# Internet Appendix for "Liquidity and Information in Limit Order Markets"

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This document includes supplementary material to the paper. In Section 1, I finish the proof of Theorem 1 in the paper, by describing in detail the equilibrium behavior of the uninformed traders. In Section 2, I endogenize the decisions of the impatient traders in the model, and show that these traders' optimal behavior coincides with the behavior assumed in the paper. In Section 3, I model explicitly the traders' information acquisition decision, and analyze the welfare implications of a reduction in the information acquisition cost. In Section 4, I define a numerical Monte Carlo procedure which estimates the information function $I$ from Definition 1 in the paper, and I verify numerically the properties of $I$ which are used in the paper. In Section 5, I introduce several extensions of the benchmark model discussed in the paper. In Section 6, I provide proofs for the various extensions of the benchmark model in the paper, and further discuss the robustness of the main results. In Section 7, I examine stationary filtering, in which the public volatility is constant over time.

# Contents

# 1 Optimal Strategies of Uninformed Patient Traders

In this section, I finish the proof of the part of Theorem 1 in the paper that describes the optimal behavior of uninformed patient traders. For that, I first prove Lemma A3 in the paper, which states that the continuation payoff of the uninformed natural buyer is:

$$\mathcal{U}^U_{\text{BLO}} \;=\; \bar{u} + \frac{S}{2} - \Delta. \tag{IA.1}$$

**Proof of Lemma A3**. I simplify notation and assume that the initial BLO is submitted at $t = 0$. In Section II in the paper, I assume that the initial belief of an uninformed trader with private valuation $\bar{u}$ is such that after submitting a BLO at $t = 0$, his posterior belief at $t = 1$ is the public density, $\psi_1 = \mathcal{N}\big(\mu_1, V^2\big)$. Formally, this is done by assuming that before trading at $t = 0$, the uninformed trader believes $v_0$ to be distributed as follows:[1]

$$v_0 \;\sim\; \mathcal{N}\big(\mu_0 + \gamma\Delta, V^2 - \sigma_I^2\big). \tag{IA.2}$$

By submitting his BLO at $t = 0$, he instantly affects the public mean according to $\mu_1 = \mu_0 + \gamma\Delta$. His belief about $v_0$, however, stays the same, since he knows he is uninformed. The asset value evolves according $v_1 = v_0 + (v_1 - v_0)$, with the increment normally distributed according to $\mathcal{N}(0, \sigma_I^2)$. Therefore, at $t = 1$ he believes $v_1 \sim \mathcal{N}(\mu_1, (V^2 - \sigma_I^2) + \sigma_I^2)$, which is the public density at $t = 1$.

As in Lemma A2 in the Appendix of the paper, denote by $\mathcal{Q}$ the set of all execution sequences $Q = \big(\mathcal{O}_0 = \text{BLO}, \mathcal{O}_1, \ldots, \mathcal{O}_{T-1}, \mathcal{O}_T = \text{SMO}\big)$ for the initial BLO. At the execution time $T$, the bid price is $\mu_T - S/2$, therefore:

$$\mathcal{U}^U_{\text{BLO}} \;=\; \bar{u} + \frac{S}{2}\sum_{Q\in\mathcal{Q}} P_0(Q) + \sum_{Q\in\mathcal{Q}} \mathsf{E}_0\Big(v_T - \mu_T \mid Q\Big)\, P_0(Q), \tag{IA.3}$$

where $\mathsf{E}_t$ is the expectation operator conditional on $\mathcal{I}_t$, the public information set just before trading at $t$; and $P_0(Q)$ is the probability that the sequence $Q$ will occur conditional on $\mathcal{I}_0$. Denote by $e(Q) = \mathsf{E}_0\big(v_T - \mu_T \mid Q\big)$. The executing order at $T$ is an SMO, therefore $e(Q) = \mathsf{E}_0\,\mathsf{E}_T\big(v_T - \mu_T \mid \text{SMO}_T\big)$. Equation (A-4) in the Appendix of the paper implies $\mathsf{E}_T\big(w_T \mid \text{SMO}_T\big) = \mathsf{E}_T\big(w_T\big) + \delta_{\text{SMO}} = \mathsf{E}_T\big(\frac{v_T - \mu_T}{V}\big) - \frac{\Delta}{V}$. Thus, $e(Q) = \mathsf{E}_0\,\mathsf{E}_T\big(v_T - \mu_T\big) - \Delta$. But $\mathsf{E}_T(v_T) = \mu_T$, therefore $e(Q) = -\Delta$.

[1]Equation (13) in the paper implies that $V^2 - \sigma_I^2 > 0$.

To finish the proof, I show that $\sum_{Q\in\mathcal{Q}} P_0(Q) = 1$, which means that the initial BLO is executed with probability 1. I use the theory of absorbing Markov chains, as described in Chapter 11 of Grinstead and Snell (2003). I briefly indicate the proof. Consider the following Markov chain with a countable number of states, where the state $j \geq 0$ indicates that the initial BLO is $j$th in the bid queue. The absorbing state $j = 0$ means that the BLO is executed. Then, from each state $j \geq 1$ the system moves to either $j - 1$ with probability 1/4 (if SMO occurs), to $j + 1$ with probability 1/4 (if BLO occurs), or remains in $j$ with probability 1/2 (if either BMO or SLO occurs). One can then check that the fundamental matrix $M$ (corresponding to the non-absorbing states) has entry $(i, j)$ given by $M_{ij} = 4 \min\{i, j\}$, for $i, j \geq 1$. The matrix of transition to the absorbing state is the column matrix $R$, whose $j$th entry is either 1/4 if $j = 1$, or 0 if $j > 1$. Theorem 11.6 from Grinstead and Snell (2003) shows that the probability of absorption (execution) starting from state $j$ is the $j$th entry in the column matrix $B = MR$. But $B$ has all entries equal to 1. This completes the proof of (IA.1).

