \phi_i\rho\sum_{j=1}^{I}\frac{(1-\phi_j)G_iG_j}{1-\phi_i\phi_j} + G_i - G_i\rho\bar{\gamma} = -\phi_i b X_{i,t-1} + G_i(1-a^-) - \sum_{j=1}^{I}\frac{\rho G_iG_j(1-\phi_i)}{1-\phi_i\phi_j}$. By assumption, $0 \leq b < 1$, hence $\phi_i b \in (-1,1)$. Lemma A.1 in the Appendix in the paper implies that $X_{i,t}$ is constant and equal to:

$$X_i \;=\; \frac{G_i(1-a^-) - \sum_{j=1}^{I}\frac{\rho G_iG_j(1-\phi_i)}{1-\phi_i\phi_j}}{1+\phi_i b}. \tag{IA.445}$$

From (IA.444), $Z_{i,t}$ is also constant and satisfies:

$$Z_i \;=\; \bar{\mu} G_i \frac{1-a^-}{1+\phi_i b} - \sum_{j=1}^{I} G_i G_j \frac{b + \frac{1-\phi_j}{1-\phi_i\phi_j}}{1+\phi_i b}. \tag{IA.446}$$

I now consider the optimization problem of IFT i, and describe a numerical procedure to solve it. In a symmetric equilibrium, IFT i assumes that the coefficients of the other IFTs are equal. To simplify notation, I eliminate the subscript i for IFT i and denote the coefficients of the other IFTs by G', Θ', and $\phi' = 1 - \Theta'$. IFT i then maximizes:

$$Z \;=\; G\frac{\bar{\mu}(1-a^-)}{1+\phi b} - (I-1)GG'\frac{b + \frac{1-\phi'}{1-\phi\phi'}}{1+\phi b} - G^2 \frac{b + \frac{1}{1+\phi}}{1+\phi b}. \tag{IA.447}$$

The first order condition for G implies that at the optimum:

$$G \;=\; \frac{\bar{\mu}(1-a^-) - (I-1)G'\big(b + \frac{1-\phi'}{1-\phi\phi'}\big)}{2\big(b+\frac{1}{1+\phi}\big)}, \qquad Z \;=\; \frac{\Big(\bar{\mu}(1-a^-) - (I-1)G'\big(b + \frac{1-\phi'}{1-\phi\phi'}\big)\Big)^2}{4\big(b+\frac{1}{1+\phi}\big)(1+\phi b)}. \tag{IA.448}$$

But in a symmetric equilibrium one has $G = G'$ and $\phi = \phi'$, therefore in equilibrium G should be related to ϕ via the function:

$$G(\phi) \;=\; \frac{\bar{\mu}(1-a^-)}{(I+1)\big(b+\frac{1}{1+\phi}\big)}. \tag{IA.449}$$

Now for each value of ϕ in a discrete grid in $(-1, 1)$ denote by $\phi' = \phi$, $G' = G(\phi)$ and consider the argument ϕ for which the function $Z(\phi, \phi', G')$ in (IA.448) attains its maximum. Denote this value by $\Phi(\phi)$. The equilibrium then corresponds to a fixed point of the function $\Phi(\cdot)$, which can be obtained numerically by minimizing $|\Phi(\phi) - \phi|$. Denote this value by ϕ^*. The above analysis then shows that $\Theta = 1 - \phi^*$ and $G = G(\phi^*)$ approximate the corresponding values in a symmetric equilibrium.

I compare two cases: (a) I IFTs, N_F FTs, and N_L lag traders, and (b) one IFT, $N_F + I - 1$ FTs, and N_L lag traders. Numerically, one sees that for the parameters considered the mean reversion coefficient Θ is larger in case (a) than in case (b), and the aggregate IFT coefficient on the current signal, which is IG, is larger in case (a) than the aggregate IFT coefficient in case (b), which is G. Thus, the intuition that works for the Cournot equilibrium is true for both coefficients Θ and G: when there is more than one IFTs, on aggregate they trade more aggressively on their signal (aggregate G is higher), and they mean revert more (Θ is higher).

5.6 One IFT and one IMT

Consider the benchmark model with $m = 2$ in Section 2 in this Internet Appendix, in which there are N_F FTs, N_M MTs, and N_S STs. These traders have, respectively, the following trading strategies:

$$\begin{aligned} \mathrm{d}x_t^F &= \gamma \mathrm{d}w_t + \mu \widetilde{\mathrm{d}w}_{t-1} + \nu \widetilde{\widetilde{\mathrm{d}w}}_{t-2}, \\ \mathrm{d}x_t^M &= \mu \widetilde{\mathrm{d}w}_{t-1} + \nu \widetilde{\widetilde{\mathrm{d}w}}_{t-2}, \\ \mathrm{d}x_t^S &= \nu' \widetilde{\widetilde{\mathrm{d}w}}_{t-2}, \end{aligned} \tag{IA.450}$$

where I consider the coefficients γ, μ, ν and ν' computed in equilibrium. The aggregate coefficients in the benchmark model are therefore:

$$\begin{aligned} \bar{\gamma} &= N_F \gamma, \qquad \bar{\mu} = N_L \mu, \qquad \bar{\nu} = N_D \nu + N_S(\nu' - \nu), \quad \text{with} \\ N_L &= N_F + N_S, \qquad N_D = N_L + N_S. \end{aligned} \tag{IA.451}$$

Suppose that I now define "pure FT" as a trader with strategy of the form $\mathrm{d}x_t = \gamma \mathrm{d}w_t$, and "pure MT" as a trader with strategy of the form $\mathrm{d}x_t = \mu \widetilde{\mathrm{d}w}_{t-1}$. Then, by inspecting the first order conditions for these traders' maximization problem, one can see that the optimal coefficients of these traders are equal to the equilibrium coefficients γ and μ, respectively.

Thus, to simplify analysis, in this subsection I analyze several departures from the benchmark model in which one pure FT or one pure MT becomes concerned about inventory and has utility with a quadratic inventory penalty, as in equation (31) in the paper. I call this trader IFT or IMT, respectively ("I" stands for "inventory"). Denote by x_t the inventory of the IFT, and by z_t the inventory of the IMT. Suppose the IFT and IMT have, respectively, trading strategies of the form:[24]

$$\mathrm{d}x_t = -\Theta x_{t-1} + G \mathrm{d}w_t, \qquad \mathrm{d}z_t = -\Omega z_{t-1} + H \widetilde{\mathrm{d}w}_t. \tag{IA.452}$$

I consider three departures from the benchmark model:

(a) One pure FT becomes IFT, with strategy $\mathrm{d}x_t = -\Theta x_{t-1} + G \mathrm{d}w_t$. The aggregate coefficients are: $\bar{\gamma} = (N_F - 1)\gamma + G$, $\bar{\mu} = N_L \mu$, $\bar{\nu} = N_D \nu + N_S(\nu' - \nu)$.

(b) One pure MT becomes IMT, with strategy $\mathrm{d}z_t = -\Omega z_{t-1} + H \widetilde{\mathrm{d}w}_t$. The aggregate

[24] In Subsection 5.3 in this Internet Appendix, I discuss trading strategies for the IFT that involve lagged coefficients, e.g., $\mathrm{d}x_t = -\Theta x_{t-1} + G \mathrm{d}w_t + M \widetilde{\mathrm{d}w}_{t-1}$. There one sees that the qualitative results remain unchanged. I thus conjecture that it is the case here as well.

coefficients are: $\bar{\gamma} = N_F \gamma$, $\bar{\mu} = (N_L - 1)\mu + H$, $\bar{\nu} = N_D \nu + N_S(\nu' - \nu)$.

(c) One IFT and one IMT with strategies as above. The aggregate coefficients are: $\bar{\gamma} = (N_F - 1)\gamma + G$, $\bar{\mu} = (N_L - 1)\mu + H$, $\bar{\nu} = N_D \nu + N_S(\nu' - \nu)$.

One IFT, no IMT

Consider one IFT with trading strategy of the form $\mathrm{d}x_t = -\Theta x_{t-1} + G\mathrm{d}w_t$. The aggregate order flow $\mathrm{d}y_t$ satisfies:

$$\mathrm{d}y_t \;=\; -\Theta x_{t-1} + \bar{\gamma}\mathrm{d}w_t + \bar{\mu}\widetilde{\mathrm{d}w}_{t-1} + \bar{\nu}\widetilde{\widetilde{\mathrm{d}w}}_{t-2} + \mathrm{d}u_t, \tag{IA.453}$$

where the only aggregate coefficient that depends on the IFT's strategy is:

$$\bar{\gamma} \;=\; \gamma^- + G. \tag{IA.454}$$

I introduce the following notation:

$$\begin{aligned} A_{11} &= \frac{\mathsf{E}\big[(\widetilde{\mathrm{d}w}_t)^2\big]}{\sigma_w^2 \mathrm{d}t}, \qquad A_{12} = \frac{\mathsf{E}\big[\widetilde{\mathrm{d}w}_t \widetilde{\widetilde{\mathrm{d}w}}_{t-1}\big]}{\sigma_w^2 \mathrm{d}t}, \qquad A_{22} = \frac{\mathsf{E}\big[(\widetilde{\widetilde{\mathrm{d}w}}_{t-1})^2\big]}{\sigma_w^2 \mathrm{d}t}, \\ X_1 &= \frac{\mathsf{E}\big(x_t \widetilde{\mathrm{d}w}_t\big)}{\sigma_w^2 \mathrm{d}t}, \quad X_2 = \frac{\mathsf{E}\big(x_t \widetilde{\widetilde{\mathrm{d}w}}_{t-1}\big)}{\sigma_w^2 \mathrm{d}t}, \quad X = \frac{\mathsf{E}\big(x_t^2\big)}{\sigma_w^2 \mathrm{d}t}, \quad Y = \frac{\mathsf{E}\big(x_{t-1} \mathrm{d}y_t\big)}{\sigma_w^2 \mathrm{d}t}, \end{aligned} \tag{IA.455}$$

where for simplicity I omit the subscript t on the left-hand side. The variables $\widetilde{\mathrm{d}w}_t = \mathrm{d}w_t - \rho \mathrm{d}y_t$, $\widetilde{\widetilde{\mathrm{d}w}}_{t-1} = \widetilde{\mathrm{d}w}_{t-1} - \rho' \mathrm{d}y_t$ and $x_t = x_{t-1} + \mathrm{d}x_t$ satisfy the following recursive equations:

$$\begin{aligned} \widetilde{\mathrm{d}w}_t &= -\rho \mathrm{d}u_t + (1-a)\mathrm{d}w_t - b\widetilde{\mathrm{d}w}_{t-1} - c\widetilde{\widetilde{\mathrm{d}w}}_{t-2} + \rho\Theta x_{t-1}, \\ \widetilde{\widetilde{\mathrm{d}w}}_{t-1} &= -\rho' \mathrm{d}u_t - a'\mathrm{d}w_t + (1-b')\widetilde{\mathrm{d}w}_{t-1} - c'\widetilde{\widetilde{\mathrm{d}w}}_{t-2} + \rho'\Theta x_{t-1}, \\ x_t &= G\mathrm{d}w_t + \phi x_{t-1}, \end{aligned} \tag{IA.456}$$

where:

$$a \;=\; \rho\bar{\gamma}, \qquad b \;=\; \rho\bar{\mu}, \qquad c \;=\; \rho\bar{\nu}, \qquad r \;=\; \frac{\rho'}{\rho}, \qquad \phi \;=\; 1 - \Theta. \tag{IA.457}$$

I now compute the IFT's expected utility. As in Proposition 6 in the paper, the IFT holds all his profits in cash, and his expected utility is the same as the expected profit. Also, from equation (35) in the paper, the IFT's normalized profit is $\tilde{\pi}_{\Theta>0} = \mathsf{E}\int_0^T x_{t-1}\mathrm{d}p_t = \lambda\,\mathsf{E}\int_0^T x_{t-1}\mathrm{d}y_t$. Using the notation in (IA.455), one gets:[25]

$$\tilde{\pi}_{\Theta>0} \;=\; \lambda \int_0^T Y_t \mathrm{d}t. \tag{IA.458}$$

Equations (IA.453) and (IA.455) imply that:

$$Y_t \;=\; \frac{\mathsf{E}\big(x_{t-1}\mathrm{d}y_t\big)}{\sigma_w^2 \mathrm{d}t} \;=\; \bar{\mu} X_{1,t-1} + \bar{\nu} X_{2,t-1} - \Theta X_{t-1}. \tag{IA.459}$$

To compute X_t and $X_{i,t}$, I analyze the recursive equations that these variables satisfy. The recursive formula for X_t is:

$$X_t \;=\; \frac{\mathsf{E}((x_t)^2)}{\sigma_w^2 \mathrm{d}t} \;=\; \frac{\mathsf{E}\big((G\mathrm{d}w_t + \phi x_{t-1})^2\big)}{\sigma_w^2 \mathrm{d}t} \;=\; G^2 + \phi^2 X_{t-1}. \tag{IA.460}$$

Using Lemma A.1 in the Appendix in the paper, it follows that X_t is constant and equal to:

$$X \;=\; \frac{G^2}{1-\phi^2}. \tag{IA.461}$$

Using (IA.456), one also computes:

$$\begin{aligned}
X_{1,t} \;&=\; \frac{\mathsf{E}\big[\big(G\mathrm{d}w_t + \phi x_{t-1}\big)\big((1-a)\mathrm{d}w_t - b\widetilde{\mathrm{d}w}_{t-1} - c\widetilde{\widetilde{\mathrm{d}w}}_{t-2} + \rho\Theta x_{t-1}\big)\big]}{\sigma_w^2 \mathrm{d}t} \\
&=\; G(1-a) - b\phi X_{1,t-1} - c\phi X_{2,t-1} + \rho\phi\Theta X_{t-1}, \\
X_{2,t} \;&=\; \frac{\mathsf{E}\big[\big(G\mathrm{d}w_t + \phi x_{t-1}\big)\big(-ra\mathrm{d}w_t + (1-b')\widetilde{\mathrm{d}w}_{t-1} - c'\widetilde{\widetilde{\mathrm{d}w}}_{t-2} + \rho'\Theta x_{t-1}\big)\big]}{\sigma_w^2 \mathrm{d}t} \\
&=\; -Gra + (1-b')\phi X_{1,t-1} - c'\phi X_{2,t-1} + \rho'\phi\Theta X_{t-1}.
\end{aligned} \tag{IA.462}$$

From (IA.461) one gets $\Theta X = \frac{G^2}{1+\phi}$. Substituting this in (IA.462), one obtains:

$$\begin{aligned}
X_{1,t} \;&=\; G(1-a) + \frac{G^2\rho\phi}{1+\phi} - b\phi X_{1,t-1} - c\phi X_{2,t-1}, \\
X_{2,t} \;&=\; -Gra + \frac{G^2\rho'\phi}{1+\phi} + (1-b')\phi X_{1,t-1} - c'\phi X_{2,t-1}.
\end{aligned} \tag{IA.463}$$

[25] Below I show that Y_t is constant, which implies that $\tilde{\pi}_{\Theta>0} = \lambda Y$. (Recall that $T = 1$.)

Denote:

$$A_X = \begin{bmatrix} -b\phi & -c\phi \\ \phi(1-b') & -c'\phi \end{bmatrix}, \qquad B_X = \begin{bmatrix} G(1-a) + \frac{G^2\rho\phi}{1+\phi} \\ -Gra + \frac{G^2\rho'\phi}{1+\phi} \end{bmatrix}. \tag{IA.464}$$

Then, if the eigenvalues of A_X are all in $(-1,1)$, Lemma A.1 in the Appendix in the paper implies that $X_{i,t}$ is constant for $i = 0,1,2$ and satisfies:

$$\begin{bmatrix} X_1 & X_2 \end{bmatrix}' = \left(I - A_X\right)^{-1} B_X \tag{IA.465}$$

Equation (IA.459) implies that Y_t is constant and equal to:

$$Y = \bar{\mu} X_1 + \bar{\nu} X_2 - \frac{G^2}{1+\phi}. \tag{IA.466}$$

The IFT's normalized expected profit is:

$$\tilde{\pi}_{\Theta>0} = \lambda Y. \tag{IA.467}$$

This finishes the proof.