I finish the discussion by analyzing the payoff of the uninformed trader in state $j \geq 0$, when the BLO is $j$th in the bid queue. By using the same argument as above, it follows that his payoff is affected only by the amount of adverse selection at the time $T$ when his order is executed. Thus, his payoff is the same regardless of $j$. This is not surprising, since the uninformed trader has a zero waiting cost. $\square$

I now finish the part of the proof of Theorem 1 in the paper that deals with uninformed patient traders.

**Proof of Theorem 1**. To simplify the description of the strategies, I use the one-stage deviation principle of subgame perfection (see Fudenberg and Tirole 1991, Section 4.2).[2] This principle implies that one need not define the strategy profile for all conceivable histories, but only for the histories that arise from at most a finite number of (out-of-equilibrium) deviations, assuming that all other players than the transgressor act according to their strategies at the time of the deviation. Thus, this principle restricts the values that certain state variables can take.

**Remark IA.1.** By the one-stage deviation principle of subgame perfection, I assume that

[2]The principle states that in order to verify that an equilibrium is subgame perfect, it suffices to check whether there is any history $h^t$ where some player $i$ can gain by deviating from the action prescribed by his strategy at $h^t$ and conforming to it thereafter—assuming all other players follow their strategies.

at all times (integer or not) the public volatility is equal to the parameter $V$.[3] At all integer times $t = 0, 1, \ldots$, the ask price is $\mu_t + S/2$ and the bid price is $\mu_t - S/2$.[4]

I now describe the complete strategy profile $\mathcal{S}$, this time including the reaction to out-of-equilibrium behavior:

(a) The uninformed buyer arriving at $t$ submits a BLO at the price $(\mu_t + \gamma\Delta) - S/2$.

(b) The uninformed buyer arriving at $t$ submits an SLO at the price $(\mu_t - \gamma\Delta) + S/2$.

(c) The informed trader who observes an asset value $v_t$ when she arrives at $t$ submits an order $\mathcal{O} \in \{\text{BMO}, \text{BLO}, \text{SLO}, \text{SMO}\}$ whenever her signal $\frac{v_t - \mu_t}{V}$ lies, respectively, in the interval $\{(\alpha, \infty), (0, \alpha), (-\alpha, 0), (-\infty, -\alpha)\}$.

(d) After the initial order submission, all traders behave as described in (e) and (f).

(e) If a BMO is submitted at $t$, then an instant later the public mean is updated to $\mu_t + \Delta$, the ask price to $\mu_t + \Delta + S/2$, and the bid price to $\mu_t + \Delta - S/2$, and all other limit traders shift their orders by $\Delta$ such that the relative ranks in the ask and bid queues are preserved. After that, no other trader moves until $t + 1$.

    The reaction to an SMO at $t$ is symmetric to the reaction to a BMO.

(f) If a BLO is submitted at $t$, then an instant later the public mean is updated to $\mu_t + \gamma\Delta$, the ask price to $\mu_t + \gamma\Delta + S/2$, and the bid price to $\mu_t + \gamma\Delta - S/2$, and all other limit traders shift their orders by $\gamma\Delta$ such that the relative ranks in the ask and bid queues are preserved. After that, no other trader moves until $t + 1$.

    The reaction to an SLO at $t$ is symmetric to the reaction to a BLO.

(g) (Out-of-equilibrium behavior) If a trader submits a limit order of different type than specified by his equilibrium strategy, then immediately he switches to a limit order of the type specified in (a), (b) or (c). Such a limit order switch does not reveal any new information about the transgressor's type.

(h) (Out-of-equilibrium behavior) If a limit order trader on the bid side deviates from the behavior above and instead of $b^*$ submits an order at $b = b^* + d$, then

[3] At $t = 0$ this is true by construction. Later, this is true both in equilibrium (see part (d) of Lemma A1 in the paper), and out-of-equilibrium, since the public density after any deviation remains equal to $V$ (see $\mathcal{S}$(g) and $\mathcal{S}$(h)).

[4] Indeed, even after deviations that modify the bid–ask spread, the instant reactions of the other traders would restore the bid and ask prices at the correct values (see $\mathcal{S}$(g)).

- If $d < 0$, the other traders do not infer any new information about the transgressor's type, and no state variables change, unless the transgressor is the first in the bid queue, in which case another limit buyer modifies his BLO at $b$.
- If $d > 0$, an instant later the other traders believe the transgressor is informed with probability 1. As a result, the transgressor's private information is revealed, and the public mean is updated to $\mu_t + d'$, with $d' > d$, while the public volatility remains $V$.[5] The transgressor remains at $b$, and the rest of the book shifts up by either $d$ or $d'$, such that the relative ranks in the bid and ask queues are the same as before the deviation.

The case when the transgressor is on the ask side is symmetric.

The belief system is described by the following rules: At $t = 0$, the uninformed investors perceive the asset value distributed according to $\mathcal{N}(0, V^2)$. Subsequently, the uninformed investors' belief about the asset value (the public density) is updated using the approximate Bayes' rule described in Section II in the paper. At the time of arrival to the market, the informed investor observes the asset value and can compute the average payoff of a limit order based on updating her belief according to the exact Bayes' rule. After the arrival, however, the informed trader cannot update her belief, and becomes essentially uninformed. In the limit order book at $t = 0$, all traders are uninformed with probability 1. At $t \geq 0$, each new trader is believed to be informed with probability $\rho$ by the other traders. Subsequently, traders' beliefs about the other investors' types are updated according to the Bayes' rule.

### Uninformed Traders

I analyze only the strategy of an uninformed buyer, since the proof for an uninformed seller is symmetric. I therefore show that the strategy of an uninformed buyer described in $\mathcal{S}$(a), $\mathcal{S}$(e)–$\mathcal{S}$(h) is optimal. Consider a (patient) uninformed buyer, with private valuation $\bar{u} > S/2$. If he arrives at $t$, I first analyze his choice among one of the following options: (i) BMO, (ii) SMO, (iii) NO (no order), (iv) BLO at $b^* = (\mu_t + \gamma\Delta) - S/2$, and later follow

[5] I also require that $d'$ is a one-to-one function of $d$. The exact value of $d' > d$ is not important, but because the transgressor gets price priority at $b = b^* + d$, some limit orders need to be shifted by more than $d$ to preserve the relative ranks in the bid and ask queue. An out-of-equilibrium belief can be arbitrary, but it should be about the transgressor's type rather than directly about the resulting public belief on $v$. Thus, the existing investors regard the transgressor as informed with probability 1, with information $v_t \sim \mathcal{N}(\mu_t + d', V^2 - \sigma_I^2)$. (Note that equation (13) in the paper implies $V > \sigma_I$.) Since $d$ is observed by the public and $d'$ is a one-to-one function of $d$, the transgressor's private information is fully revealed to the public. Thus, the public at $t$ also has the belief $\mathcal{N}(\mu_t + d', V^2 - \sigma_I^2)$. Because the asset value has a normal increment with density $\mathcal{N}(0, \sigma_I^2)$, the public density at $t + 1$ becomes $\mathcal{N}(\mu_t + d', V^2)$.

$\mathcal{S}$, or (v) SLO at $a^* = (\mu - \gamma\Delta) + S/2$, and later follow $\mathcal{S}$. I can rule out option (v), since it is essentially equivalent to option (iv): $\mathcal{S}$(g) requires that the uninformed buyer immediately reverses his SLO to a BLO, while the other traders do not infer any new information regarding the transgressor's type.

I first show that, as specified by $\mathcal{S}$, the option (iv) dominates all the other options when the limit prices in (iv) have the equilibrium values, $b^*$ and $a^*$, respectively. Then, I show that option (iv) is less profitable when the limit prices have different values. Moreover, I show that option (v) is less profitable when the SLO is switched to BLO after a lag—the proof presented below only works for an infinite lag (the SLO is never switched to BLO), but the argument is similar for a finite lag. In conclusion, the uninformed buyer always submits a BLO at the equilibrium price, which proves that $\mathcal{S}$(a) is optimal. Moreover, I rule out subsequent one-stage deviations after (iv) or (v) are chosen, which proves that $\mathcal{S}$(e) and $\mathcal{S}$(f) are optimal for the uninformed trader (as well as $\mathcal{S}$(g)). Finally, if another trader later deviates by submitting a limit order at a non-equilibrium price, then the uninformed buyer optimally reacts as specified in $\mathcal{S}$(h).

Let $\mathcal{U}^U_{\mathcal{O}}$ be the expected payoff from submitting $\mathcal{O} \in \{\mathrm{BMO}, \mathrm{BLO}, \mathrm{NO}, \mathrm{SLO}, \mathrm{SMO}\}$ and later following $\mathcal{S}$, except that in the case of $\mathcal{O}^U_{\mathrm{SLO}}$, I assume that the SLO is not switched to BLO. As explained in Remark IA.1, I assume that the current limit order book is such that the ask price is $a_t = \mu_t + S/2$, and the bid price is $b_t = \mu_t - S/2$. I show that $\mathcal{U}^U_{\mathrm{BLO}} > \mathcal{U}^U_{\mathrm{BMO}} > \mathcal{U}^U_{\mathrm{NO}} > \mathcal{U}^U_{\mathrm{SLO}} > \mathcal{U}^U_{\mathrm{SMO}}$. Recall that an uninformed buyer who arrives at $t$ believes the asset value $v_t$ to be distributed according to $\mathcal{N}(\mu_0 + \gamma\Delta, V^2 - \sigma_I^2)$ (see the proof of Lemma A3). Then, Lemma A3 shows that $\mathcal{U}^U_{\mathrm{BLO}} = S/2 - \Delta + \bar{u}$, and also rules out SLO, since this has a lower payoff than BLO. One also computes $\mathcal{U}^U_{\mathrm{BMO}} = \mathsf{E}_t(v_t) - a_t + \bar{u} = (\mu_t + \gamma\Delta) - (\mu_t + S/2) + \bar{u} = \bar{u} + \gamma\Delta - S/2$, $\mathcal{U}^U_{\mathrm{SMO}} = b_t - \mathsf{E}_t(v_t) - \bar{u} = (\mu_t - S/2) - (\mu_t + \gamma\Delta) - \bar{u} = -S/2 - \gamma\Delta - \bar{u}$, and $\mathcal{U}^U_{\mathrm{NO}} = 0$. Collecting these formulas, one obtains:

$$\text{(IA.4)} \qquad \mathcal{U}^U_{\mathrm{BMO}} \;=\; \bar{u} + \gamma\Delta - \frac{S}{2}, \qquad \mathcal{U}^U_{\mathrm{SMO}} \;=\; -\bar{u} - \gamma\Delta - \frac{S}{2}, \qquad \mathcal{U}^U_{\mathrm{NO}} \;=\; 0.$$

By inspection, SMO and NO can be ruled out, because BMO clearly yields a larger payoff ($\bar{u} > S/2$). To rule out BMO, note that condition (A-17) implies $\alpha - I(\rho, \alpha) > \frac{\rho}{\beta}(1 + \gamma)$, which if one multiplies by $V = \beta\rho^{-1}\Delta$ implies $s > \Delta(1 + \gamma)$, and hence $\mathcal{U}^U_{\mathrm{BMO}} < \mathcal{U}^U_{\mathrm{BLO}}$.