One IMT, no IFT

Consider one IMT with trading strategy of the form $\mathrm{d}z_t = -\Omega z_{t-1} + H\widetilde{\mathrm{d}w}_{t-1}$. The aggregate order flow $\mathrm{d}y_t$ satisfies:

$$\mathrm{d}y_t = -\Omega z_{t-1} + \bar{\gamma}\mathrm{d}w_t + \bar{\mu}\widetilde{\mathrm{d}w}_{t-1} + \bar{\nu}\widetilde{\widetilde{\mathrm{d}w}}_{t-2} + \mathrm{d}u_t, \tag{IA.468}$$

where the only aggregate coefficient that depends on the IMT's strategy is:

$$\bar{\mu} = \mu^- + H. \tag{IA.469}$$

I introduce the following notation:

$$\begin{aligned} A_{11} &= \frac{\mathsf{E}\big[(\widetilde{\mathrm{d}w}_t)^2\big]}{\sigma_w^2\mathrm{d}t}, \qquad A_{12} = \frac{\mathsf{E}\big[\widetilde{\mathrm{d}w}_t\widetilde{\widetilde{\mathrm{d}w}}_{t-1}\big]}{\sigma_w^2\mathrm{d}t}, \qquad A_{22} = \frac{\mathsf{E}\big[(\widetilde{\widetilde{\mathrm{d}w}}_{t-1})^2\big]}{\sigma_w^2\mathrm{d}t}, \\ Z_1 &= \frac{\mathsf{E}\big(z_t\widetilde{\mathrm{d}w}_t\big)}{\sigma_w^2\mathrm{d}t}, \quad Z_2 = \frac{\mathsf{E}\big(z_t\widetilde{\widetilde{\mathrm{d}w}}_{t-1}\big)}{\sigma_w^2\mathrm{d}t}, \quad Z = \frac{\mathsf{E}\big(z_t^2\big)}{\sigma_w^2\mathrm{d}t}, \quad Y = \frac{\mathsf{E}\big(z_{t-1}\mathrm{d}y_t\big)}{\sigma_w^2\mathrm{d}t}, \end{aligned} \tag{IA.470}$$

where for simplicity I omit the subscript t on the left-hand side. The variables $\widetilde{\mathrm{d}w}_t = \mathrm{d}w_t - \rho \mathrm{d}y_t$, $\widetilde{\widetilde{\mathrm{d}w}}_{t-1} = \widetilde{\mathrm{d}w}_{t-1} - \rho' \mathrm{d}y_t$ and $z_t = z_{t-1} + \mathrm{d}z_t$ satisfy the following recursive equations:

$$\begin{aligned} \widetilde{\mathrm{d}w}_t &= -\rho \mathrm{d}u_t + (1-a)\mathrm{d}w_t - b\widetilde{\mathrm{d}w}_{t-1} - c\widetilde{\widetilde{\mathrm{d}w}}_{t-2} + \rho\Omega z_{t-1}, \\ \widetilde{\widetilde{\mathrm{d}w}}_{t-1} &= -\rho' \mathrm{d}u_t - a'\mathrm{d}w_t + (1-b')\widetilde{\mathrm{d}w}_{t-1} - c'\widetilde{\widetilde{\mathrm{d}w}}_{t-2} + \rho'\Omega z_{t-1}, \\ z_t &= H\widetilde{\mathrm{d}w}_{t-1} + \psi z_{t-1}, \end{aligned} \tag{IA.471}$$

where:

$$\begin{aligned} a &= \rho\bar{\gamma}, \qquad b = \rho\bar{\mu}, \qquad c = \rho\bar{\nu}, \qquad r = \frac{\rho'}{\rho}, \\ a' &= ra, \qquad b' = rb, \qquad c' = c', \qquad \psi = 1 - \Omega. \end{aligned} \tag{IA.472}$$

I now compute the IMT's expected utility. As in Proposition 6 in the paper, the IMT holds all his profits in cash, and therefore his expected utility is the same as the expected profit. Also, from equation (35) in the paper, the IMT's normalized profit is $\tilde{\pi}_{\Omega>0} = \mathsf{E}\int_0^T z_{t-1}\mathrm{d}p_t = \lambda\,\mathsf{E}\int_0^T z_{t-1}\mathrm{d}y_t$. Using equations (IA.468) and (IA.470), one obtains:

$$\tilde{\pi}_{\Omega>0} = \lambda \int_0^T Y_t \mathrm{d}t, \quad \text{with} \quad Y_t = \frac{\mathsf{E}\big(z_{t-1}\mathrm{d}y_t\big)}{\sigma_w^2 \mathrm{d}t} = \bar{\mu} Z_{1,t-1} + \bar{\nu} Z_{2,t-1} - \Omega Z_{t-1}. \tag{IA.473}$$

To compute $A_{ij,t}$, $Z_{i,t}$, and Z_t, I consider the recursive equations these variables satisfy. For simplicity, I omit the subscript t on the left-hand side, and the subscript $t-1$ on the right-hand side of these equations:

$$
\begin{aligned}
A_{11} &= \frac{\mathsf{E}\left[\left(-\rho \mathrm{d}u_t + (1-a)\mathrm{d}w_t - b\widetilde{\mathrm{d}w}_{t-1} - c\widetilde{\widetilde{\mathrm{d}w}}_{t-2} + \rho\Omega z_{t-1}\right)^2\right]}{\sigma_w^2 \mathrm{d}t} \\
&= \rho^2\tilde{\sigma}_u^2 + (1-a)^2 + b^2 A_{11} + 2bcA_{12} + c^2 A_{22} - 2b\rho\Omega Z_1 - 2c\rho\Omega Z_2 + \rho^2\Omega^2 Z, \\
A_{12} &= \frac{\mathsf{E}\left[\left(-\rho \mathrm{d}u_t + (1-a)\mathrm{d}w_t - b\widetilde{\mathrm{d}w}_{t-1} - c\widetilde{\widetilde{\mathrm{d}w}}_{t-2} + \rho\Omega z_{t-1}\right)\left(-\rho' \mathrm{d}u_t - a'\mathrm{d}w_t + (1-b')\widetilde{\mathrm{d}w}_{t-1} - c'\widetilde{\widetilde{\mathrm{d}w}}_{t-2} + \rho'\Omega z_{t-1}\right)\right]}{\sigma_w^2 \mathrm{d}t} \\
&= \rho\rho'\tilde{\sigma}_u^2 - a'(1-a) - b(1-b')A_{11} - c(1-2b')A_{12} + cc'A_{22} + (1-2b')\rho\Omega Z_1 - 2c'\rho\Omega Z_2 + \rho\rho'\Omega^2 Z, \\
A_{22} &= \frac{\mathsf{E}\left[\left(-\rho' \mathrm{d}u_t - a'\mathrm{d}w_t + (1-b')\widetilde{\mathrm{d}w}_{t-1} - c'\widetilde{\widetilde{\mathrm{d}w}}_{t-2} + \rho'\Omega z_{t-1}\right)^2\right]}{\sigma_w^2 \mathrm{d}t} \\
&= \rho'^2\tilde{\sigma}_u^2 + a'^2 + (1-b')^2 A_{11} - 2c'(1-b')A_{12} + c'^2 A_{22} + 2(1-b')\rho'\Omega Z_1 - 2c'\rho'\Omega Z_2 + \rho'^2\Omega^2 Z, \\
Z_1 &= \frac{\mathsf{E}\left[\left(H\widetilde{\mathrm{d}w}_{t-1} + \psi z_{t-1}\right)\left(-b\widetilde{\mathrm{d}w}_{t-1} - c\widetilde{\widetilde{\mathrm{d}w}}_{t-2} + \rho\Omega z_{t-1}\right)\right]}{\sigma_w^2 \mathrm{d}t} \\
&= -HbA_{11} - HcA_{12} + (H\rho\Omega - \psi b)Z_1 - \psi c Z_2 + \psi\rho\Omega Z, \\
Z_2 &= \frac{\mathsf{E}\left[\left(H\widetilde{\mathrm{d}w}_{t-1} + \psi z_{t-1}\right)\left((1-b')\widetilde{\mathrm{d}w}_{t-1} - c'\widetilde{\widetilde{\mathrm{d}w}}_{t-2} + \rho'\Omega z_{t-1}\right)\right]}{\sigma_w^2 \mathrm{d}t} \\
&= H(1-b')A_{11} - Hc'A_{12} + \left(H\rho'\Omega + \psi(1-b')\right)Z_1 - \psi c' Z_2 + \psi\rho'\Omega Z, \\
Z &= \frac{\mathsf{E}\left((H\widetilde{\mathrm{d}w}_{t-1} + \psi z_{t-1})^2\right)}{\sigma_w^2 \mathrm{d}t} = H^2 A_{11} + 2H\psi Z_1 + \psi^2 Z.
\end{aligned}
\tag{IA.474}
$$

Denote:

$$
\begin{aligned}
A_z &= \begin{bmatrix}
b^2 & 2bc & c^2 & -2b\rho\Omega & -2c\rho\Omega & \rho^2\Omega^2 \\
-b(1-b') & -c(1-2b') & cc' & (1-2b')\rho\Omega & -2c'\rho\Omega & \rho\rho'\Omega^2 \\
(1-b')^2 & -2c'(1-b') & c'^2 & 2(1-b')\rho'\Omega & -2c'\rho'\Omega & \rho'^2\Omega^2 \\
-Hb & -Hc & 0 & H\rho\Omega - b\psi & -c\psi & \rho\psi\Omega \\
H(1-b') & -Hc' & 0 & H\rho'\Omega + (1-b')\psi & -c'\psi & \rho'\psi\Omega \\
H^2 & 0 & 0 & 2H\psi & 0 & \psi^2
\end{bmatrix}, \\
B_z &= \begin{bmatrix} \rho^2\tilde{\sigma}_u^2 + (1-a)^2 & \rho\rho'\tilde{\sigma}_u^2 - a'(1-a) & \rho'^2\tilde{\sigma}_u^2 + a'^2 & 0 & 0 & 0 \end{bmatrix}'.
\end{aligned}
\tag{IA.475}
$$

Then, if the eigenvalues of A_z are all in $(-1, 1)$, Lemma A.1 in the Appendix in the paper implies that all the variables involved are constant and satisfy:

$$\begin{bmatrix} A_{11} & A_{12} & A_{22} & Z_1 & Z_2 & Z \end{bmatrix}' = \left(I - A_z\right)^{-1} B_z. \tag{IA.476}$$

Equation (IA.473) implies that Y_t is constant and equal to:

$$Y = \bar{\mu} Z_1 + \bar{\nu} Z_2 - \Omega Z. \tag{IA.477}$$

The IMT's normalized expected profit is:

$$\tilde{\pi}_{\Omega>0} = \lambda Y. \tag{IA.478}$$

This finishes the proof.

One IMT, one IFT

Consider one IFT with trading strategy $\mathrm{d}x_t = -\Theta x_{t-1} + G \widetilde{\mathrm{d}w}_{t-1}$ and one IMT with trading strategy $\mathrm{d}z_t = -\Omega z_{t-1} + H \widetilde{\mathrm{d}w}_{t-1}$. The aggregate order flow $\mathrm{d}y_t$ satisfies:

$$\mathrm{d}y_t = -\Theta x_{t-1} - \Omega z_{t-1} + \bar{\gamma} \mathrm{d}w_t + \bar{\mu} \widetilde{\mathrm{d}w}_{t-1} + \bar{\nu} \widetilde{\widetilde{\mathrm{d}w}}_{t-2} + \mathrm{d}u_t, \tag{IA.479}$$

where the aggregate coefficient that depend on the strategies of IFT and IMT are:

$$\bar{\gamma} = \gamma^- + G, \qquad \bar{\mu} = \mu^- + H. \tag{IA.480}$$

I introduce the following notation:

$$\begin{aligned}
A_{11} &= \frac{\mathsf{E}\big[(\widetilde{\mathrm{d}w}_t)^2\big]}{\sigma_w^2 \mathrm{d}t}, & A_{12} &= \frac{\mathsf{E}\big[\widetilde{\mathrm{d}w}_t \widetilde{\widetilde{\mathrm{d}w}}_{t-1}\big]}{\sigma_w^2 \mathrm{d}t}, & A_{22} &= \frac{\mathsf{E}\big[(\widetilde{\widetilde{\mathrm{d}w}}_{t-1})^2\big]}{\sigma_w^2 \mathrm{d}t}, \\
X_1 &= \frac{\mathsf{E}\big(x_t \widetilde{\mathrm{d}w}_t\big)}{\sigma_w^2 \mathrm{d}t}, & X_2 &= \frac{\mathsf{E}\big(x_t \widetilde{\widetilde{\mathrm{d}w}}_{t-1}\big)}{\sigma_w^2 \mathrm{d}t}, & X &= \frac{\mathsf{E}\big(x_t^2\big)}{\sigma_w^2 \mathrm{d}t}, \\
Z_1 &= \frac{\mathsf{E}\big(z_t \widetilde{\mathrm{d}w}_t\big)}{\sigma_w^2 \mathrm{d}t}, & Z_2 &= \frac{\mathsf{E}\big(z_t \widetilde{\widetilde{\mathrm{d}w}}_{t-1}\big)}{\sigma_w^2 \mathrm{d}t}, & Z &= \frac{\mathsf{E}\big(z_t^2\big)}{\sigma_w^2 \mathrm{d}t}, \\
W &= \frac{\mathsf{E}\big(x_{t-1} z_{t-1}\big)}{\sigma_w^2 \mathrm{d}t}, & Y^x &= \frac{\mathsf{E}\big(x_{t-1} \mathrm{d}y_t\big)}{\sigma_w^2 \mathrm{d}t}, & Y^z &= \frac{\mathsf{E}\big(z_{t-1} \mathrm{d}y_t\big)}{\sigma_w^2 \mathrm{d}t}.
\end{aligned} \tag{IA.481}$$

The variables $\widetilde{\mathrm{d}w}_t = \mathrm{d}w_t - \rho \mathrm{d}y_t$, $\widetilde{\widetilde{\mathrm{d}w}}_{t-1} = \widetilde{\mathrm{d}w}_{t-1} - \rho' \mathrm{d}y_t$ and $z_t = z_{t-1} + \mathrm{d}z_t$ satisfy the following recursive equations:

$$\begin{aligned} \widetilde{\mathrm{d}w}_t &= -\rho \mathrm{d}u_t + (1-a)\mathrm{d}w_t - b\widetilde{\mathrm{d}w}_{t-1} - c\widetilde{\widetilde{\mathrm{d}w}}_{t-2} + \rho\Theta x_{t-1} + \rho\Omega z_{t-1}, \\ \widetilde{\widetilde{\mathrm{d}w}}_{t-1} &= -\rho' \mathrm{d}u_t - a'\mathrm{d}w_t + (1-b')\widetilde{\mathrm{d}w}_{t-1} - c'\widetilde{\widetilde{\mathrm{d}w}}_{t-2} + \rho'\Theta x_{t-1} + \rho'\Omega z_{t-1}, \\ x_t &= G\mathrm{d}w_t + \phi x_{t-1}, \qquad z_t = H\widetilde{\mathrm{d}w}_{t-1} + \psi z_{t-1}, \end{aligned} \tag{IA.482}$$

where:

$$\begin{aligned} a &= \rho\bar{\gamma}, \qquad b = \rho\bar{\mu}, \qquad c = \rho\bar{\nu}, \qquad r = \frac{\rho'}{\rho}, \\ a' &= ra, \qquad b' = rb, \qquad c' = rc, \qquad \phi = 1-\Theta, \qquad \psi = 1-\Omega. \end{aligned} \tag{IA.483}$$

I now compute the expected utility of the IFT and IMT. As in Proposition 6 in the paper, the IFT and IMT hold all their profits in cash, and thus their expected utility is the same as the expected profit. Also, from equation (35) in the paper, the IFT's normalized profit is $\tilde{\pi}_{\Theta>0} = \mathsf{E}\int_0^T x_{t-1}\mathrm{d}p_t = \lambda\,\mathsf{E}\int_0^T x_{t-1}\mathrm{d}y_t$, and the IMT's normalized profit is $\tilde{\pi}_{\Omega>0} = \mathsf{E}\int_0^T z_{t-1}\mathrm{d}p_t = \lambda\,\mathsf{E}\int_0^T z_{t-1}\mathrm{d}y_t$. Using the notation in (IA.481), one obtains (recall that $T=1$):

$$\tilde{\pi}_{\Theta>0} = \lambda\int_0^T Y_t^x \mathrm{d}t, \qquad \tilde{\pi}_{\Omega>0} = \lambda\int_0^T Y_t^z \mathrm{d}t. \tag{IA.484}$$

Equations (IA.479) and (IA.481) imply that:

$$\begin{aligned} Y_t^x &= \frac{\mathsf{E}\big(x_{t-1}\mathrm{d}y_t\big)}{\sigma_w^2 \mathrm{d}t} = \bar{\mu}X_{1,t-1} + \bar{\nu}X_{2,t-1} - \Theta X_{t-1} - \Omega W_{t-1}, \\ Y_t^z &= \frac{\mathsf{E}\big(z_{t-1}\mathrm{d}y_t\big)}{\sigma_w^2 \mathrm{d}t} = \bar{\mu}Z_{1,t-1} + \bar{\nu}Z_{2,t-1} - \Theta W_{t-1} - \Omega Z_{t-1}. \end{aligned} \tag{IA.485}$$