I now show that the continuation payoff for the uninformed buyer from submitting or maintaining his BLO at the equilibrium price $b^*$ is at last as large as the payoff obtained by choosing BLO at either $b > b^*$ or $b < b^*$.

(D1) I first rule out BLO at $b > b^*$. According to $\mathcal{S}(\mathrm{h})$, overshooting a bid leads to a shift in the public mean by a positive quantity. But the trader knows that he is in fact uninformed and that the correct public mean is lower. This amounts to getting a negative shift in the mean of his belief about the fundamental value. Condition (A-17) then implies that a negative shift in mean for the trader's density brings a decrease in expected payoff.

(D2) I also rule out BLO at $b < b^*$. According to $\mathcal{S}(\mathrm{h})$, undershooting a bid at the bid price does not change the public mean, but if the transgressor is the first in the bid queue, it prompts another trader in the bid queue to immediately modify his BLO at $b^*$. Then, the transgressor loses his first rank in the bid queue, which according to Lemma A3 does not change his payoff.

Next, I prove the optimality of $\mathcal{S}(\mathrm{h})$, which describes the response of the uninformed buyer to another trader who deviates from the equilibrium by choosing a limit order at $b = b^* + d$ instead of $b^*$. Note that according to $\mathcal{S}(\mathrm{h})$, the out-of-equilibrium belief specifies that the public mean moves up by a quantity $d' > d$. Then, the uninformed buyer finds himself in the same situation as before, when he had to decide whether to submit his bid at the equilibrium price or not. A similar argument as in the cases (D1) and (D2) above shows that it is indeed optimal for the uninformed buyer to also shift his bid by the amount specified by $\mathcal{S}(\mathrm{h})$.

**Informed Traders**

The proof is in the Appendix of the paper. $\square$

## 2 Optimal Strategies of Impatient Traders

### 2.1 Model Setup

Recall that traders in the model described in Section II in the paper have a type defined by two preference parameters: the private valuation $u \in \{-\bar{u}, 0, \bar{u}\}$, and the waiting coefficient $r \in \{0, \bar{r}\}$. I introduce the following notation: PNB for patient natural buyer ($u = \bar{u}$ and $r = 0$), PNS for patient natural seller ($u = -\bar{u}$ and $r = 0$), INB for impatient natural buyer ($u = \bar{u}$ and $r = \bar{r}$), INS for impatient natural seller ($u = -\bar{u}$ and $r = \bar{r}$), PS for patient speculator ($u = 0$ and $r = 0$), and IS for impatient speculator ($u = 0$ and $r = \bar{r}$).

In the model described in Section II in the paper, it is assumed that the impatient traders make the following automatic decisions:

- An INB always submits a BMO (buy market order).
- An INS always submits an SMO (sell market order).
- An IS does not submit any order, and exits the model.

In this section, I endogenize these decisions, taking as given the information acquisition decision described in Section 3 in this Internet Appendix. I thus assume that each PS acquires information, in the sense of observing the asset value at the time of arrival, while each IS remains uninformed.

I describe briefly the trading game, which is the same as in Section II in the paper. At each integer time $t = 1, 2, \ldots$ (corresponding to clock times $\frac{1}{\lambda}, \frac{2}{\lambda}, \ldots$) the asset value changes according to $v_t = v_{t-1} + \sigma_I \varepsilon_t$, where $\sigma_I = \frac{\sigma_v}{\sqrt{\lambda}}$ is the inter-arrival volatility, and $\varepsilon_t \sim \mathcal{N}(0, 1)$. At each integer time $t \geq 0$, a player called "Nature" draws a new trader. The trader's type is as follows: (i) with probability $\rho$, the trader is a speculator ($u = 0$), in which case with probability 1 she is a PS and becomes informed, or with probability 0 is an IS and remains uninformed,[6] or (ii) with probability $1 - \rho$ the trader is a natural buyer or seller, in which case with equal probability the trader is a PNB, PNS, INB, or INS.

The only difference is that the impatient traders now can submit any type of orders. Thus, a trader that arrives to the market at $t$ either submits no order (NO), or chooses an order of type $\{\text{BMO}, \text{BLO}, \text{SLO}, \text{SMO}\}$ for 1 unit of the asset. In the case of no order, the trader exits the model forever, and Nature immediately draws another trader from the pool.

[6] Even though the probability of an impatient speculator arriving is 0, I need to specify her strategy to show that her optimal behavior is to submit no order and exit the model.

At non-integer times the game is played with the existing traders in the limit order book. The game is set in continuous time, based on the framework of Bergin and MacLeod (1993). Thus, the game allows for instantaneous responses, by completing the space of strategies with respect to the response time.

## 2.2 Equilibrium with Impatient Traders

In this section, I show that the stationary equilibrium of the benchmark model in the paper extends to the different types of impatient traders in the way assumed in Section II in the paper.

As in Theorem 1 in the paper, I show that there exists a stationary MPE of the model if the conditions stated in Result 1 are satisfied. The conditions refer to the same information function $I$ from Definition 1 in the paper. To prove Theorem IA.1 below, there is no new condition on $I$, but I introduce a condition on the waiting coefficient of the impatient traders, $\bar{r}$. Recall that the conditions of Result 1 are verified numerically in Section 4.

**Theorem IA.1.** *Suppose the information function $I$ satisfies analytically the conditions from Result 1 in the paper, and the investor preference parameters satisfy $\bar{u} \geq \frac{S}{2}$, $\omega \geq \gamma\Delta$, and $\bar{r} > \lambda S$. Then, there exists a stationary Markov perfect equilibrium (MPE) of the game. In equilibrium, the* INB *submits a buy market order, the* INS *a sell market order, and the* IS *submits no order and exits the model.*

## 2.3 Proofs of Results

As in the proof of Theorem 1, I define a game assessment for a perfect Bayesian equilibrium (PBE), which is the collection of a strategy profile and a belief system which are compatible withe each other. To show that the PBE is in fact an MPE, note that the strategy profile (defined below) is Markov, meaning that the strategies depend only on the current value of the following state variables:

- Public variables: the public density, given by the public mean ($\mu_t$) and the public volatility ($\sigma_t$); and the limit order book, given by the bid price ($b_t$), the ask price ($a_t$), and the bid and ask queues.[7]

- Private variable for informed traders at the time of arrival: the asset value ($v_t$).

[7] Because in the model traders can submit orders only for 1 unit, the limit prices for orders other than the first ones in the bid and ask queues are not relevant.

Because I am interested in a stationary equilibrium, I assume that an instant before $t = 0$ the initial public density is $\mathcal{N}(0, V^2)$. The ask price is $S/2$, the bid price is $-S/2$, where $S$ is the parameter from (7) in the paper, while the initial limit order book has countably many limit orders on each side. As in Remark IA.1 from Section 1 in this Internet Appendix, I use the one-stage deviation principle of subgame perfection to assume that at all times (integer or not) the public volatility is equal to the parameter $V$. Also, at all integer times $t = 0, 1, \ldots$, the ask price is $\mu_t + S/2$ and the bid price is $\mu_t - S/2$.

To define the strategy profile $\mathcal{S}$, I first describe the action of a new trader who arrives at $t$. Then, I describe the reaction of the other traders remaining in the limit order book to the new arrival at $t$. Finally, I describe the reaction of the existing traders to any out-of-equilibrium deviation that might occur from either the new trader or an existing trader. As discussed before, I take the results of Section 3 in this Internet Appendix as given, and assume that only the PS acquires information, while all the other types of traders remain uninformed.

**The Game Assessment**

Recall that a game assessment is the collection of a strategy profile and a belief system which are compatible withe each other. The strategy profile $\mathcal{S}$ is given by the set of rules $\mathcal{S}$(a)–$\mathcal{S}$(h) in Section 1, plus the following rules:

(i) The INB arriving at $t$ submits a BMO and exits the model.

(j) The INS arriving at $t$ submits an SMO and exits the model.

(k) The IS arriving at $t$ submits NO (no order) and exits the model.

The belief system is described by the following rules: At $t = 0$, the uninformed investors perceive the asset value distributed according to $\mathcal{N}(0, V^2)$. Subsequently, the uninformed investors' belief about the asset value (the public density) is updated using the approximate Bayes' rule described in Section II in the paper. At the time of arrival to the market, the informed investor observes the asset value and can compute the average payoff of a limit order based on updating her belief according to the exact Bayes' rule. After the arrival, however, the informed trader cannot update her belief, and becomes essentially uninformed. In the limit order book at $t = 0$, all traders on the bid side are PNBs, and all traders on the ask side are PNSs. At $t \geq 0$, each new trader is believed to be of the following type:

PS with probability $\rho$, IS with probability 0, PNB with probability $(1-\rho)/4$, PNS with probability $(1-\rho)/4$, INB with probability $(1-\rho)/4$, or INS with probability $(1-\rho)/4$. Subsequently, traders' beliefs about the other investors' types are updated according to the Bayes' rule.

Because the strategy profile $\mathcal{S}$ defined above depends only on the current value of the state variables, the strategies are indeed Markov. As in Theorem 1 in the paper, I now show that the strategy of each type of impatient investor is a best response to the other investors' strategies.

**Impatient Natural Buyers and Sellers**

I show that conditional on the INB (impatient natural buyer) not acquiring information, his strategy specified in $\mathcal{S}$(i) and $\mathcal{S}$(g) is optimal. The proof is symmetric for the INS (impatient natural seller).

Consider an INB who arrives at $t$ and has waiting cost $\bar{r} > \lambda S$ and private valuation $\bar{u} > S/2$. To prove the optimality of $\mathcal{S}$ for the INB, I first show that at any time the INB prefers a market order to a limit order in the same direction. Second, I show that among BMO, BLO and NO, the INB prefers the BMO.