To compute $A_{ij,t}$, $X_{i,t}$, $Z_{i,t}$, X_t, Z_t, and W_t I consider the recursive equations these variables satisfy. For simplicity, I omit the subscript t on the left-hand side, and the subscript $t-1$ on the right-hand side of these equations:

$$\begin{aligned}
A_{11} &= \rho^2\tilde{\sigma}_u^2 + (1-a)^2 + b^2 A_{11} + 2bcA_{12} + c^2 A_{22} - 2b\rho\Theta X_1 - 2c\rho\Theta X_2 + \rho^2\Theta^2 X \\
&\quad - 2b\rho\Omega Z_1 - 2c\rho\Omega Z_2 + \rho^2\Omega^2 Z + 2\rho^2\Theta\Omega W, \\
A_{12} &= \rho\rho'\tilde{\sigma}_u^2 - a'(1-a) - b(1-b')A_{11} - c(1-2b')A_{12} + cc'A_{22} + (1-2b')\rho\Theta X_1 \\
&\quad - 2c'\rho\Theta X_2 + \rho\rho'\Theta^2 X + (1-2b')\rho\Omega Z_1 - 2c'\rho\Omega Z_2 + \rho\rho'\Omega^2 Z + 2\rho\rho'\Theta\Omega W, \\
A_{22} &= \rho'^2\tilde{\sigma}_u^2 + a'^2 + (1-b')^2 A_{11} - 2c'(1-b')A_{12} + c'^2 A_{22} + 2(1-b')\rho'\Theta X_1 \\
&\quad - 2c'\rho'\Theta X_2 + \rho'^2\Theta^2 X + 2(1-b')\rho'\Omega Z_1 - 2c'\rho'\Omega Z_2 + \rho'^2\Omega^2 Z + 2\rho'^2\Theta\Omega W, \\
X_1 &= G(1-a) - b\phi X_1 - c\phi X_2 + \rho\phi\Theta X + \rho\phi\Omega W, \\
X_2 &= -Ga' + (1-b')\phi X_1 - c'\phi X_2 + \rho'\phi\Theta X + \rho'\phi\Omega W, \\
X &= G^2 + \phi^2 X, \\
Z_1 &= -HbA_{11} - HcA_{12} + H\rho\Theta X_1 + (H\rho\Omega - b\psi)Z_1 - c\psi Z_2 + \rho\psi\Omega Z + \rho\psi\Theta W, \\
Z_2 &= H(1-b')A_{11} - Hc'A_{12} + H\rho'\Theta X_1 + \big(H\rho'\Omega + (1-b')\psi\big)Z_1 \\
&\quad - c'\psi Z_2 + \rho'\psi\Omega Z + \rho'\psi\Theta W, \\
Z &= H^2 A_{11} + 2H\psi Z_1 + \psi^2 Z, \\
W &= H\phi X_1 + \phi\psi W.
\end{aligned} \tag{IA.486}$$

To simplify formulas, I use the recursive formulas for X_t together with the restriction $\phi \in (-1, 1)$. Applying Lemma A.1 in the Appendix in the paper, it follows that X_t is constant and equal to:

$$X = \frac{G^2}{1-\phi^2}. \tag{IA.487}$$

From this, I obtain the formula $\Theta X = \frac{G^2}{1+\phi}$, and substitute it in (IA.486). Denote by:

$$A = \begin{bmatrix} b^2 & 2bc & c^2 & -2b\rho\Theta & -2c\rho\Theta & -2b\rho\Omega & -2c\rho\Omega & \rho^2\Omega^2 & 2\rho^2\Theta\Omega \\ -b(1-b') & -c(1-2b') & cc' & (1-2b')\rho\Theta & -2c'\rho\Theta & (1-2b')\rho\Omega & -2c'\rho\Omega & \rho\rho'\Omega^2 & 2\rho\rho'\Theta\Omega \\ (1-b')^2 & -2c'(1-b') & c'^2 & 2(1-b')\rho'\Theta & -2c'\rho'\Theta & 2(1-b')\rho'\Omega & -2c'\rho'\Omega & \rho'^2\Omega^2 & 2\rho'^2\Theta\Omega \\ 0 & 0 & 0 & -b\phi & -c\phi & 0 & 0 & 0 & \rho\phi\Omega \\ 0 & 0 & 0 & (1-b')\phi & -c'\phi & 0 & 0 & 0 & \rho'\phi\Omega \\ -Hb & -Hc & 0 & H\rho\Theta & 0 & H\rho\Theta - b\psi & -c\psi & \rho\psi\Omega & \rho\psi\Theta \\ H(1-b') & -Hc' & 0 & H\rho'\Theta & 0 & H\rho'\Omega + (1-b')\psi & -c'\psi & \rho'\psi\Omega & \rho'\psi\Theta \\ H^2 & 0 & 0 & 0 & 0 & 2H\psi & 0 & \psi^2 & 0 \\ 0 & 0 & 0 & H\phi & 0 & 0 & 0 & 0 & \phi\psi \end{bmatrix},$$

$$B = \begin{bmatrix} \rho^2\tilde{\sigma}_u^2 + (1-a)^2 + \frac{\rho^2\Theta G^2}{1+\phi} \\ \rho\rho'\tilde{\sigma}_u^2 - a'(1-a) + \frac{\rho\rho'\Theta G^2}{1+\phi} \\ \rho'^2\tilde{\sigma}_u^2 + a'^2 + \frac{\rho'^2\Theta G^2}{1+\phi} \\ G(1-a) + \frac{\rho\phi G^2}{1+\phi} \\ -Ga' + \frac{\rho'\phi G^2}{1+\phi} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{IA.488}$$

Then, if the eigenvalues of A are all in $(-1, 1)$, Lemma A.1 in the Appendix in the paper implies that all the variables involved are constant and satisfy:

$$\begin{bmatrix} A_{11} & A_{12} & A_{22} & X_1 & X_2 & Z_1 & Z_2 & Z & W \end{bmatrix}' = \big(I - A\big)^{-1} B. \tag{IA.489}$$

Equation (IA.485) implies that Y_t^x and Y_t^z are constant and equal, respectively, to:

$$\begin{aligned} Y^x &= \bar{\mu}X_1 + \bar{\nu}X_2 - \Theta X - \Omega W, \\ Y^z &= \bar{\mu}Z_1 + \bar{\nu}Z_2 - \Omega Z - \Theta W. \end{aligned} \tag{IA.490}$$

The normalized expected profits of the IFT and IMT are equal, respectively, to:

$$\tilde{\pi}_{\Theta>0} = \lambda Y^x, \qquad \tilde{\pi}_{\Omega>0} = \lambda Y^z. \tag{IA.491}$$

This finishes the proof.

Numerical results

I consider the three cases examined thus far: (a) one IFT and no IMT, (b) one IMT and no IFT, and (c) one IFT and one IMT. I solve numerically for the optimum in all three cases.[26] I am interested in the coefficients of the IFT's strategy $\mathrm{d}x_t = G\mathrm{d}w_t - \Theta x_{t-1}$ and the IMT's strategy $\mathrm{d}z_t = H\mathrm{d}w_t - \Omega z_{t-1}$.

First, I compare the optimal coefficients of the IFT in cases (a) and (c). For all parameter values considered, the coefficient G becomes lower when the IMT is present (in case (c)), while the coefficient Θ remains in both cases equal to 0_+.[27] Intuitively, when one goes from case (a) to case (c) a MT is replaced by an IMT who therefore trades less intensely on his signal (see Theorem 2 in the paper, which shows that the optimal G is smaller than the FTs' coefficient γ). As there is now less trading coming from slower traders, the benefit of trading on his signal decreases, and as a result the IFT's optimal G decreases.

I now compare the optimal coefficients of the IMT in cases (b) and (c). For all parameter values considered, one sees that the coefficient H is higher when the IFT is present (in case (c)), while Ω is lower when the IFT is present. The intuition in both cases comes from understanding the effect of the IFT. Recall that at $t-1$ the IFT trades on $\mathrm{d}w_{t-1}$, and at time t he reduces his inventory by Θx_{t-1}, thus providing liquidity to the slower traders (including to the IMT) who trade on $\widetilde{\mathrm{d}w}_{t-1}$. But at $t-1$ the IMT also reduces his inventory by Ωz_{t-1}. Therefore, in case (c) the IMT faces at $t-1$ a lower effective price impact for the component $H\widetilde{\mathrm{d}w}_{t-1}$ compared to the component $-\Omega z_{t-1}$. This implies that compared to case (b), in case (c) the IMT trades with a higher coefficient H but with a lower coefficient Ω.

[26] To avoid the matrix A being singular, I pick a relatively small total of traders, e.g., $N_F + N_M + N_S < 15$.

[27] Recall that $\Theta = 0_+$ is the lowest value of the mean reversion coefficient Θ in the quick regime. Note that $\Theta = 0_+$ should not to be confused $\Theta = 0$ (the neutral regime), because there is a discontinuity at zero.

6 Smooth inventory management

In this section, I examine in more detail the "smooth regime" from the model with inventory management from Section 4 in the paper, in which the IFT has a trading strategy of the form: $\mathrm{d}x_t = -\theta\, x_t \mathrm{d}t + G\,\mathrm{d}w_t$, with $\theta \in [0, \infty)$. In Subsection 6.1, I prove that the IFT's expected utility changes continuously from the smooth regime to the quick regime (see Section 5 in this Internet Appendix), and then show that the smooth regime is never optimal. The proofs are given in Subsection 6.2. In Subsection 6.3, I show that the same results hold if the IFT has a more general strategy, of the form: $\mathrm{d}x_t = -\theta\, x_t \mathrm{d}t + G\,\mathrm{d}w_t + M\,\widetilde{\mathrm{d}w}_{t-1}$.

6.1 Equilibrium with smooth inventory management

In this subsection, I solve for a partial equilibrium of the model with inventory management from Section 4 in the paper, in which the IFT chooses the smooth regime, i.e., has a trading strategy of the form:

$$\mathrm{d}x_t \;=\; -\theta\, x_t \mathrm{d}t + G\,\mathrm{d}w_t, \quad \text{with} \quad \theta \;\in\; [0, \infty). \tag{IA.492}$$

Recall that in the quick regime, the IFT has a trading strategy of the form:

$$\mathrm{d}x_t \;=\; -\Theta x_{t-1} + G\,\mathrm{d}w_t, \quad \text{with} \quad \Theta \;\in\; [0, 2). \tag{IA.493}$$

I call strategies of type (IA.492) "smooth strategies," and strategies of type (IA.493) "quick strategies." A smooth strategy can be considered a particular case of a quick strategy if the coefficient Θ is infinitesimal: $\Theta = \theta \mathrm{d}t$.[28]

A result that I prove below is that the IFT's expected utility changes continuously from the smooth regime to the quick regime. The connection is made by the right limit of $\theta \in [0, \infty)$, which coincides with the left limit of $\Theta \in (0, 2)$, which I write as $\Theta = 0_+$. Therefore, I make the equivalence:

$$\theta \;=\; +\infty \qquad \Longleftrightarrow \qquad \Theta \;=\; 0_+. \tag{IA.494}$$

The agents in the model are:

- One IFT, who chooses a smooth strategy of the form (IA.492) with $\theta \in [0, \infty)$ and

[28] In calculus, $\mathrm{d}t$ is considered positive but smaller than any positive real number (and with $\mathrm{d}t^2 = 0$).

$G \in \mathbb{R}$. The IFT maximizes the expected utility U given by (31):

$$U = \mathsf{E}\left(\int_0^T (v_T - p_t)\mathrm{d}x_t\right) - C_I\,\mathsf{E}\left(\int_0^T x_t^2\mathrm{d}t\right), \qquad \text{(IA.495)}$$

where $T = 1$, and $C_I > 0$ is the IFT's inventory aversion coefficient;

- N_F FTs, with trading strategy $\mathrm{d}x_t^F = \gamma\mathrm{d}w_t$, with $\gamma \geq 0$;
- N_L STs, with trading strategy $\mathrm{d}x_t^S = \mu(\mathrm{d}w_{t-1} - \rho\mathrm{d}y_{t-1})$, with $\mu \geq 0$;
- A dealer who sets a linear pricing rule $\mathrm{d}p_t = \lambda\mathrm{d}y_t$;
- Exogenous noise traders, whose order flow is $\mathrm{d}u_t$.

I introduce the following coefficients:

$$\begin{aligned} R &= \frac{\lambda}{\rho}, \quad \gamma^- = N_F\gamma, \quad \bar{\gamma} = \gamma^- + G, \quad \bar{\mu} = N_L\mu, \\ a^- &= \rho\gamma^-, \quad a = \rho\bar{\gamma}, \quad b = \rho\bar{\mu}. \end{aligned} \qquad \text{(IA.496)}$$

The coefficients satisfy $\gamma^- \geq 0$, $\bar{\mu} \geq 0$, $\rho > 0$, $\lambda > 0$.

As usual, tilde notation denotes normalization by σ_w or σ_w^2. For instance, the normalized expected utility of the IFT is denoted by:

$$\tilde{U} = \frac{U}{\sigma_w^2}. \qquad \text{(IA.497)}$$

IFT's expected utility

For any smooth strategy of the IFT (not necessarily optimal), I compute the IFT's expected utility, while taking the behavior of the others as given. First, define the following function of $\theta \in (0, \infty)$:

$$F_\theta = \int_0^1 \left(1 - \mathrm{e}^{-\theta t}\right)\mathrm{d}t = 1 - \frac{1 - \mathrm{e}^{-\theta}}{\theta}. \qquad \text{(IA.498)}$$

This function is strictly increasing in θ and has well defined limits at the interval endpoints: $\lim_{\theta\to 0} F_\theta = 0$ and $\lim_{\theta\to\infty} F_\theta = 1$. Therefore, by abuse of notation I define F_θ for the whole interval $\theta \in [0, \infty]$.

Proposition IA.10. *In the model described above, suppose $b = \rho\bar{\mu} < 1$. Then, the normalized expected utility of the IFT with a trading strategy as in (IA.492) is:*

$$\tilde{U}_{\theta} \;=\; G(1-\lambda\bar{\gamma})\big(1-F_\theta\big) + \bar{\mu}\frac{\lambda G(1-\rho\bar{\gamma})}{1+\rho\bar{\mu}}F_\theta - \frac{\lambda G^2}{2(1+\rho\bar{\mu})}F_{2\theta} - \frac{C_I G^2}{2\theta}F_{2\theta}. \qquad \text{(IA.499)}$$

Proposition IA.10 shows that the normalized maximum utility of the IFT in the smooth regime ($\tilde{U}_{\theta}$) varies continuously from $\theta = 0$ to $\theta = \infty$. The next result shows that:

- The limit when $\theta \to 0$ of $\tilde{U}_{\theta}$ coincides with $\tilde{U}_{\Theta=0}$, the normalized maximum utility of the IFT in the neutral regime ($\Theta = 0$).
- The limit when $\theta \to \infty$ of $\tilde{U}_{\theta}$ coincides with $\tilde{U}_{\Theta=0_+}$, the left limit when $\Theta \to 0$ of the normalized maximum utility of the IFT in the quick regime ($\Theta > 0$).

Corollary IA.3. *The normalized expected utility of the IFT in the smooth regime various continuously from $\theta = 0$ to $\theta = \infty$. It has the following limits at the endpoints:*

$$\begin{aligned} \lim_{\theta\to 0}\tilde{U}_{\theta} \;&=\; \tilde{U}_{\theta=0} \;=\; \tilde{U}_{\Theta=0} \;=\; G(1-\lambda\bar{\gamma}), \\ \lim_{\theta\to\infty}\tilde{U}_{\theta} \;&=\; \tilde{U}_{\Theta=0_+} \;=\; \bar{\mu}\frac{\lambda G(1-\rho\bar{\gamma})}{1+\rho\bar{\mu}} - \frac{\lambda G^2}{2(1+\rho\bar{\mu})}. \end{aligned} \qquad \text{(IA.500)}$$

Also, when $\theta \to \infty$, the IFT's (normalized) inventory costs converge to zero:

$$\lim_{\theta\to\infty}\frac{1}{\sigma_w^2}\,C_I\,\mathsf{E}\left(\int_0^T x_t^2 \mathrm{d}t\right) \;=\; 0. \qquad \text{(IA.501)}$$

Optimal smooth inventory management

I now take a partial equilibrium approach, and solve for the optimal behavior of the IFT in the smooth regime, while taking the behavior of other agents as given. I show that this problem translates into an optimization problem in one variable, which can be solved numerically. The main conclusion is that the optimal trading strategy of the IFT in the smooth regime occurs either at $\theta = 0$ or at $\theta = \infty$. This result is obtained in two steps.

In the first step, I fix $\theta \in [0,\infty]$ and compute the maximum expected utility of the IFT when the coefficient G varies. Denote this utility by $U_{\theta}^{\max}$. In the second step, I numerically search for the $\theta \in [0,\infty]$ that maximizes $U_{\theta}^{\max}$, and find that the optimum θ is either 0 or ∞.

Proposition IA.11 provides a formula for $U_{\theta}^{\max}$.

Proposition IA.11. *For a fixed $\theta \in [0, \infty)$ denote by $U_{\theta}^{\max} = \tilde{U}_{\theta}^{\max}(C_I)$ the maximum normalized expected utility of the IFT in the smooth regime when G varies. One computes:*

$$\tilde{U}_{\theta}^{\max} \;=\; \frac{1}{2}\,\frac{\Big(\big(1 - Ra^-\big) - F_\theta\Big(1 - R\frac{a^-+b}{1+b}\Big)\Big)^2}{2\lambda\left(1 - \frac{F_\theta}{1+b}\right) + F_{2\theta}\left(\frac{\lambda}{1+b} + \frac{C_I}{\theta}\right)}, \tag{IA.502}$$

where F_θ is defined as in equation (IA.498).

Corollary IA.4 provides formulas for $\tilde{U}_{\theta}^{\max}$ when $\theta = 0$ and $\theta = \infty$.