Let $\mathcal{U}_{\mathrm{BMO}}^{\mathrm{INB}}$ be the expected payoff from submitting a BMO at $t$, and let $\mathcal{U}_{\mathrm{BLO}}^{\mathrm{INB}}$ be the expected payoff from a one-stage deviation by submitting a BLO at $t$ and switching to BMO at $t+1$. (The same argument works for a shorter time lag between BLO and BMO, but it is simpler for a lag of 1.) From Corollary 3 in the paper, at $t+1$ all orders in $\{\mathrm{BMO}, \mathrm{BLO}, \mathrm{SLO}, \mathrm{SMO}\}$ have equal probability 1/4. However, only the SMO leads to the execution of the BLO, which produces an instant expected payoff of $\frac{S}{2} - \Delta + \bar{u}$ (see Lemma A3 in the Appendix of the paper). All the other types of order (BMO, BLO, and SLO) at $t+1$ imply that the BLO of the INB is not executed, and hence by switching to a BMO at $t+1$ he gets a payoff of $\mathcal{U}_{\mathrm{BMO}}^{\mathrm{INB}} = \bar{u} - \frac{S}{2}$. Moreover, the INB incurs a waiting cost equal to $\frac{\bar{r}}{\lambda}$, because between $t$ and $t+1$ he waits a clock time equal to $\frac{1}{\lambda}$. One thus obtains:

$$\text{(IA.5)} \qquad \mathcal{U}_{\mathrm{BMO}}^{\mathrm{INB}} \;=\; \bar{u} - \frac{S}{2}, \qquad \mathcal{U}_{\mathrm{BLO}}^{\mathrm{INB}} \;=\; \left( \frac{3}{4}\,\mathcal{U}_{\mathrm{BMO}}^{\mathrm{INB}} + \frac{1}{4}\left( \frac{S}{2} - \Delta + \bar{u} \right) \right) - \frac{\bar{r}}{\lambda}.$$

But $\mathcal{U}_{\mathrm{BMO}}^{\mathrm{INB}} > \mathcal{U}_{\mathrm{BLO}}^{\mathrm{INB}}$ is equivalent to $\frac{\bar{r}}{\lambda} > \frac{1}{4}(s-\Delta)$, which is true since $\bar{r} > \lambda S$.

By the same argument, it also follows that the INB prefers BMO to SLO (the argument is stronger, since it leads to a loss of the private valuation $\bar{u}$).

One now computes:

$$\text{(IA.6)} \qquad \mathcal{U}^{\text{INB}}_{\text{BMO}} \;=\; \bar{u} - \frac{S}{2}, \qquad \mathcal{U}^{\text{INB}}_{\text{SMO}} \;=\; -\bar{u} - \frac{S}{2}, \qquad \mathcal{U}^{\text{INB}}_{\text{NO}} \;=\; 0.$$

Since $\bar{u} > S/2$, it follows that the INB prefers BMO to SMO and NO.

**Impatient Speculators**

I show that conditional on the IS (impatient speculator) not acquiring information, his strategy specified in $\mathcal{S}$(k), and $\mathcal{S}$(g) is optimal.

The analysis for IS is similar to the analysis for the INB, with the only difference that the private valuation of the IS is 0. Recall that for the INB the only positive payoff comes from a BMO, and it equals $\bar{u} - S/2$. For the IS, the payoff from either a BMO or an SMO equals $-S/2$, which is negative. This shows that, as specified by $\mathcal{S}$, the IS submits NO (no order), which has zero payoff, and exits the model.

The proof of Theorem IA.1 is now complete.

# 3 Information Acquisition and Welfare

In this section, I model explicitly the information acquisition decision, and discuss its welfare implications. Throughout this section, I follow the principle that the number of natural buyers and sellers is fixed, but the number of speculators is determined by the condition that their expected payoff minus the information acquisition costs must be 0. Otherwise, more informed speculators would arrive to the market, until they drive down their expected profit to 0.

The zero profit principle implies that the total trading activity ($\lambda$) is no longer an exogenous parameter, as assumed in Section II in the paper. Instead, the total trading activity depends on the number of informed traders, which in turn depends on the information acquisition cost. Thus, to study the effect of the information cost on the equilibrium, the cost of acquiring information needs to be treated as exogenous.

In presenting the model, however, I treat the number of informed traders as exogenous (in Sections 3.1 and 3.2), and I wait until Section 3.3 to endogenize it, when I discuss the effect of an exogenous change in the information cost on aggregate trader welfare.

## 3.1 Model Setup

Recall that each trader has a type $(u, r)$, which consists of a private valuation $u$ for the asset and a waiting coefficient $r$. The private valuation $u$ can take 3 possible values, $\{-\bar{u}, 0, \bar{u}\}$, where $\bar{u} > 0$. A trader is a "natural buyer" if $u = \bar{u}$, a "natural seller" if $u = -\bar{u}$, or "speculator" if $u = 0$. Traders incur a waiting cost of the form $r \times \tau$, where $\tau$ is the expected waiting time, and $r$ is a constant coefficient. The waiting coefficient $r$ can take two possible values, $\{0, \bar{r}\}$, where $\bar{r} > 0$. A trader is "patient" if $r = 0$, or "impatient" if $r = \bar{r}$.

I now discuss in more detail the arrival process of the various types of traders, and show that the process that results from explicitly modeling the information acquisition decision is identical to the process exogenously assumed in Section II in the paper. At each time $t = 1, 2, \ldots$, a trader is drawn randomly from the pool of traders. The pool of traders is divided into two main subpools: (i) a mass $N_{\bar{u}}$ of traders with $u = \pm\bar{u}$, further divided equally into natural buyers ($u = \bar{u}$) and natural sellers ($u = -\bar{u}$), and (ii) a mass $N_0$ of traders with $u = 0$ (speculators). Furthermore, each subpool of traders is divided equally into patient and impatient traders. If a trader decides to stay out of the market—in which case exits the model forever,—another trader is immediately drawn from the same subpool and replaces him.[8] In Theorem IA.2 below, I show that impatient speculators stay out of the market. Therefore, I assume directly that in the speculator pool there are only patient speculators.

Consider the following exogenous parameters: the mass of speculators $N_0$, the mass of natural buyers and sellers $N_{\bar{u}}$, and the fundamental volatility of the asset $\sigma_v$. Define:

$$\rho \;=\; \frac{N_0}{N_0 + N_{\bar{u}}}, \qquad \lambda \;=\; \ell\,(N_0 + N_{\bar{u}}), \tag{IA.7}$$

where $\ell > 0$ is an exogenous quantity called the "liquidity parameter." As in Section II in the paper, I call $\rho$ the "informed share," and $\lambda$ the "total trading activity." Given the constants $\alpha$, $\beta$, $\gamma$ and the information function $I$ from Section III.B in the paper, define:

$$\Delta \;=\; \sqrt{\tfrac{2}{1+\gamma^2}}\,\frac{\sigma_v}{\sqrt{\lambda}}, \qquad V \;=\; \beta\rho^{-1}\Delta, \qquad S \;=\; \big(\alpha - I(\rho,\alpha)\big)\,V. \tag{IA.8}$$

where $\phi(\cdot)$ is the standard normal density.

I now describe information acquisition. I assume that when a trader arrives to the market, he must decide whether to pay a cost $C$ to acquire information. If he pays $C$, he observes the asset value only once, at the time of entry; otherwise, he remains uninformed.[9] The cost $C$ is increasing in the absolute value of his private valuation $u$. This reflects the fact that natural buyers or sellers have different motivation for trading, such as hedging or liquidity reasons, therefore it is plausible that they have a higher information cost than the speculators.[10] For simplicity, I assume a cost function of the form linear in the absolute valuation:

$$C \;=\; C(u) \;=\; C_0 + |u|, \quad \text{with} \quad C_0 \;=\; 2\left(\phi(\alpha) + \int_0^{\alpha} I(\rho,w)\phi(w)\,\mathrm{d}w\right)V. \tag{IA.9}$$

As shown in Theorem IA.2 below, the formula for the base cost $C_0$ implies a zero profit condition for the (patient) speculators.

[8] This assumption is made for tractability, and it implies that time between trades is not informative.

[9] Intuitively, it is not optimal to acquire information after the initial decision has been made. This is because information is expensive, in the sense that the cost of information is such that the expected payoff of an informed trader equals the information cost (see Corollary IA.1 below). Thus, for simplicity I assume that the decision can be made only once, at the beginning.

[10] If $C$ is the same for all types of traders, all the results in Theorem IA.2 go through, except that the impatient natural buyers and sellers might then prefer to acquire information.

### 3.2 Equilibrium with Information Acquisition

In this section, I show that there exists a stationary MPE of the model with information acquisition. In equilibrium, I show that the exogenous assumptions made in Section II in the paper are true when traders decide whether to acquire information. In particular, the natural buyers and sellers (patient or impatient) remain uninformed, the patient speculators acquire information, while the impatient speculators do not acquire information. Given these results, the equilibrium behavior of all types of traders coincides with the equilibrium behavior described in Section III in the paper.

For simplicity, consider a commitment parameter equal to $\omega = \gamma\Delta$.[11] As in Section III in the paper, the existence of an MPE depends on certain conditions that the information function $I$ must satisfy. To this end, however, I define a slightly more general information function.

**Definition IA.1.** *In the context of Definition 1 in the paper, denote by $I(\rho, g_1, j)$ and $J(\rho, g_1, j)$ the two functions that satisfy the same formulas as the functions $I$ and $J$, respectively, except that the density $g_1$ is now considered arbitrary.*

Recall that the functions $I(\rho, w, j)$ and $J(\rho, w, j)$ are defined in the paper by starting from a particular density: $g_1 = \mathcal{N}\left(w - \gamma\frac{\rho}{\beta}, \rho^2\frac{1+\gamma^2}{2\beta^2}\right)$. Thus, in this section, the new notation $I(\rho, g_1, j)$ is used to indicate starting from an arbitrary density, while the old notation $I(\rho, w, j)$ is used to indicate starting from the density $\mathcal{N}\left(w - \gamma\frac{\rho}{\beta}, \rho^2\frac{1+\gamma^2}{2\beta^2}\right)$. As in the paper, when $j = 1$, the argument $j$ can be omitted.

To state the conditions for an MPE, consider the conditions from Result 1 in the paper:

$$\begin{aligned}
&I(\rho, w),\ w - I(\rho, w),\ \text{and } I(\rho, w) - I(\rho, -w) \text{ are strictly increasing in } w,\\
&S \ > \ \Delta(1+\gamma), \qquad \frac{S}{2} + 2I(\rho, 0)V + \gamma\Delta \ > \ 0,\\
&I = I(\rho, w, j) \text{ decreases in } j \text{ for all } w > 0, \text{ and}\\
&J(\rho, w, j) = 1 \text{ for all } w \text{ and } j \geq 1.
\end{aligned} \tag{IA.10}$$

[11] Note that in order to prove Theorem 1 in the paper, the commitment parameter must be above a threshold: $\omega \geq \gamma\Delta$. In this section, for simplicity the parameter is set equal to the threshold: $\omega = \gamma\Delta$.