Corollary IA.4. *One has the following formulas:*

$$\tilde{U}_{0}^{\max} \;=\; \frac{1}{2}\,\frac{\big(1 - Ra^-\big)^2}{2\lambda + C_I}, \qquad \tilde{U}_{\infty}^{\max} \;=\; \frac{\big(Rb(1 - a^-)\big)^2}{4\lambda(1 + b)\big(b + \frac{1}{2}\big)}. \tag{IA.503}$$

The value of C_I that makes $\tilde{U}_{0}^{\max} = \tilde{U}_{\infty}^{\max}$ is:

$$C_I^s \;=\; 2\lambda\left(\frac{(1 - Ra^-)^2(1 + b)\big(b + \frac{1}{2}\big)}{R^2 b^2 (1 - a^-)^2} - 1\right). \tag{IA.504}$$

Moreover, when $C_I = 0$ and $\theta = 0$, the maximum expected of the IFT is:

$$\tilde{U}_0 \;=\; \tilde{U}_{0, C_I = 0}^{\max} \;=\; \frac{\big(1 - Ra^-\big)^2}{4\lambda}. \tag{IA.505}$$

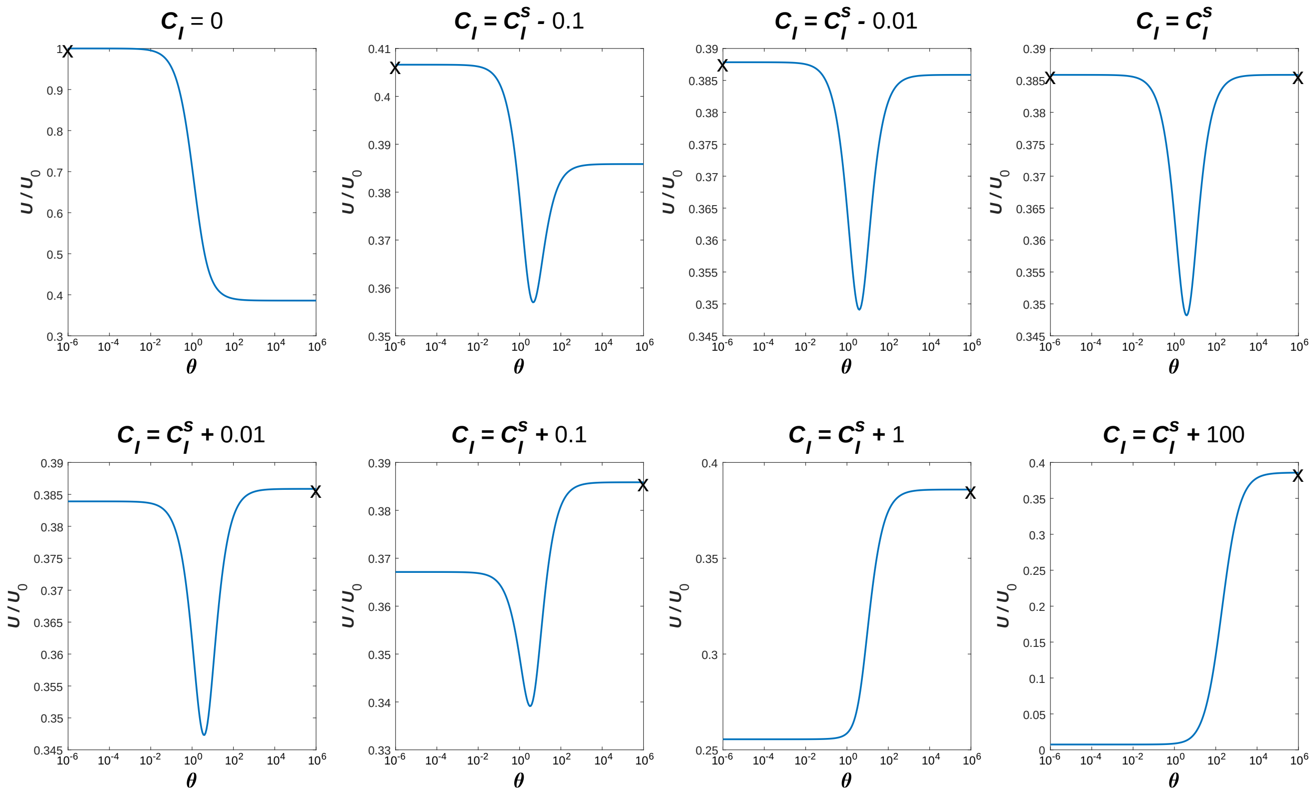
In the second step, I show numerically that the maximum $U_{\theta}^{\max}$ occurs either at $\theta = 0$ or at $\theta = \infty$, but not at an interior point in $(0, \infty)$. This results holds for all the parameter values considered.

Result IA.1. *Suppose the model coefficients arise from the inventory management equilibrium of Theorem 3. Then, the expression $U_{\theta}^{\max}$ in (IA.502) never attains its maximum value at an interior point $\theta \in (0, \infty)$.*

To understand this numerical result, I consider a particular example, with $N_F = 5$ fast traders, and $N_S = 5$ slow traders. In this case, equation (IA.504) implies that the value of the cutoff is $C_I^s = 1.2038$. This means that when $C_I = C_I^s$, the expected utility difference at the two endpoints $(U_0 - U_\infty)$ switches sign. Figure IA.4 shows the IFT's maximum expected utility as a function of θ, given several values of C_I around the cutoff. The maximum expected utility U is computed according to equation (IA.502), and reported in the graph as a ratio $\frac{U}{U_0}$, where U_0 is the expected utility in (IA.505)

Figure IA.4: Optimal IFT smooth trading strategies

This figure shows $\tilde{U}_\theta^{\max}(C_I)$, the maximum expected utility of the IFT in the smooth regime as a function of θ, for various values of the inventory aversion C_I. Each graph corresponds to an inventory aversion coefficient C_I, which in certain cases is reported relative to the cutoff value $C_I^s = 1.2038$. In each graph, the expected utility U is normalized by the value U_0 that coresponds to $\theta = 0$ and $C_I = 0$. In each graph, the maximum utility is marked with an "x". The parameter values are $N_F = 5$, $N_L = 5$, $\sigma_w = 1$, and $\sigma_u = 1$.



that corresponds to $\theta = 0$ and $C_I = 0$ (no inventory management, and zero inventory costs).

As shown in Figure IA.4, there are two sharply distinctly regimes, depending on whether the inventory aversion coefficient C_I is above the threshold value C_I^s:

- If $C_I < C_I^s$, the IFT optimally chooses $\theta = 0$;
- If $C_I > C_I^s$, the IFT optimally chooses $\theta = \infty$.

Thus, the smooth regime is never optimal, and I can just study what happens at the extremities, $\theta = 0$ and $\theta = \infty$.

6.2 Proofs

Proof of Proposition IA.10. Denote:

$$\begin{aligned} \Omega_t^{xx} &= \frac{\mathsf{E}\big(x_t^2\big)}{\sigma_w^2}, \qquad \Omega_t^{xe} = \frac{\mathsf{E}\big(x_t(w_t - p_t)\big)}{\sigma_w^2}, \qquad E_t = \frac{\mathsf{E}\big((w_t - p_t)\widetilde{\mathrm{d}w_t}\big)}{\sigma_w^2\,\mathrm{d}t}, \\ \Omega_t^{xw} &= \frac{\mathsf{E}\big(x_t w_t)\big)}{\sigma_w^2}, \qquad \Omega_t^{xp} = \frac{\mathsf{E}\big(x_t p_t\big)}{\sigma_w^2}. \end{aligned} \tag{IA.506}$$

Note first that:

$$\Omega_t^{xp} \;=\; \Omega_t^{xw} - \Omega_t^{xe}. \tag{IA.507}$$

From the definition of Ω^{xw}, one obtains:

$$\begin{aligned} \frac{\mathrm{d}\Omega_t^{xw}}{\mathrm{d}t} &= \frac{1}{\sigma_w^2 \mathrm{d}t}\,\mathsf{E}\big(\mathrm{d}x_t w_{t-1} + x_{t-1}\mathrm{d}w_t + \mathrm{d}x_t \mathrm{d}w_t\big) \\ &= -\theta\Omega_{t-1}^{xw} + G. \end{aligned} \tag{IA.508}$$

As there is no initial inventory, $\Omega_0^{xw} = 0$. Thus, the solution for the differential equation (IA.508) is:

$$\Omega_t^{xw} \;=\; G\,\frac{1 - \mathrm{e}^{-\theta t}}{\theta}. \tag{IA.509}$$

In order to compute Ω_t^{xx} and Ω_t^{xe}, one needs to define additional covariances.[29] Denote:

$$X_t = \frac{\mathsf{E}\big(x_t\widetilde{\mathrm{d}w_t}\big)}{\sigma_w^2\,\mathrm{d}t}, \quad W_t = \frac{\mathsf{E}\big(w_t\widetilde{\mathrm{d}w_t}\big)}{\sigma_w^2\,\mathrm{d}t}, \quad P_t = \frac{\mathsf{E}\big(p_t\widetilde{\mathrm{d}w_t}\big)}{\sigma_w^2\,\mathrm{d}t}, \quad E_t = \frac{\mathsf{E}\big((w_t - p_t)\widetilde{\mathrm{d}w_t}\big)}{\sigma_w^2\,\mathrm{d}t}. \tag{IA.510}$$

To compute these covariances, I derive recursive formulas for them. Note that the aggregate order flow at t is of the form:

$$\mathrm{d}y_t \;=\; -\theta x_{t-1}\mathrm{d}t + \bar{\gamma}\mathrm{d}w_t + \bar{\mu}\widetilde{\mathrm{d}w_{t-1}} + \mathrm{d}u_t. \tag{IA.511}$$

To simplify notation, denote:

$$a \;=\; \rho\bar{\gamma}, \qquad b \;=\; \rho\bar{\mu}, \qquad A = 1 - a, \qquad B \;=\; \frac{1-a}{1+b}. \tag{IA.512}$$

[29]The inventory management term is of the order of $\mathrm{d}t$, and thus it does not affect any instantaneous covariances with infinitesimal terms, such as $\frac{\mathsf{E}\left((\widetilde{\mathrm{d}w_t})^2\right)}{\sigma_w^2 \mathrm{d}t} = A$. However, covariances with aggregate measures such as x_t, w_t, and p_t are affected by the slow accumulation of the $\mathrm{d}t$ term. For instance, as the formula (IA.517) for W_t shows, the equation $\frac{\mathsf{E}\left(w_t\widetilde{\mathrm{d}w_t}\right)}{\sigma_w^2 \mathrm{d}t} = B$ is no longer true here.

Then, write $\widetilde{\mathrm{d}w}_t = \mathrm{d}w_t - \rho \mathrm{d}y_t$ as follows:

$$\widetilde{\mathrm{d}w}_t \;=\; \rho\theta x_{t-1}\mathrm{d}t + \mathrm{d}w_t(1-a) - b\widetilde{\mathrm{d}w}_{t-1} - \rho \mathrm{d}u_t. \tag{IA.513}$$

The recursive formula for X_t is:

$$\begin{aligned} X_t \;&=\; \frac{1}{\sigma_w^2 \mathrm{d}t}\,\mathsf{E}\Big(\big(x_{t-1}+\mathrm{d}x_t\big)\big(\rho\theta x_{t-1}\mathrm{d}t + \mathrm{d}w_t(1-a) - b\widetilde{\mathrm{d}w}_{t-1} - \rho \mathrm{d}u_t\big)\Big) \\ &=\; \rho\theta\frac{\mathsf{E}\big((x_{t-1})^2\big)}{\sigma_w^2\mathrm{d}t} - b\frac{\mathsf{E}\big(x_{t-1}\widetilde{\mathrm{d}w}_{t-1}\big)}{\sigma_w^2\mathrm{d}t} + (1-a)\frac{\mathsf{E}\big(\mathrm{d}x_t\mathrm{d}w_t\big)}{\sigma_w^2\mathrm{d}t} - b\frac{\mathsf{E}\big(\mathrm{d}x_t\widetilde{\mathrm{d}w}_{t-1}\big)}{\sigma_w^2\mathrm{d}t} \\ &=\; \rho\theta\Omega_{t-1}^{xx} - bX_{t-1} + (1-a)G. \end{aligned} \tag{IA.514}$$

Thus, $X_t + bX_{t-1} = \rho\theta\Omega_{t-1}^{xx} + (1-a)G$. Since by assumption $b < 1$ (and $b \geq 0$), I use Lemma A.1 in the Appendix in the paper to obtain the following formula:[30]

$$X_t \;=\; \frac{\rho\theta\Omega_t^{xx} + (1-a)G}{1+b} \;=\; \frac{\rho\theta\Omega_t^{xx}}{1+b} + BG. \tag{IA.515}$$

Similarly, the recursive formula for W_t is:

$$\begin{aligned} W_t \;&=\; \frac{1}{\sigma_w^2 \mathrm{d}t}\,\mathsf{E}\Big(\big(w_{t-1}+\mathrm{d}w_t\big)\big(\rho\theta x_{t-1}\mathrm{d}t + \mathrm{d}w_t(1-a) - b\widetilde{\mathrm{d}w}_{t-1} - \rho \mathrm{d}u_t\big)\Big) \\ &=\; \rho\theta\frac{\mathsf{E}\big(w_{t-1}x_{t-1}\big)}{\sigma_w^2\mathrm{d}t} - b\frac{\mathsf{E}\big(w_{t-1}\widetilde{\mathrm{d}w}_{t-1}\big)}{\sigma_w^2\mathrm{d}t} + (1-a)\frac{\mathsf{E}\big(\mathrm{d}w_t\mathrm{d}w_t\big)}{\sigma_w^2\mathrm{d}t} \\ &=\; \rho\theta\Omega_{t-1}^{xw} - bW_{t-1} + (1-a). \end{aligned} \tag{IA.516}$$

Thus, $W_t + bW_{t-1} = \rho\theta\Omega_{t-1}^{xw} + (1-a)$. As above, I use Lemma A.1 in the Appendix in the paper to get:

$$W_t \;=\; \frac{\rho\theta\Omega_t^{xw} + (1-a)}{1+b} \;=\; \frac{\rho\theta\Omega_t^{xw}}{1+b} + B. \tag{IA.517}$$

The recursive formula for P_t is:

$$\begin{aligned} P_t \;&=\; \frac{1}{\sigma_w^2 \mathrm{d}t}\,\mathsf{E}\Big(\big(p_{t-1}+\lambda\mathrm{d}y_t\big)\big(\rho\theta x_{t-1}\mathrm{d}t + \mathrm{d}w_t(1-a) - b\widetilde{\mathrm{d}w}_{t-1} - \rho \mathrm{d}u_t\big)\Big) \\ &=\; \rho\theta\frac{\mathsf{E}\big(p_{t-1}x_{t-1}\big)}{\sigma_w^2\mathrm{d}t} - b\frac{\mathsf{E}\big(p_{t-1}\widetilde{\mathrm{d}w}_{t-1}\big)}{\sigma_w^2\mathrm{d}t} + (1-a)\lambda\frac{\mathsf{E}\big(\mathrm{d}y_t\mathrm{d}w_t\big)}{\sigma_w^2\mathrm{d}t} - b\lambda\frac{\mathsf{E}\big(\mathrm{d}y_t\widetilde{\mathrm{d}w}_{t-1}\big)}{\sigma_w^2\mathrm{d}t} - \lambda\rho\tilde{\sigma}_u^2 \\ &=\; \rho\theta\Omega_{t-1}^{xp} - bP_{t-1} + (1-a)\lambda\bar{\gamma} - b\lambda\bar{\mu}A - \lambda\rho\tilde{\sigma}_u^2. \end{aligned} \tag{IA.518}$$

[30]The difference between Ω_t^{xx} and Ω_{t-1}^{xx} is infinitesimal, hence it can be ignored. In other words, one can use Lemma A.1 either for α_t or for α_{t-1}, and obtain the same result.

Define:

$$M = (1-a)\lambda\bar{\gamma} - b\lambda\bar{\mu}A - \lambda\rho\tilde{\sigma}_u^2. \tag{IA.519}$$

One can check that $M = 0$ when a and b have the equilibrium values from Theorem 1. This simply reflects the fact that $\widetilde{\mathrm{d}w}_t$ is orthogonal to p_t in the absence of inventory management, i.e., $P_t = 0$ when $\theta = 0$.[31] As in the case of X_t, I use Lemma A.1 in the Appendix in the paper to obtain:

$$P_t = \frac{\rho\theta\Omega_t^{xp} + M}{1+b} = \frac{\rho\theta(\Omega_t^{xw} - \Omega_t^{xe}) + M}{1+b}. \tag{IA.520}$$

From (IA.517) and (IA.520) one also obtains:

$$E_t = W_t - P_t = \frac{\rho\theta\Omega_t^{xe} - M}{1+b} + B. \tag{IA.521}$$

I now compute Ω_t^{xx}. From its definition, one obtains:

$$\begin{aligned} \frac{\mathrm{d}\Omega_t^{xx}}{\mathrm{d}t} &= \frac{1}{\sigma_w^2\mathrm{d}t}\,\mathsf{E}\big(2x_{t-1}\mathrm{d}x_t + (\mathrm{d}x_t)^2\big) \\ &= \frac{1}{\sigma_w^2\mathrm{d}t}\,\mathsf{E}\Big(2x_{t-1}\big(-\theta x_{t-1}\mathrm{d}t + \gamma\mathrm{d}w_t + \mu\widetilde{\mathrm{d}w}_{t-1}\big) + (\mathrm{d}x_t)^2\Big). \\ &= -2\theta\Omega_{t-1}^{xx} + G^2. \end{aligned} \tag{IA.522}$$

This is a first order ODE with solution:

$$\Omega_t^{xx} = G^2\,\frac{1-\mathrm{e}^{-2\theta t}}{2\theta}. \tag{IA.523}$$

Finally, one computes Ω_t^{xe}. Since $\mathrm{d}w_t - \mathrm{d}p_t = \lambda\theta x_{t-1}\mathrm{d}t + (1-\lambda\bar{\gamma})\mathrm{d}w_t - \lambda\bar{\mu}\widetilde{\mathrm{d}w}_{t-1} - \lambda\mathrm{d}u_t$, from the definition of Ω_t^{xe}, one obtains:

$$\begin{aligned} \frac{\mathrm{d}\Omega_t^{xe}}{\mathrm{d}t} &= \frac{1}{\sigma_w^2\mathrm{d}t}\,\mathsf{E}\big((w_{t-1}-p_{t-1})\mathrm{d}x_t + x_{t-1}(\mathrm{d}w_t - \mathrm{d}p_t) + (\mathrm{d}w_t - \mathrm{d}p_t)\mathrm{d}x_t\big) \\ &= -\theta\Omega_{t-1}^{xe} + \lambda\theta\Omega_{t-1}^{xx} - \lambda\bar{\mu}X_{t-1} + (1-\lambda\bar{\gamma})G. \end{aligned} \tag{IA.524}$$

From (IA.515), one has $X_{t-1} = \frac{\rho\theta\Omega_{t-1}^{xx}}{1+b} + BG$. one obtains:

$$\begin{aligned} \frac{\mathrm{d}\Omega_t^{xe}}{\mathrm{d}t} &= -\theta\Big(1 - \frac{\rho\mu}{1+b}\Big)\Omega_{t-1}^{xe} + \lambda\theta\Big(1 - \frac{\rho\bar{\mu}}{1+b}\Big)\Omega_{t-1}^{xx} - \lambda\bar{\mu}BG + (1-\lambda\bar{\gamma})G \\ &= -\theta\Omega_{t-1}^{xe} + \lambda\theta\frac{1}{1+b}\,\Omega_{t-1}^{xx} - \lambda\bar{\mu}BG + \tilde{\pi}_0, \end{aligned} \tag{IA.525}$$

[31] Indeed, using $\rho^2\tilde{\sigma}_u^2 = (1-a)(a-b^2)$, one computes $\frac{\lambda}{\rho}\big((1-a)a - (1-a)b^2 - (1-a)(a-b^2)\big) = 0$.

where $\tilde{\pi}_0$ is the IFT's normalized expected profit when $\theta = 0$:

$$\tilde{\pi}_0 \;=\; (1-\lambda\bar{\gamma})G. \tag{IA.526}$$

From (IA.523), $\lambda\theta\frac{1}{1+b}\,\Omega^{xx}_{t-1} = \frac{\lambda\theta}{1+b}\,G^2\,\frac{1-\mathrm{e}^{-2\theta t}}{2\theta} = \frac{\lambda G^2}{2(1+b)}\,(1-\mathrm{e}^{-2\theta t})$. The differential equation for Ω^{xe}_t becomes:

$$\begin{aligned}\frac{\mathrm{d}\Omega^{xe}_t}{\mathrm{d}t} \;&=\; -\theta\Omega^{xe}_{t-1} + D_1\big(1-\mathrm{e}^{-2\theta t}\big) + D_2, \quad \text{with}\\ D_1 \;&=\; \frac{\lambda G^2}{2(1+b)}, \qquad D_2 \;=\; \tilde{\pi}^0 - \lambda\bar{\mu}BG.\end{aligned} \tag{IA.527}$$

This is a first order ODE with solution:

$$\Omega^{xe}_t \;=\; \big(D_1 + D_2\big)\frac{1-\mathrm{e}^{-\theta t}}{\theta} + D_1\,\frac{\mathrm{e}^{-\theta t}-\mathrm{e}^{-2\theta t}}{\theta}. \tag{IA.528}$$

Next, one computes the IFT's expected profit in the smooth regime:

$$\begin{aligned}\tilde{\pi}_\theta \;&=\; \frac{1}{\sigma_w^2}\,\mathsf{E}\int_0^T (w_t - p_t)\mathrm{d}x_t\\ &=\; \frac{1}{\sigma_w^2}\,\mathsf{E}\int_0^T \big(w_{t-1} - p_{t-1} + \mathrm{d}w_t - \lambda\mathrm{d}y_t\big)\Big(G\mathrm{d}w_t - \theta x_{t-1}\mathrm{d}t\Big)\\ &=\; \int_0^T \Big(-\theta\Omega^{xe}_{t-1} + G - \lambda G\bar{\gamma}\Big)\mathrm{d}t.\end{aligned} \tag{IA.529}$$

Therefore:

$$\tilde{\pi}_\theta \;=\; \tilde{\pi}_0 - \int_0^T \theta\Omega^{xe}_t\mathrm{d}t. \tag{IA.530}$$

From (IA.528):

$$\theta\Omega^{xe}_t \;=\; \frac{\lambda G^2}{2(1+b)}\Big(1-\mathrm{e}^{-2\theta t}\Big) + \big(\tilde{\pi}_0 - \lambda\bar{\mu}BG\big)\Big(1-\mathrm{e}^{-\theta t}\Big), \tag{IA.531}$$

with D_1 and D_2 as in (IA.527). One computes:

$$\tilde{\pi}_\theta \;=\; \tilde{\pi}_0\int_0^T \mathrm{e}^{-\theta t}\,\mathrm{d}t + \lambda\bar{\mu}BG\int_0^T\big(1-\mathrm{e}^{-\theta t}\big)\mathrm{d}t - \frac{\lambda G^2}{2(1+b)}\int_0^T\big(1-\mathrm{e}^{-2\theta t}\big)\mathrm{d}t. \tag{IA.532}$$

This is the first line in equation (IA.499). Since the normalized expected utility of the IFT satisfies:

$$\tilde{U}_\theta \;=\; \tilde{\pi}_\theta - \frac{1}{\sigma_w^2}\,C_I\,\mathsf{E}\left(\int_0^T x_t^2\mathrm{d}t\right), \tag{IA.533}$$

to prove the second part of equation (IA.499), one only has to show that:

$$\frac{1}{\sigma_w^2} C_I \operatorname{E}\left(\int_0^T x_t^2 \mathrm{d}t\right) \;=\; C_I \int_0^T \Omega_t^{xx} \mathrm{d}t \;=\; \frac{C_I G^2}{2\theta} \int_0^T \big(1 - \mathrm{e}^{-2\theta t}\big) \mathrm{d}t. \qquad \text{(IA.534)}$$

But equation (IA.523) implies that $\Omega_t^{xx} = G^2 \frac{1-\mathrm{e}^{-2\theta t}}{2\theta}$. This completes the proof of Proposition IA.10. ■

Proof of Corollary IA.3. Note that for $t > 0$ one has:

$$\lim_{\theta \to 0} \frac{1 - \mathrm{e}^{-2\theta t}}{2\theta} \;=\; t \quad \Longrightarrow \quad \lim_{\theta \to 0} \frac{1}{2\theta} \int_0^T \big(1 - \mathrm{e}^{-2\theta t}\big) \mathrm{d}t \;=\; \int_0^T t \mathrm{d}t \;=\; \frac{1}{2}. \qquad \text{(IA.535)}$$

Equation (IA.499) implies that when $\theta \to 0$, $\tilde{U}_{\theta}$ converges to $G(1 - \lambda\bar{\gamma}) - \frac{C_I}{2} G^2$. But by summing (75) and (76) from the proof of Theorem 2, one obtains that $\tilde{U}_{\Theta=0} = G(1 - \lambda\bar{\gamma}) - \frac{C_I}{2} G^2$. Therefore, $\tilde{U}_{\Theta=0} = \tilde{U}_{\theta=0}$.

Equation (IA.499) also implies that when $\theta \to \infty$, $\tilde{U}_{\theta}$ converges to $\bar{\mu} \frac{\lambda G(1-\rho\bar{\gamma})}{1+\rho\bar{\mu}} - \frac{\lambda G^2}{2(1+\rho\bar{\mu})}$. But equation (IA.295) from the proof of Proposition IA.7 computes $\tilde{U}_{\Theta=0_+} = \frac{\lambda\bar{\mu} G(1-\rho\bar{\gamma})}{1+\rho\bar{\mu}} - \frac{\lambda G^2}{2(1+\rho\bar{\mu})}$. Therefore, $\tilde{U}_{\Theta=0_+} = \tilde{U}_{\theta=\infty}$.

To show that the inventory costs approach zero when $\theta \to \infty$, note that in equation (IA.499), $\mathrm{e}^{-\theta t}$ converges uninformly to zero (for $t \in [0, T]$). ■

Proof of Proposition IA.11. In Proposition IA.10, I have already computed the normalized expected utility of the IFT:

$$\tilde{U}_{\theta} \;=\; G(1 - \lambda\bar{\gamma})\big(1 - F_\theta\big) + \bar{\mu} \frac{\lambda G(1 - \rho\bar{\gamma})}{1 + \rho\bar{\mu}} F_\theta - \frac{\lambda G^2}{2(1 + \rho\bar{\mu})} F_{2\theta} - \frac{C_I G^2}{2\theta} F_{2\theta}, \qquad \text{(IA.536)}$$

where as in equation (IA.498) $F_\theta = \int_0^1 \big(1 - \mathrm{e}^{-\theta t}\big) \mathrm{d}t \;=\; 1 - \frac{1-\mathrm{e}^{-\theta}}{\theta}$. One verifies that $\frac{F_\theta}{\theta}$ is a well-defined analytical function, and it satisfies:

$$\lim_{\theta \to 0} \frac{F_\theta}{\theta} \;=\; \frac{1}{2}. \qquad \text{(IA.537)}$$

I rewrite equation (IA.536) as $\tilde{U}_{\theta} = G(1 - \lambda\bar{\gamma})(1 - F_\theta) + \frac{RbG(1-\rho\bar{\gamma})}{1+b} F_\theta - \frac{\lambda G^2}{2(1+b)} F_{2\theta} - \frac{C_I G^2}{2\theta} F_{2\theta}$. Since $\bar{\gamma} = \gamma^- + G$, one computes:

$$\tilde{U}_{\theta} \;=\; G\left((1 - Ra^-) - F_\theta \left(1 - R\frac{a^- + b}{1 + b}\right) \right) - \frac{G^2}{2} \left(2\lambda \left(1 - \frac{F_\theta}{1 + b}\right) + F_{2\theta} \left(\frac{\lambda}{1 + b} + \frac{C_I}{\theta}\right) \right). \qquad \text{(IA.538)}$$

Fix $\theta \in [0, \infty]$. Then, the first order condition in G implies:

$$G = \frac{\left(1 - Ra^-\right) - F_\theta\left(1 - R\frac{a^- + b}{1+b}\right)}{2\lambda\left(1 - \frac{F_\theta}{1+b}\right) + F_{2\theta}\left(\frac{\lambda}{1+b} + \frac{C_I}{\theta}\right)}. \tag{IA.539}$$

The second order condition for a maximum is also clearly satisfied. Hence, for a given θ, the maximum normalized expected utility of the IFT when G varies is:

$$\tilde{U}_\theta^{\max} = \frac{1}{2}\frac{\left(\left(1 - Ra^-\right) - F_\theta\left(1 - R\frac{a^- + b}{1+b}\right)\right)^2}{2\lambda\left(1 - \frac{F_\theta}{1+b}\right) + F_{2\theta}\left(\frac{\lambda}{1+b} + \frac{C_I}{\theta}\right)}. \tag{IA.540}$$

This proves equation (IA.502). ■

Proof of Corollary IA.4. I use the formula for $U_\theta^{\max}$ from Proposition IA.11. When $\theta = 0$, one has $\lim\limits_{\theta \to 0} \frac{F_{2\theta}}{2\theta} = \frac{1}{2}$, hence one obtains the first equation in (IA.503). When $\theta \to \infty$, one has $\lim\limits_{\theta \to \infty} F_\theta = 1$, hence:

$$\tilde{U}_\infty = \frac{1}{2}\frac{R^2 b^2 \frac{(1-a^-)^2}{(1+b)^2}}{2\lambda\frac{b}{1+b} + \frac{\lambda}{1+b}} = \frac{\left(Rb(1-a^-)\right)^2}{4\lambda(1+b)\left(b+\frac{1}{2}\right)}, \tag{IA.541}$$

which proves the second equation in (IA.503). One can now solve directly for the C_I that makes $U_0^{\max} = U_\infty^{\max}$.

Finally, equation (IA.505) follows from the formula for $\tilde{U}_0^{\max}$ in (IA.503) by setting $C_I = 0$. ■

6.3 General smooth strategies

In this subsection, I solve for a partial equilibrium of the model with inventory management in which the IFT trades in the smooth regime, but with a more general trading strategy:

$$\mathrm{d}x_t = -\theta\, x_t \mathrm{d}t + G\,\mathrm{d}w_t + M\,\widetilde{\mathrm{d}w}_{t-1}, \quad \text{with} \quad \theta \in [0, \infty). \tag{IA.542}$$

Define the following coefficients:

$$\begin{aligned} R &= \frac{\lambda}{\rho}, \qquad \gamma^- = N_F \gamma, \qquad \bar{\gamma} = \gamma^- + G, \qquad \bar{\mu} = N_L \mu, \\ a^- &= \rho\gamma^-, \qquad a = \rho\bar{\gamma}, \qquad b = \rho\bar{\mu}. \end{aligned} \tag{IA.543}$$

The coefficients satisfy $\gamma^- \geq 0$, $\bar{\mu} \geq 0$, $\rho > 0$, $\lambda > 0$.

As usual, tilde notation denotes normalization by σ_w or σ_w^2. For instance, the normalized expected utility of the IFT is:

$$\tilde{U} \;=\; \frac{U}{\sigma_w^2}. \tag{IA.544}$$

For any smooth strategy of the IFT (not necessarily optimal), one computes the IFT's expected utility, while taking the behavior of the others as given. First, define the following function of $\theta \in (0, \infty)$:

$$F_\theta \;=\; \int_0^1 \big(1 - \mathrm{e}^{-\theta t}\big)\mathrm{d}t \;=\; 1 - \frac{1 - \mathrm{e}^{-\theta}}{\theta}. \tag{IA.545}$$

This function is strictly increasing in θ and has well defined limits at the interval endpoints: $\lim_{\theta\to 0} F_\theta = 0$ and $\lim_{\theta\to\infty} F_\theta = 1$. Therefore, by abuse of notation, I define F_θ for the whole interval $\theta \in [0, \infty]$. I introduce further notation:

$$\begin{aligned}
A \;&=\; \frac{(1-a)^2 + \rho^2\tilde{\sigma}_u^2}{1 - b^2}, \qquad B \;=\; \frac{1-a}{1+b}, \\
G' \;&=\; G + MB, \qquad G''^2 \;=\; G'^2 + M^2\frac{\rho^2\tilde{\sigma}_u^2}{(1+b)^2}, \\
D_1 \;&=\; \frac{\lambda G''^2}{2(1+b^-)}, \qquad D_2 \;=\; G'(1-\lambda\bar{\gamma}) - \lambda\bar{\mu}BG' + \frac{\lambda\rho M\tilde{\sigma}_u^2}{(1+b)^2}, \\
D_3 \;&=\; G'(1-\lambda\bar{\gamma}) + \frac{\lambda\rho M\tilde{\sigma}_u^2 - \lambda\bar{\mu}MA}{1+b}.
\end{aligned} \tag{IA.546}$$

Proposition IA.12. *Suppose $b = \rho\bar{\mu} < 1$. Then, the normalized expected utility of the IFT with a trading strategy as in (IA.542) is:*

$$\tilde{U}_\theta \;=\; -D_1 F_{2\theta'} - D_2 F_{\theta'} + D_3 - C_I G''^2 \frac{F_{2\theta'}}{2\theta'}. \tag{IA.547}$$

Proof. Recall that:

$$\begin{aligned}
\mathrm{d}x_t \;&=\; -\theta x_{t-1}\mathrm{d}t + G\mathrm{d}w_t + M\widetilde{\mathrm{d}w}_{t-1}, \quad \text{with} \quad \theta \;\in\; [0,\infty), \\
\mathrm{d}y_t \;&=\; -\theta x_{t-1}\mathrm{d}t + \bar{\gamma}\mathrm{d}w_t + \bar{\mu}\widetilde{\mathrm{d}w}_{t-1} + \mathrm{d}u_t, \quad \text{with} \\
&\quad \bar{\gamma} \;=\; G + \gamma^-, \quad \bar{\mu} \;=\; M + \mu^-, \\
\widetilde{\mathrm{d}w}_t \;&=\; \rho\theta x_{t-1}\mathrm{d}t + \mathrm{d}w_t(1-a) - b\widetilde{\mathrm{d}w}_{t-1} - \rho\mathrm{d}u_t, \quad \text{with} \\
&\quad a \;=\; \rho\bar{\gamma} \;=\; a^- + \rho G, \qquad b \;=\; \rho\bar{\mu} \;=\; b^- + \rho M.
\end{aligned} \tag{IA.548}$$

Denote:

$$
\begin{aligned}
\Omega_t^{xx} &= \frac{\mathsf{E}\big(x_t^2\big)}{\sigma_w^2}, \qquad \Omega_t^{xw} = \frac{\mathsf{E}\big(x_t w_t\big)}{\sigma_w^2}, \qquad \Omega_t^{xp} = \frac{\mathsf{E}\big(x_t p_t\big)}{\sigma_w^2}, \\
\Omega_t^{xe} &= \Omega_t^{xw} - \Omega_t^{xp}, \qquad A_t = \frac{\mathsf{E}\big(\widetilde{\mathrm{d}w}_t^{\,2}\big)}{\sigma_w^2\mathrm{d}t}, \qquad X_t = \frac{\mathsf{E}\big(x_t\widetilde{\mathrm{d}w}_t\big)}{\sigma_w^2\,\mathrm{d}t}, \\
W_t &= \frac{\mathsf{E}\big(w_t\widetilde{\mathrm{d}w}_t\big)}{\sigma_w^2\,\mathrm{d}t}, \qquad P_t = \frac{\mathsf{E}\big(p_t\widetilde{\mathrm{d}w}_t\big)}{\sigma_w^2\,\mathrm{d}t}, \qquad E_t = \frac{\mathsf{E}\big((w_t - p_t)\widetilde{\mathrm{d}w}_t\big)}{\sigma_w^2\,\mathrm{d}t}, \\
B &= \frac{1-a}{1+b}, \qquad \theta' = \theta\Big(1 - \frac{\rho M}{1+b}\Big) = \theta\,\frac{1+b^-}{1+b}, \qquad G' = G + MB.
\end{aligned} \tag{IA.549}
$$

The IFT's expected profit in the smooth regime is:

$$
\begin{aligned}
\tilde{\pi}_{\theta} &= \frac{1}{\sigma_w^2}\,\mathsf{E}\int_0^T (w_t - p_t)\mathrm{d}x_t \\
&= \frac{1}{\sigma_w^2}\,\mathsf{E}\int_0^T \big(w_{t-1} - p_{t-1} + \mathrm{d}w_t - \lambda\mathrm{d}y_t\big)\Big(-\theta x_{t-1}\mathrm{d}t + G\mathrm{d}w_t + M\widetilde{\mathrm{d}w}_{t-1}\Big) \\
&= \int_0^T \Big(-\theta\Omega_{t-1}^{xe} + G - \lambda G\bar{\gamma} + ME_{t-1} - \lambda M\bar{\mu}A_{t-1}\Big)\mathrm{d}t.
\end{aligned} \tag{IA.550}
$$

One computes the covariances involved in the formula above. The recursive formula for A_t is:

$$
\begin{aligned}
A_t &= \frac{1}{\sigma_w^2\mathrm{d}t}\,\mathsf{E}\Big(\big(\rho\theta x_{t-1}\mathrm{d}t + \mathrm{d}w_t(1-a) - b\widetilde{\mathrm{d}w}_{t-1} - \rho\mathrm{d}u_t\big)^2\Big) \\
&= (1-a)^2 + b^2A_{t-1} + \rho^2\tilde{\sigma}_u^2.
\end{aligned} \tag{IA.551}
$$

Lemma A.1 in the Appendix in the paper implies that A_t is constant and equal to:

$$
A = \frac{(1-a)^2 + \rho^2\tilde{\sigma}_u^2}{1-b^2}. \tag{IA.552}
$$

The recursive formula for W_t is:

$$
\begin{aligned}
W_t &= \frac{1}{\sigma_w^2\mathrm{d}t}\,\mathsf{E}\Big(\big(w_{t-1} + \mathrm{d}w_t\big)\big(\rho\theta x_{t-1}\mathrm{d}t + \mathrm{d}w_t(1-a) - b\widetilde{\mathrm{d}w}_{t-1} - \rho\mathrm{d}u_t\big)\Big) \\
&= \rho\theta\Omega_{t-1}^{xw} - bW_{t-1} + (1-a).
\end{aligned} \tag{IA.553}
$$

Thus, $W_t + bW_{t-1} = \rho\theta\Omega_{t-1}^{xw} + (1-a)$. Lemma A.1 in the Appendix in the paper implies that:[32]

$$W_t \;=\; \frac{\rho\theta\Omega_t^{xw} + (1-a)}{1+b} \;=\; \frac{\rho\theta\Omega_t^{xw}}{1+b} + B. \qquad \text{(IA.554)}$$

From the definition of Ω^{xw}, one obtains:

$$\begin{aligned}\frac{\mathrm{d}\Omega_t^{xw}}{\mathrm{d}t} \;&=\; \frac{1}{\sigma_w^2 \mathrm{d}t}\,\mathsf{E}\big(w_{t-1}\mathrm{d}x_t + x_{t-1}\mathrm{d}w_t + \mathrm{d}x_t\mathrm{d}w_t\big) \\ &=\; -\theta\Omega_{t-1}^{xw} + MW_{t-1} + G \;=\; -\theta\Omega_{t-1}^{xw}\Big(1 - \frac{\rho M}{1+b}\Big) + G + MB \\ &-\theta'\Omega_{t-1}^{xw} + G'.\end{aligned} \qquad \text{(IA.555)}$$

The initial inventory is zero, which implies $\Omega_0^{xw} = 0$. Thus, the solution for the differential equation (IA.555) is:

$$\Omega_t^{xw} \;=\; G'\,\frac{1 - \mathrm{e}^{-\theta' t}}{\theta'}. \qquad \text{(IA.556)}$$

The recursive formula for X_t is:

$$\begin{aligned}X_t \;&=\; \frac{1}{\sigma_w^2 \mathrm{d}t}\,\mathsf{E}\Big(\big(x_{t-1} + \mathrm{d}x_t\big)\big(\rho\theta x_{t-1}\mathrm{d}t + \mathrm{d}w_t(1-a) - b\widetilde{\mathrm{d}w}_{t-1} - \rho\mathrm{d}u_t\big)\Big) \\ &=\; \rho\theta\frac{\mathsf{E}\big((x_{t-1})^2\big)}{\sigma_w^2} - b\frac{\mathsf{E}\big(x_{t-1}\widetilde{\mathrm{d}w}_{t-1}\big)}{\sigma_w^2\mathrm{d}t} + (1-a)\frac{\mathsf{E}\big(\mathrm{d}x_t\mathrm{d}w_t\big)}{\sigma_w^2\mathrm{d}t} - b\frac{\mathsf{E}\big(\mathrm{d}x_t\widetilde{\mathrm{d}w}_{t-1}\big)}{\sigma_w^2\mathrm{d}t} \\ &=\; \rho\theta\Omega_{t-1}^{xx} - bX_{t-1} + (1-a)G - bMA_{t-1}.\end{aligned} \qquad \text{(IA.557)}$$

Thus, $X_t + bX_{t-1} = \rho\theta\Omega_{t-1}^{xx} + (1-a)G - bMA$. Lemma A.1 in the Appendix in the paper implies:

$$X_t \;=\; \frac{\rho\theta\Omega_t^{xx} + (1-a)G - bMA}{1+b} \;=\; \frac{\rho\theta\Omega_t^{xx}}{1+b} + BG - \frac{bMA}{1+b}. \qquad \text{(IA.558)}$$

From the definition of Ω_t^{xx}, one obtains:

$$\begin{aligned}\frac{\mathrm{d}\Omega_t^{xx}}{\mathrm{d}t} \;&=\; \frac{1}{\sigma_w^2\mathrm{d}t}\,\mathsf{E}\big(2x_{t-1}\mathrm{d}x_t + (\mathrm{d}x_t)^2\big) \\ &=\; -2\theta\Omega_{t-1}^{xx} + 2MX_{t-1} + G^2 + M^2A_{t-1} \\ &=\; -2\theta\Omega_{t-1}^{xx}\Big(1 - \frac{\rho M}{1+b}\Big) + G^2 + 2MBG + M^2A\frac{1-b}{1+b} \\ &=\; -2\theta\Omega_{t-1}^{xx}\Big(1 - \frac{\rho M}{1+b}\Big) + (G+MB)^2 + M^2\frac{\rho^2\tilde{\sigma}_u^2}{(1+b)^2} \\ &=\; -2\theta'\Omega_{t-1}^{xx} + G''^2,\end{aligned} \qquad \text{(IA.559)}$$

[32]The difference between W_t and W_{t-1} is infinitesimal, hence it can be ignored. In other words, one can use Lemma A.1 either for α_t or for α_{t-1}, and obtain the same result.

where:

$$G''^2 \;=\; G'^2 + M^2\frac{\rho^2\tilde{\sigma}_u^2}{(1+b)^2}. \tag{IA.560}$$

This is a first order ODE with solution:

$$\Omega_t^{xx} \;=\; G''^2\,\frac{1-\mathrm{e}^{-2\theta' t}}{2\theta'}. \tag{IA.561}$$

The recursive formula for P_t is:

$$\begin{aligned}
P_t \;&=\; \frac{1}{\sigma_w^2\mathrm{d}t}\,\mathsf{E}\Big(\big(p_{t-1}+\lambda\mathrm{d}y_t\big)\big(\rho\theta x_{t-1}\mathrm{d}t+\mathrm{d}w_t(1-a)-b\widetilde{\mathrm{d}w}_{t-1}-\rho\mathrm{d}u_t\big)\Big)\\
&=\; \rho\theta\Omega_{t-1}^{xp}-bP_{t-1}+(1-a)\lambda\frac{\mathsf{E}\big(\mathrm{d}y_t\mathrm{d}w_t\big)}{\sigma_w^2\mathrm{d}t}-b\lambda\frac{\mathsf{E}\big(\mathrm{d}y_t\widetilde{\mathrm{d}w}_{t-1}\big)}{\sigma_w^2\mathrm{d}t}-\lambda\rho\tilde{\sigma}_u^2\\
&=\; \rho\theta\Omega_{t-1}^{xp}-bP_{t-1}+(1-a)\lambda\bar{\gamma}-b\lambda\bar{\mu}A-\lambda\rho\tilde{\sigma}_u^2.
\end{aligned} \tag{IA.562}$$

Recall the recursive formula $W_t=\rho\theta\Omega_{t-1}^{xw}-bW_{t-1}+(1-a)$. The difference between the recursive formulas for W_t and P_t is:

$$E_t \;=\; \rho\theta\Omega_{t-1}^{xe}-bE_{t-1}+(1-a)(1-\lambda\bar{\gamma})+b\lambda\bar{\mu}A+\lambda\rho\tilde{\sigma}_u^2. \tag{IA.563}$$

Thus, $E_t+bE_{t-1}=\rho\theta\Omega_{t-1}^{xe}+(1-a)(1-\lambda\bar{\gamma})+b\lambda\bar{\mu}A+\lambda\rho\tilde{\sigma}_u^2$. Lemma A.1 in the Appendix in the paper implies that:

$$E_t \;=\; \frac{\rho\theta\Omega_t^{xe}+(1-a)(1-\lambda\bar{\gamma})+b\lambda\bar{\mu}A+\lambda\rho\tilde{\sigma}_u^2}{1+b}. \tag{IA.564}$$

Note that $\mathrm{d}w_t-\mathrm{d}p_t=\lambda\theta x_{t-1}\mathrm{d}t+(1-\lambda\bar{\gamma})\mathrm{d}w_t-\lambda\bar{\mu}\widetilde{\mathrm{d}w}_{t-1}-\lambda\mathrm{d}u_t$. By the definition of Ω_t^{xe}, one obtains:

$$\begin{aligned}
\frac{\mathrm{d}\Omega_t^{xe}}{\mathrm{d}t} \;&=\; \frac{1}{\sigma_w^2\mathrm{d}t}\,\mathsf{E}\big((w_{t-1}-p_{t-1})\mathrm{d}x_t+x_{t-1}(\mathrm{d}w_t-\mathrm{d}p_t)+(\mathrm{d}w_t-\mathrm{d}p_t)\mathrm{d}x_t\big)\\
&=\; -\theta\Omega_{t-1}^{xe}+ME_{t-1}+\lambda\theta\Omega_{t-1}^{xx}-\lambda\bar{\mu}X_{t-1}+(1-\lambda\bar{\gamma})G-\lambda\bar{\mu}MA_{t-1}\\
&=\; -\theta\Big(1-\frac{\rho M}{1+b}\Big)\Omega_{t-1}^{xe}+M\,\frac{(1-a)(1-\lambda\bar{\gamma})+b\lambda\bar{\mu}A+\lambda\rho\tilde{\sigma}_u^2}{1+b}\\
&\quad+\lambda\theta\Big(1-\frac{\rho\bar{\mu}}{1+b}\Big)\Omega_{t-1}^{xx}-\lambda\bar{\mu}BG+\lambda\bar{\mu}\frac{bMA}{1+b}+(1-\lambda\bar{\gamma})G-\lambda\bar{\mu}MA\\
&=\; -\theta'\Omega_{t-1}^{xe}+\frac{\lambda\theta}{1+b}\Omega_{t-1}^{xx}+G'(1-\lambda\bar{\gamma})-\lambda\bar{\mu}BG'+\frac{\lambda\rho M\tilde{\sigma}_u^2}{(1+b)^2}.
\end{aligned} \tag{IA.565}$$

From (IA.561), $\frac{\lambda\theta}{1+b}\,\Omega^{xx}_{t-1} = \frac{\lambda\theta}{1+b}\,G''^2\,\frac{1-\mathrm{e}^{-2\theta' t}}{2\theta'} = \frac{\lambda G''^2}{2(1+b^-)}\,(1-\mathrm{e}^{-2\theta' t})$. Denote:

$$\begin{aligned} D_1 &= \frac{\lambda G''^2}{2(1+b^-)}, \qquad D_2 \;=\; G'(1-\lambda\bar{\gamma}) - \lambda\bar{\mu}BG' + \frac{\lambda\rho M\tilde{\sigma}_u^2}{(1+b)^2}, \\ D_3 &= G'(1-\lambda\bar{\gamma}) + \frac{\lambda\rho M\tilde{\sigma}_u^2 - \lambda\bar{\mu}MA}{1+b}. \end{aligned} \tag{IA.566}$$

The differential equation for Ω^{xe}_t can be written as follows:

$$\frac{\mathrm{d}\Omega^{xe}_t}{\mathrm{d}t} \;=\; -\theta'\Omega^{xe}_{t-1} + D_1\big(1-\mathrm{e}^{-2\theta' t}\big) + D_2. \tag{IA.567}$$

This is a first order ODE with solution:

$$\Omega^{xe}_t \;=\; D_1\,\frac{1-\mathrm{e}^{-2\theta' t}}{\theta'} + D_2\frac{1-\mathrm{e}^{-\theta' t}}{\theta'}. \tag{IA.568}$$

I now compute the normalized expected profit of IFT, using formula (IA.564) for E_t:

$$\begin{aligned} \tilde{\pi}_\theta &= \int_0^T \Big(-\theta\Omega^{xe}_{t-1} + G - \lambda G\bar{\gamma} + ME_{t-1} - \lambda M\bar{\mu}A\Big)\mathrm{d}t \\ &= \int_0^T \Big(-\theta'\Omega^{xe}_{t-1} + G'(1-\lambda\bar{\gamma}) + \frac{\lambda\rho M\tilde{\sigma}_u^2 - \lambda\bar{\mu}MA}{1+b}\Big)\mathrm{d}t \\ &= \int_0^T \Big(-D_1\big(1-\mathrm{e}^{-2\theta' t}\big) - D_2\big(1-\mathrm{e}^{-\theta' t}\big) + D_3\Big)\mathrm{d}t. \end{aligned} \tag{IA.569}$$

When $\theta = 0$, one obtains:

$$\tilde{\pi}_0 \;=\; D_3 \;=\; G'(1-\lambda\bar{\gamma}) + \frac{\lambda\rho M\tilde{\sigma}_u^2 - \lambda\bar{\mu}MA}{1+b}. \tag{IA.570}$$

When $\theta = \infty$, one gets $\tilde{\pi}_\infty = D_3 - D_1 - D_2$, from which one computes:

$$\tilde{\pi}_\infty \;=\; \frac{\lambda}{2(1+b^-)}\Big(-G^2 + 2\mu^-(1-a)G - MA(2b\mu^- + M)\Big). \tag{IA.571}$$

This coincides with the formula in (IA.410).

The normalized expected utility of the IFT satisfies:

$$\begin{aligned} \tilde{U}_\theta &= \tilde{\pi}_\theta - \frac{1}{\sigma_w^2}\,C_I\,\mathsf{E}\left(\int_0^T x_t^2\mathrm{d}t\right) \;=\; \tilde{\pi}_\theta - C_I\int_0^T \Omega^{xx}_t\mathrm{d}t \\ &= \int_0^T \Big(-D_1\big(1-\mathrm{e}^{-2\theta' t}\big) - D_2\big(1-\mathrm{e}^{-\theta' t}\big) + D_3 - C_I G''^2\frac{1-\mathrm{e}^{-2\theta' t}}{2\theta'}\Big)\mathrm{d}t. \end{aligned} \tag{IA.572}$$

Recall from (IA.545) that the function $F_\theta = \int_0^1 \bigl(1 - \mathrm{e}^{-\theta t}\bigr)\mathrm{d}t = 1 - \frac{1-\mathrm{e}^{-\theta}}{\theta}$ is well defined for $\theta \in [0, \infty]$, and also that the ratio $\frac{F_\theta}{\theta}$ is also well defined for $\theta \in [0, \infty]$. one obtains:

$$\tilde{U}_\theta \;=\; -D_1 F_{2\theta'} - D_2 F_{\theta'} + D_3 - C_I G''^2 \frac{F_{2\theta'}}{2\theta'}, \tag{IA.573}$$

which proves Proposition IA.12. ■

Numerical results

Numerically, when the IFT can choose among trading strategies of the type $\mathrm{d}x_t = -\Theta x_{t-1} + G\mathrm{d}w_t + M\widetilde{\mathrm{d}w}_{t-1}$, the results are qualitatively the same as when the IFT can choose only strategies with $M = 0$. The latter case is examined in detail in Subsection 6.1, and therefore I do not report the results for unconstrained M. In conclusion, even when the IFT can choose more general smooth strategies, it is never optimal to choose an interior point $\theta \in (0, \infty)$.

7 Fast and slow traders in discrete time

In this section, I analyze a discrete-time version of the benchmark model with FTs and STs in the paper. I denote this discrete-time version by $\mathcal{D}_1$, just as in continuous time I denote its counterpart by $\mathcal{M}_1$. It is useful to analyze how the discrete-time model $\mathcal{D}_1$ compares in the limit to the continuous-time model $\mathcal{M}_1$. I show that although the model $\mathcal{D}_1$ does not converge to its continuous-time counterpart $\mathcal{M}_1$, the difference is quite small.

I attribute this difference to the assumption that in $\mathcal{M}_1$ the speculators's choice of weights has no effect on the covariance structure of the dealer's signals (see equation (13)). By contrast, I conjecture that in the continuous-time limit of $\mathcal{D}_1$ the speculators take this effect into account.[33] If this conjecture is correct, the results of this section allow us to analyze the equilibrium effect of changing this assumption. This effect turns out to be quite small: see Figure IA.5.

7.1 Model

I first describe the model $\mathcal{D}_1$. Trading occurs at intervals of length Δt apart, at times $t_1 = \Delta t$, $t_2 = 2\Delta t$, $\ldots$, $t_T = T\Delta t$. To simplify notation, I refer to these times as $1, 2, \ldots, T$. The liquidation value of the asset is:

$$v_T = \sum_{t=1}^{T} \Delta v_t, \quad \text{with} \quad \Delta v_t = v_t - v_{t-1} = \sigma_v \Delta B_t^v, \tag{IA.574}$$

where B_t^v is a standard Brownian motion. The risk-free rate is assumed to be zero.

There are three types of market participants: (a) $N \geq 1$ risk-neutral speculators, who observe the flow of information at different speeds, as described below; (b) noise traders; and (c) one competitive risk-neutral dealer, who sets the price at which trading takes place.

[33] Some evidence that this conjecture is correct is the speculator's behavior in the continuous version of Kyle (1985). Indeed, in that model the speculator chooses his optimal weight by taking into account his effect on the covariance matrix $\Sigma_t = \mathsf{Var}(v - p_t)$. In this model, signals are used only for a finite number of lags, and therefore I conjecture that the this effect is much weaker than in Kyle (1985).

7.1.1 Information

At $t = 0$, there is no information asymmetry between the speculators and the dealer. Subsequently, each speculator receives the following flow of signals:

$$\Delta s_t \;=\; \Delta v_t + \Delta\eta_t, \quad \text{with} \quad \Delta\eta_t \;=\; \sigma_\eta \mathrm{d}B_t^\eta, \tag{IA.575}$$

where $t = 1, \ldots, T$ and B_t^η is a standard Brownian motion independent from all other variables. Define the speculators' forecast by:

$$w_t \;=\; \mathsf{E}\big(v_T \;\big|\; \{s_\tau\}_{\tau \le t}\big). \tag{IA.576}$$

Its increment is $\Delta w_t = \frac{\sigma_v^2}{\sigma_v^2 + \sigma_\eta^2} \Delta s_t$. Also, $w_0 = 0$.

Speculators obtain their signal with a lag $\ell = 0, 1$. A 0-speculator, also called a FT, is a trader who at $t = 1, \ldots, T$ observes the signal Δs_t. A 1-speculator, also called a ST, is a trader who at $t = 2, \ldots, T$ observes the signal Δs_{t-1}. Denote by N_F the number of FTs, and by N_S the number of STs. Denote the total number of traders by N_L:

$$N_L \;=\; N_F + N_S. \tag{IA.577}$$

This is also the number of "lag traders," i.e., the number of traders that use their lagged signals.

7.1.2 Trading

At each $t \in (0, T]$, denote by Δx_t^i the market order submitted by speculator $i = 1, \ldots, N$ at t, and by Δu_t the market order submitted by the noise traders, which is of the form $\Delta u_t = \sigma_u \Delta B_t^u$, where B_t^u is a standard Brownian motion independent from all other variables. Then, the aggregate order flow executed by the dealer at t is:

$$\Delta y_t \;=\; \sum_{i=1}^{N} \Delta x_t^i + \Delta u_t. \tag{IA.578}$$

Because the dealer is risk-neutral and competitive, she executes the order flow at a price equal to her expectation of the liquidation value conditional on her information. Let $\mathcal{I}_t = \{y_\tau\}_{\tau < t}$ be the dealer's information set just before trading at t. Thus, the order flow at date t, Δy_t, executes at:

$$p_t \;=\; \mathsf{E}\big(v_T \mid \mathcal{I}_t \cup \Delta y_t\big). \tag{IA.579}$$

7.1.3 Equilibrium Definition

A trading strategy for an ℓ-speculator is a process for his position in the risky asset, x_t, measurable with respect to his information set $\mathcal{J}_t^{(\ell)} = \{y_\tau\}_{\tau<t} \cup \{s_\tau\}_{\tau \le t-\ell}$. Denote by E_t^ℓ the expectation of an ℓ-speculator, conditional on $\mathcal{J}_t^{(\ell)}$; and denote by E_t the expectation of the dealer, conditional on the public information:

$$\mathsf{E}_t^\ell(\,\cdot\,) \;=\; \mathsf{E}\big(\,\cdot \mid \mathcal{J}_t^{(\ell)}\big), \qquad \mathsf{E}_t(\,\cdot\,) \;=\; \mathsf{E}\big(\,\cdot \mid \mathcal{I}_t\big). \tag{IA.580}$$

For a given trading strategy, the speculator's expected profit π_τ, from date τ onwards, is:

$$\pi_\tau \;=\; \mathsf{E}_t^\ell\left(\sum_{t=\tau}^{T}(v_T - p_t)\Delta x_t\right). \tag{IA.581}$$

The unpredictable part of the lagged signal Δw_{t-1} at t is defined by:

$$\widetilde{\Delta w}_{t-1} \;=\; \Delta w_{t-1} - \mathsf{E}_t\big(\Delta w_{t-1}\big). \tag{IA.582}$$

As in continuous time, I consider only the trading strategies of the FTs and STs which are linear in the current and lagged signals, that is, only the trading strategies of the form:

$$\Delta x_t \;=\; \gamma_t \Delta w_t + \mu_t \widetilde{\Delta w}_{t-1}, \tag{IA.583}$$

where γ_t must be zero for the STs.

As in Kyle (1985), one can show that the dealer sets linear pricing rules of the form:

$$\Delta p_t \;=\; \lambda_t \Delta y_t, \qquad \mathsf{E}_t\big(\Delta w_{t-1}\big) \;=\; \rho_t \Delta y_{t-1}, \tag{IA.584}$$

where Δy_t is the total order flow at t.

A linear equilibrium is such that: (i) at every date t, each speculator's trading strategy (11) maximizes his expected trading profit (IA.581) given the dealer's pricing policy, and (ii) the dealer's pricing policy given by (IA.579) and (IA.584) is consistent with the equilibrium speculators' trading strategies.

As in continuous time, I simplify notation and normalize covariances and variances using the tilde notation. For instance:

$$\widetilde{\mathsf{Cov}}(\Delta w_t, \Delta w_t) \;=\; \frac{\mathsf{Cov}(\Delta w_t, \Delta w_t)}{\sigma_w^2 \Delta t} \;=\; 1. \tag{IA.585}$$

7.2 Equilibrium

The main result of this section, Theorem IA.4, reduces solving for the equilibrium of $\mathcal{D}_1$ to the solution of a discrete system of equations. First, I prove a lemma that computes the speculators' expected profit.

Lemma IA.2. *In the context of Theorem IA.4, the FT computes:*

$$\mathsf{E}\Big(w_t - p_{t-1} \,\Big|\, \mathcal{I}_t, \Delta w_t, \widetilde{\Delta w}_{t-1}\Big) \;=\; \Delta w_t + C_t \widetilde{\Delta w}_{t-1}; \tag{IA.586}$$

the coefficient C_t *is given by:*

$$C_t \;=\; \frac{B_t}{A_t}, \tag{IA.587}$$

where B_t*,* D_t*,* A_t *satisfy the following recursive formulas:*

$$\begin{aligned} B_{t+1} &= 1 - N_F \lambda_t \gamma_t^* - N_F \rho_t \gamma_t^* - \rho_t (N_F \mu_t^* + N_S \nu_t^*) B_t + \lambda_t \rho_t D_t, \\ D_{t+1} &= (N_F \gamma_{t+1}^*)^2 + (N_F \mu_{t+1}^* + N_S \nu_{t+1}^*)^2 A_{t+1} + \tilde{\sigma}_u^2, \\ A_{t+1} &= 1 - 2 N_F \rho_t \gamma_t^* + \rho_t^2 D_t, \end{aligned} \tag{IA.588}$$

and γ^**,* μ^**, and* ν^* *are the equilibrium values of the corresponding coefficients.*

The ST computes:

$$\mathsf{E}\Big(w_t - p_{t-1} \,\Big|\, \mathcal{I}_t, \widetilde{\Delta w}_{t-1}\Big) \;=\; C_t \widetilde{\Delta w}_{t-1}, \tag{IA.589}$$

with C_t *as above.*

Proof. Since all the variables involved are jointly multivariate normal, the conditional expectation in (IA.586) is of the form:

$$\mathsf{E}\Big(w_t - p_{t-1} \,\Big|\, \mathcal{I}_t, \Delta w_t, \widetilde{\Delta w}_{t-1}\Big) \;=\; c_{1,t} \Delta w_t + c_{2,t} \widetilde{\Delta w}_{t-1} + c_{0,t}, \quad \text{with} \quad c_{i,t} \in \mathcal{I}_t. \tag{IA.590}$$

Because $w_t - p_{t-1}$, Δw_t and $\widetilde{\Delta w}_{t-1}$ are orthogonal to $\mathcal{I}_t$, one obtains:

$$c_{0,t} \;=\; 0, \quad c_{1,t} \;=\; \frac{\mathsf{Cov}\big(w_t - p_{t-1}, \Delta w_t\big)}{\mathsf{Var}\big(\Delta w_t\big)}, \quad c_{2,t} \;=\; \frac{\mathsf{Cov}\big(w_t - p_{t-1}, \widetilde{\Delta w}_{t-1}\big)}{\mathsf{Var}\big(\widetilde{\Delta w}_{t-1}\big)}. \tag{IA.591}$$

Since $p_{t-1} \in \mathcal{I}_t$, one has $c_{1,t} = 1$. Denote:

$$B_t \;=\; \widetilde{\mathsf{Cov}}\big(w_t - p_{t-1}, \widetilde{\Delta w}_{t-1}\big), \quad D_t \;=\; \widetilde{\mathsf{Var}}\big(\Delta y_t\big), \quad A_t \;=\; \widetilde{\mathsf{Var}}\big(\widetilde{\Delta w}_{t-1}\big). \tag{IA.592}$$

I now give a recursive formula for:

$$C_t \;=\; c_{2,t} \;=\; \frac{B_t}{A_t}. \tag{IA.593}$$

The aggregate order flow has the form:

$$\Delta y_t \;=\; N_F \gamma_t^* \Delta w_t + (N_F \mu_t^* + N_S \nu_t^*) \widetilde{\Delta w}_{t-1} + \Delta u_t. \tag{IA.594}$$

Therefore, one computes ($\tilde{\sigma}_u^2 = \frac{\sigma_u^2}{\sigma_w^2}$):

$$\begin{aligned} A_{t+1} &= \widetilde{\mathsf{Var}}\big(\Delta w_t - \rho_t \Delta y_t\big) \;=\; 1 - 2N_F \rho_t \gamma_t + \rho_t^2 D_t \\ D_{t+1} &= (N_F \gamma_{t+1}^*)^2 + (N_F \mu_{t+1}^* + N_S \nu_{t+1}^*)^2 A_{t+1} + \tilde{\sigma}_u^2, \\ B_{t+1} &= \widetilde{\mathsf{Cov}}\big(w_t - p_{t-1} - \lambda_t \Delta y_t, \Delta w_t - \rho_t \Delta y_t\big) \\ &= 1 - \rho_t N_F \gamma_t^* - \rho_t (N_F \mu_t^* + N_S \nu_t^*) B_t - \lambda_t N_F \gamma_t^* + \lambda_t \rho_t D_t. \end{aligned} \tag{IA.595}$$

These are the desired formulas. Finally, for the ST one has the same computation as for the FT. ■

I now state the main result of this section.

Theorem IA.4. *Consider the discrete model with N_F fast traders and N_S slow traders, and let $N_L = N_F + N_S$. Then, the equilibrium reduces to the following system of equations:*

$$\begin{aligned} a_t &= N_F \rho_t \gamma_t, \qquad b_t \;=\; N_L \rho_t \mu_t, \qquad R_t \;=\; \frac{\lambda_t}{\rho_t}, \\ a_t &= \frac{1 - 2\alpha_t \rho_t}{\frac{N_F+1}{N_F} R_t - 2\alpha_t \rho_t}, \qquad b_t \;=\; \frac{C_t}{\frac{N_L+1}{N_L} R_t - 2\alpha_t \rho_t}, \\ C_t &= \frac{B_t}{1 - a_{t-1}}, \qquad a_t \;=\; a_t^2 + b_t^2 (1 - a_{t-1}) + \rho_t^2 \tilde{\sigma}_u^2, \\ R_t &= 1 + \frac{b_t B_t}{a_t}, \qquad B_{t+1} \;=\; 1 - a_t - b_t B_t \;=\; 1 - R_t a_t, \\ \alpha_{t-1} &= b_t^2 \left(\frac{R_t}{N_L^2 \rho_t} + \left(1 - \frac{2}{N_L}\, \alpha_t\right) \right). \end{aligned} \tag{IA.596}$$

Proof. I start by computing the speculators' optimal strategies, taking the dealer's pricing rules as given. Then, I derive the dealer's pricing rules taking the speculator's optimal strategies as given. Finally, I put together the equilibrium conditions to determine the system of equations satisfied by the equilibrium coefficients.

Speculators' optimal strategies

I now proceed with computing the FT's value function. Denote:

$$\mathsf{E}^F_t(X) \;=\; \mathsf{E}(X|\mathcal{I}_t, \Delta w_t, \widetilde{\Delta w}_{t-1}), \tag{IA.597}$$

the expectation from the point of view of the FT at t. Then, the FT's value function at t satisfies the Bellman equation:

$$V^F_t \;=\; \max_{\Delta x}\Big(\mathsf{E}^F_t\big((v_T - p_t)\Delta x\big) \;+\; V^F_{t+1}\Big). \tag{IA.598}$$

As in the general case, I conjecture a value function for the FT that is quadratic in the current signals:

$$V^F_t \;=\; \alpha^0_{t-1} + \alpha_{t-1}\big(\widetilde{\Delta w}_{t-1}\big)^2 + 2\alpha'_{t-1}\big(\widetilde{\Delta w}_{t-1}\big)\big(\Delta w_t\big) + \alpha''_{t-1}\big(\Delta w_t\big)^2. \tag{IA.599}$$

Then, the Bellman equation becomes:

$$\begin{aligned} V^F_t \;=\; & \max_{\Delta x} \mathsf{E}^F_t \Bigg(\big(w_t - p_{t-1} - \lambda_t \Delta y_t\big)\Delta x \\ & + \alpha^0_t + \alpha_t\big(\Delta w_t - \rho_t \Delta y_t\big)^2 + 2\alpha'_t\big(\Delta w_t - \rho_t\Delta y_t\big)\big(\Delta w_{t+1}\big) + \alpha''_t\big(\Delta w_{t+1}\big)^2\Bigg), \end{aligned} \tag{IA.600}$$

where Δy_t is assumed by the FT to be of the form:

$$\Delta y_t \;=\; \Delta x + (N_F - 1)\gamma^*_t \Delta w_t + \big((N_F-1)\mu^*_t + N_S\nu^*_t\big)\widetilde{\Delta w}_{t-1} + \Delta u_t. \tag{IA.601}$$

From equation (IA.586), one computes $\mathsf{E}^F_t\big(w_t - p_{t-1}\big) = \Delta w_t + C_t\widetilde{\Delta w}_{t-1}$, with C_t satisfying certain equations described in Lemma IA.2 above. Therefore:

$$\begin{aligned} V^F_t \;=\; & \max_{\Delta x} \mathsf{E}^F_t \Bigg(\big(\Delta w_t + C_t\widetilde{\Delta w}_{t-1} - \lambda_t\Delta y_t\big)\Delta x + \\ & \alpha^0_t + \alpha_t\big(\Delta w_t - \rho_t\Delta y_t\big)^2 + \alpha''_t\sigma^2_w\Delta t\Bigg), \end{aligned} \tag{IA.602}$$

The terms can be rearranged:

$$V^F_t \;=\; \max_{\Delta x}\big(W_1 - \lambda_t\Delta x\big) + \alpha_t\big(W_2 - \rho_t\Delta x)^2 + Z, \tag{IA.603}$$

where:

$$
\begin{aligned}
W_1 &= W_{11}\Delta w_t + W_{12}\widetilde{\Delta w}_{t-1}, \quad \text{with} \\
& \qquad W_{11} \;=\; 1-(N_F-1)\lambda_t\gamma_t^*, \qquad W_{12} \;=\; C_t - \lambda_t\big((N_F-1)\mu_t^* + N_S\nu_t^*\big), \\
W_2 &= W_{21}\Delta w_t + W_{22}\widetilde{\Delta w}_{t-1}, \quad \text{with} \\
& \qquad W_{21} \;=\; 1-(N_F-1)\rho_t\gamma_t^*, \qquad W_{22} \;=\; -\rho_t\big((N_F-1)\mu_t^* + N_S\nu_t^*\big), \\
Z &= \alpha_t^0 + \alpha_t''\sigma_w^2\Delta t + \alpha_t\sigma_u^2\Delta t.
\end{aligned}
\tag{IA.604}
$$

The first order condition with respect to Δx is:

$$
W_1 - 2\lambda_t\Delta x - 2\alpha_t\rho_t(W_2 - \rho_t\Delta x) \;=\; 0. \tag{IA.605}
$$

Denote:

$$
\tilde{\lambda}_t \;=\; \lambda_t - \alpha_t\rho_t^2. \tag{IA.606}
$$

Then, the first order condition implies

$$
\Delta x \;=\; \frac{W_1 - 2\alpha_t\rho_t W_2}{2\tilde{\lambda}_t} \;=\; \frac{W_{11} - 2\alpha_t\rho_t W_{21}}{2\tilde{\lambda}_t}\Delta w_t + \frac{W_{21} - 2\alpha_t\rho_t W_{22}}{2\tilde{\lambda}_t}\widetilde{\Delta w}_{t-1}. \tag{IA.607}
$$

The second order condition for a maximum is:

$$
\tilde{\lambda}_t \;>\; 0. \tag{IA.608}
$$

By identifying the coefficients of V_t^F, one obtains:

$$
\begin{aligned}
\alpha_{t-1}^0 &= Z, \\
\alpha_{t-1} &= \frac{\big(W_{12} - 2\alpha_t\rho_t W_{22}\big)^2}{4\tilde{\lambda}_t} + \alpha_t W_{22}^2, \\
\alpha_{t-1}' &= \frac{\big(W_{11} - 2\alpha_t\rho_t W_{21}\big)\big(W_{12} - 2\alpha_t\rho_t W_{22}\big)}{4\tilde{\lambda}_t} + \alpha_t W_{11}W_{21}, \\
\alpha_{t-1}'' &= \frac{\big(W_{11} - 2\alpha_t\rho_t W_{21}\big)^2}{4\tilde{\lambda}_t} + \alpha_t W_{11}^2.
\end{aligned}
\tag{IA.609}
$$

Note that only α_t is involved in a recursive equation, while all the other coefficients can be computed using α_t (and the equilibrium coefficients). I write the equation for α_t

more explicitly:

$$\alpha_{t-1} \;=\; \frac{\Big(C_t - \big(\lambda_t - 2\alpha_t\rho_t^2\big)\big((N_F-1)\mu_t^* + N_S\nu_t^*\big)\Big)^2}{4\tilde{\lambda}_t} + \alpha_t\rho_t^2\big((N_F-1)\mu_t^* + N_S\nu_t^*\big)^2. \tag{IA.610}$$

From (IA.607), one obtains the equations for the coefficients γ_t and μ_t for the FT:

$$\begin{aligned}\gamma_t &= \frac{1 - 2\alpha_t\rho_t - \big(\lambda_t - 2\alpha_t\rho_t^2\big)(N_F-1)\gamma_t^*}{2\tilde{\lambda}_t},\\ \mu_t &= \frac{C_t - \big(\lambda_t - 2\alpha_t\rho_t^2\big)\big((N_F-1)\mu_t^* + N_S\nu_t^*\big)}{2\tilde{\lambda}_t}.\end{aligned} \tag{IA.611}$$

I now proceed in a similar way to compute the ST's value function. Denote by $\mathsf{E}_t^S(X) = \mathsf{E}(X|\mathcal{I}_t, \widetilde{\Delta w}_{t-1})$, the expectation from the point of view of the ST at t. Then, the ST's value function at t satisfies the Bellman equation:

$$V_t^S \;=\; \max_{\Delta x}\Big(\mathsf{E}_t^S\big((v_T - p_t)\Delta x\big) \;+\; V_{t+1}^S\Big). \tag{IA.612}$$

I conjecture a value function for the ST that is quadratic in the current signal:

$$V_t^F \;=\; \beta_{t-1}^0 + \beta_{t-1}\big(\widetilde{\Delta w}_{t-1}\big)^2. \tag{IA.613}$$

With a similar computation as for the FT, β_t satisfies the recursive equation:

$$\beta_{t-1} \;=\; \frac{\Big(C_t - \big(\lambda_t - 2\beta_t\rho_t^2\big)\big(N_F\mu_t^* + (N_S-1)\nu_t^*\big)\Big)^2}{4\lambda_t'} + \beta_t\rho_t^2\big(N_F\mu_t^* + (N_S-1)\nu_t^*\big)^2, \tag{IA.614}$$

where $\lambda_t' = \lambda_t - \beta_t\rho_t^2$. One also obtains:

$$\nu_t \;=\; \frac{C_t - \big(\lambda_t - 2\beta_t\rho_t^2\big)\big(N_F\mu_t^* + (N_S-1)\nu_t^*\big)}{2\lambda_t'}. \tag{IA.615}$$

Note that, if $\mu_t = \nu_t$, then α_t and β_t satisfy the same equation. Thus, I search for an equilibrium in which:

$$\mu_t \;=\; \nu_t, \qquad \alpha_t \;=\; \beta_t. \tag{IA.616}$$

In that equilibrium, $\gamma_t = \gamma_t^*$, $\mu_t = \nu_t = \mu_t^* = \nu_t^*$, and $\alpha_t = \beta_t$. One obtains the following equations:

$$
\begin{aligned}
\gamma_t &= \frac{1 - 2\alpha_t \rho_t}{(N_F + 1)\lambda_t - 2N_F \alpha_t \rho_t^2}, \\
\mu_t &= \frac{C_t}{(N_L + 1)\lambda_t - 2N_L \alpha_t \rho_t^2}, \\
\alpha_{t-1} &= \mu_t^2 \Big(\lambda_t + (N_L^2 - 2N_L)\alpha_t \rho_t^2 \Big).
\end{aligned}
\tag{IA.617}
$$

Dealer's pricing rules

The dealer takes the speculator's strategies as given, which means that she assumes the aggreate order flow to be of the form:

$$
\Delta y_t = N_F \gamma_t \Delta w_t + N_L \mu_t \widetilde{\Delta w}_{t-1} + \mathrm{d}u_t. \tag{IA.618}
$$

Therefore, she sets λ_t and ρ_t according to the usual formulas:

$$
\begin{aligned}
\lambda_t &= \frac{\mathsf{Cov}_t(v_T, \Delta y_t)}{\mathsf{Var}_t(\Delta y_t)} = \frac{\widetilde{\mathsf{Cov}}\big(w_t - p_{t-1}, \Delta y_t\big)}{\widetilde{\mathsf{Var}}(\Delta y_t)} = \frac{N_F \gamma_t + N_L \mu_t C_t}{D_t}, \\
\rho_t &= \frac{\mathsf{Cov}_t(\Delta v_t, \Delta y_t)}{\mathsf{Var}_t(\Delta y_t)} = \frac{\widetilde{\mathsf{Cov}}\big(\Delta w_t, \Delta y_t\big)}{\widetilde{\mathsf{Var}}(\Delta y_t)} = \frac{N_F \gamma_t}{D_t}.
\end{aligned}
\tag{IA.619}
$$

I now rewrite the equations from Lemma IA.2 above, using the equation I derived above: $\rho_t D_t = N_F \gamma_t$:

$$
\begin{aligned}
B_{t+1} &= 1 - N_F \rho_t \gamma_t - N_L \rho_t \mu_t B_t, \\
A_{t+1} &= 1 - N_F \rho_t \gamma_t, \\
D_{t+1} &= (N_F \gamma_{t+1})^2 + (N_L \mu_{t+1})^2 (1 - N_F \rho_t \gamma_t) + \tilde{\sigma}_u^2, \\
C_t &= \frac{B_t}{A_t} = \frac{B_t}{1 - N_F \rho_{t-1} \gamma_{t-1}}.
\end{aligned}
\tag{IA.620}
$$

Equilibrium conditions

To mirror the continuous-time version of the model, I define the following variables:

$$
a_t = N_F \rho_t \gamma_t, \qquad b_t = N_L \rho_t \mu_t, \qquad R_t = \frac{\lambda_t}{\rho_t}. \tag{IA.621}
$$

From $\rho D_t = N_F\gamma_t$, one obtains $N_F\rho_t\gamma_t = \rho_t^2 D_t = (N_F\rho_t\gamma_t)^2+(N_L\rho_t\mu_t)^2(1-N_F\rho_{t-1}\gamma_{t-1})+\rho^2\tilde{\sigma}_u^2$. With the new notation, this equation becomes:

$$a_t \;=\; a_t^2 + b_t^2(1-a_{t-1}) + \rho_t^2\tilde{\sigma}_u^2. \tag{IA.622}$$

Also, one computes:

$$R_t \;=\; \frac{\lambda_t}{\rho_t} \;=\; \frac{a_t + b_t B_t}{\rho^2 D_t} \;=\; \frac{a_t + b_t B_t}{a_t} \;=\; 1 + \frac{b_t B_t}{a_t}. \tag{IA.623}$$

I put together the equations that determine the equilibrium:

$$\begin{aligned}
a_t &= N_F\rho_t\gamma_t, \qquad b_t \;=\; N_L\rho_t\mu_t, \qquad R_t \;=\; \frac{\lambda_t}{\rho_t},\\
a_t &= \frac{1-2\alpha_t\rho_t}{\frac{N_F+1}{N_F}R_t - 2\alpha_t\rho_t}, \qquad b_t \;=\; \frac{C_t}{\frac{N_L+1}{N_L}R_t - 2\alpha_t\rho_t},\\
C_t &= \frac{B_t}{1-a_{t-1}}, \qquad a_t \;=\; a_t^2 + b_t^2(1-a_{t-1}) + \rho_t^2\tilde{\sigma}_u^2,\\
R_t &= 1+\frac{b_t B_t}{a_t}, \qquad B_{t+1} \;=\; 1 - a_t - b_t B_t \;=\; 1 - R_t a_t,\\
\alpha_{t-1} &= b_t^2\left(\frac{R_t}{N_L^2\rho_t} + \left(1-\frac{2}{N_L}\,\alpha_t\right)\right).
\end{aligned} \tag{IA.624}$$

This proves (IA.596). ■

7.3 Numerical results

Theorem IA.4 show that finding the equilibrium reduces to solving a discrete system of equations:

$$\begin{aligned}
a_t &= N_F\rho_t\gamma_t, \qquad b_t \;=\; N_L\rho_t\mu_t, \qquad R_t \;=\; \frac{\lambda_t}{\rho_t},\\
a_t &= \frac{1-2\alpha_t\rho_t}{\frac{N_F+1}{N_F}R_t - 2\alpha_t\rho_t}, \qquad b_t \;=\; \frac{C_t}{\frac{N_L+1}{N_L}R_t - 2\alpha_t\rho_t},\\
C_t &= \frac{B_t}{1-a_{t-1}}, \qquad a_t \;=\; a_t^2 + b_t^2(1-a_{t-1}) + \rho_t^2\tilde{\sigma}_u^2,\\
R_t &= 1+\frac{b_t B_t}{a_t}, \qquad B_{t+1} \;=\; 1 - a_t - b_t B_t \;=\; 1 - R_t a_t,\\
\alpha_{t-1} &= b_t^2\left(\frac{R_t}{N_L^2\rho_t} + \left(1-\frac{2}{N_L}\,\alpha_t\right)\right).
\end{aligned} \tag{IA.625}$$

This system can be solved numerically. For all the parameter values considered, the solutions are numerically very close to a constant, except when t is either close to 0, or close to T. This suggests that it is a good idea to analyze the behavior of these coefficients when the number of trading rounds becomes large. In this continuous-time limit, using Lemma IA.1 in the Appendix in the paper, one expects that all these coefficients become constant.

Therefore, I consider a constant solution of (IA.625) with all coefficients constant. For instance, from the recursive equation for B, one has $B = \frac{1-a}{1+b}$, which coincides with the value of B from the continuous-time version. One obtains the following equations:

$$\begin{aligned} B \;&=\; \frac{1-a}{1+b}, \quad C \;=\; \frac{1}{1+b}, \quad R \;=\; \frac{a+b}{a(1+b)}, \quad \rho^2\tilde{\sigma}_u^2 \;=\; (a-b^2)(1-a), \\ \alpha \;&=\; \frac{b^2 R}{N_L^2 \rho} + b^2\left(1 - \frac{2}{N_L}\right)\alpha, \quad a \;=\; \frac{1-2\alpha\rho}{\frac{N_F+1}{N_F}R - 2\alpha\rho}, \quad b \;=\; \frac{C}{\frac{N_L+1}{N_L}R - 2\alpha\rho}. \end{aligned} \tag{IA.626}$$

Solving the equation for α, and multiplying by 2ρ, one obtains:

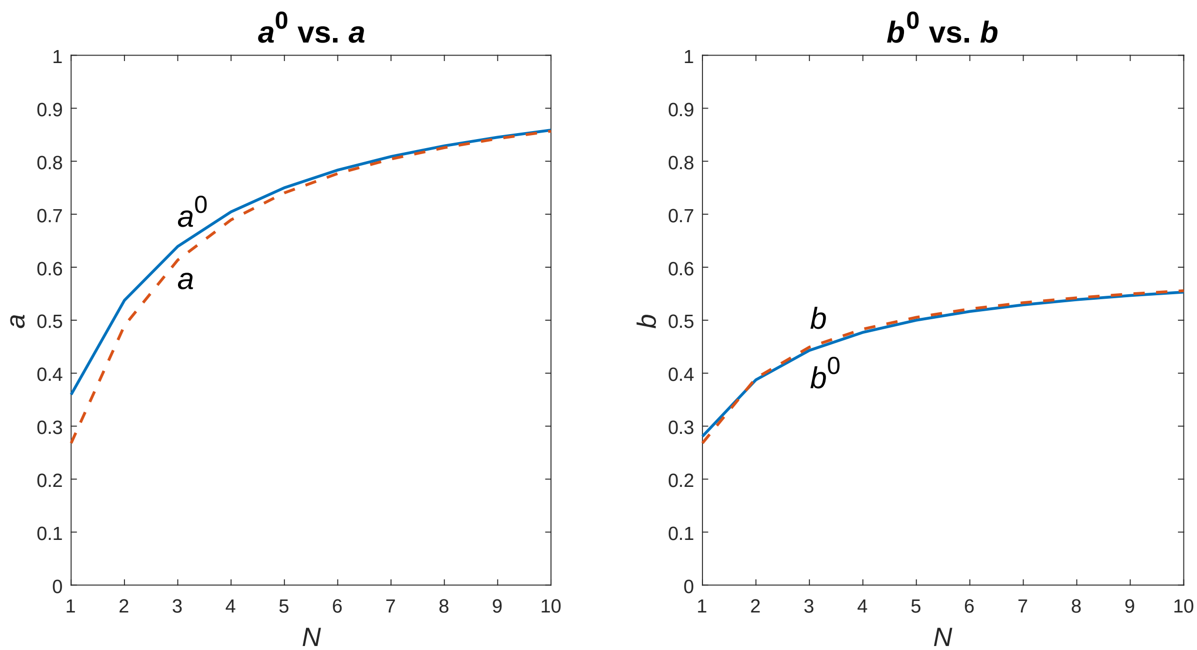
$$2\alpha\rho \;=\; \frac{2b^2 R}{1 - b^2\left(1 - \frac{2}{N_L}\right)} \, \frac{1}{N_L^2}. \tag{IA.627}$$

Note that the first four equations in (IA.626) coincide with the corresponding ones in the continuous-time case. However, the last two equations (for a and b) differ from the continuous-time value by the term $2\alpha\rho$. But all the terms in (IA.627), other than $\frac{1}{N_L^2}$, are of order one, hence the term $2\alpha\rho$ is of the order of $\frac{1}{N_L^2}$:

$$2\alpha\rho \;=\; O_{N_L}\left(\tfrac{1}{N_L^2}\right). \tag{IA.628}$$

I now describe a numerical procedure that computes with high accuracy a solution (a, b) of the system above. Denote by a^0 and b^0 the corresponding equilibrium values from Theorem 1, in which the choice of weights does not affect the covariance structure. Then, starting with (a^0, b^0), one computes $R^0 = \frac{a^0+b^0}{a^0(1+b^0)}$, and then $2\alpha^0\rho^0 = \frac{2(b^0)^2 R^0}{1-(b^0)^2\left(1-\frac{2}{N_L}\right)} \frac{1}{N_L^2}$ using (IA.627). Using (IA.626), one recomputes the values of (a, b). Denote them by (a^1, b^1). Iterate the procedure until it converges. Then, define $(a, b) = \lim_{n\to\infty}(a^n, b^n)$. Then, (a, b) satisfy the system of equations in (IA.626). Figure IA.5 shows the solution for the case when there are only FTs, and their number is $N \in \{1, \ldots, 10\}$. (The introduction of STs makes the approximation even better, since $N_L = N_F + N_S$ increases.) From the figure, one sees that the approximation is good even for low N.

Figure IA.5: Comparison of equilibrium weights The figure compares the equilibrium modified weights a and b, which are a solution of the system of equations (IA.626), with the modified weights a^0 and b^0 from the benchmark model, in which the choice of weights does not affect the covariance structure (see Theorem 1 in the paper). In each model, there are $N = 1, 2, \ldots, 10$ identical speculators.



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