I introduce the following additional conditions:

$$I(\rho,w)+I(\rho,-w) \text{ is strictly decreasing in } w \text{ whenever } w>0, \tag{IA.11}$$

$$\frac{b}{2}+\hat{\omega}+I\left(\rho,-\left(\tfrac{b}{2}-\hat{\omega}\right)\right) \;>\; 0, \tag{IA.12}$$

$$\frac{\hat{S}}{2}+\frac{I(\rho,\alpha)+I(\rho,-\alpha)}{2}+\hat{\omega} \;>\; 0 \;\text{ whenever } D(\alpha)>\hat{S}, \tag{IA.13}$$

$$\frac{\rho}{\beta}+\frac{\hat{S}}{4}-\frac{b}{2}-\Phi(-a_*)b-\big(\phi(\alpha)-\phi(a_*)\big)+F_1-D_1 \;<\; 0, \tag{IA.14}$$

$$2f\left(\tfrac{\hat{S}}{2}-\hat{\omega}\right)-\hat{\omega}-\hat{C}_0 \;<\; 0, \text{ where } f(x)=\phi(x)-x\Phi(-x), \tag{IA.15}$$

$$\frac{\hat{S}}{2}+I(\rho,g_1,1) \;<\; 0, \quad \text{with} \quad g_1 \;=\; \mathcal{N}\left(-\gamma\frac{\Delta}{V},\frac{V^2+\sigma_I^2}{V^2}\right), \tag{IA.16}$$

$$F:(0,1)\to(0,\infty) \text{ is one-to-one and strictly decreasing in } \rho, \tag{IA.17}$$

where I define:

(IA.18)

$$A(\rho,w) \;=\; w-I(\rho,w), \qquad B(\rho,w) \;=\; w-I(\rho,-w), \qquad D(\rho,w) \;=\; I(\rho,w)-I(\rho,-w),$$

$$F(\rho) \;=\; \frac{2\beta}{\rho}\left(\phi(\alpha)+\int_0^\alpha I(\rho,w)\phi(w)\,\mathrm{d}w\right), \qquad \hat{\omega} \;=\; \frac{\omega}{V} \;=\; \frac{\gamma\rho}{\beta}, \qquad \hat{S} \;=\; \alpha-I(\rho,\alpha),$$

$$\hat{C}_0 \;=\; 2\left(\phi(\alpha)+\int_0^\alpha I(\rho,w)\phi(w)\,\mathrm{d}w\right), \qquad b \;=\; \max\big(B(\alpha),2\hat{S}\big), \qquad a_* \;=\; B^{-1}(b),$$

$$F_1 \;=\; \int_\alpha^{a_*} I(\rho,-w)\phi(w)\,\mathrm{d}w, \qquad D_1 \;=\; \int_0^\alpha D(w)\phi(w)\,\mathrm{d}w.$$

The new conditions are verified numerically in Section 4 in this Internet Appendix.

**Result IA.1.** The conditions (IA.11)–(IA.17) are satisfied for all $\rho\in(0,1)$.

I now state the main result of this section.

**Theorem IA.2.** *If the information function $I$ from Definition IA.1 satisfies analytically the conditions* (IA.10)–(IA.17)*, and the investor preference parameters satisfy $\bar{u}\geq\frac{S}{2}$, $\omega=\gamma\Delta$, and $\bar{r}>\lambda S$, then there exists a stationary MPE of the game. In equilibrium, the natural buyers and sellers (patient or impatient) remain uninformed, the patient speculators acquire information, while the impatient speculators do not acquire information.*

An important property of the equilibrium is that for an informed trader arriving at $t$, her continuation payoff after acquiring information, but before observing the asset value, is equal to the information acquisition cost $C_0$ from equation (IA.9). Equivalently, the informed trader's ex-ante expected utility is 0. This is not surprising, because $C_0$ was chosen precisely so that the zero profit condition for the informed trader holds in equilibrium.

**Corollary IA.1.** *In equilibrium, the informed trader has zero ex-ante expected utility.*

Another property is that the information cost $C_0$ is decreasing in the information $\rho$, or equivalently, since $\rho = \frac{N_0}{N_0 + N_{\bar{u}}}$, the information cost is decreasing in the mass of speculators $N_0$. I state this as a numerical result, and I verify it later, in Section 4 in this Internet Appendix.

**Result IA.2.** The function $F(\rho)$ satisfies the following approximation:

$$F(\rho) \;\approx\; \beta^2 \left( \frac{1}{\rho} - 1 \right). \tag{IA.19}$$

Moreover, the function:

$$H(\rho) \;=\; F(\rho)\sqrt{1-\rho} \;:\; (0,1) \;\to\; (0,\infty) \tag{IA.20}$$

is one-to-one and strictly decreasing in $\rho$.

Intuitively, if there are more informed traders ($N_0$ is larger), the informed share $\rho$ is larger, and the information cost $C_0$ is smaller. Alternatively, if I consider an exogenous decrease in the cost of information ($C_0$ decreases), then there is an increase in the arrival rate of informed traders, as well as an increase in the informed share $\rho$. In the next section, I discuss the effects of such an exogenous change in the information cost on trader welfare.

### 3.3 Trader Welfare

In this section, I study the effect on aggregate trader welfare of an exogenous change in the cost of information $C_0$. As mentioned at the beginning of Section 3.1, I follow the principle that the number of natural buyers and sellers is fixed ($N_{\bar{u}}$), but the number of speculators ($N_0$) is determined by the condition that their ex-ante payoff from trading on their information is equal to the information acquisition cost ($C_0$).

Thus, in this section I treat as exogenous the number of natural buyers and sellers ($N_{\bar{u}}$, the information acquisition cost ($C_0$), and the fundamental volatility of the asset ($\sigma_v$). From Result IA.2, it follows that the function $H(\rho) = F(\rho)\sqrt{1-\rho}$ is one-to-one and strictly decreasing in $\rho$. Thus, numerically the inverse function $H^{-1} : (0,\infty) \to (0,1)$ exists and is strictly decreasing in its argument. From Result 2, $\frac{S}{\Delta} = \frac{\alpha - I(\rho,\alpha)}{\rho}\,\beta$ is strictly decreasing in $\rho$.

**Proposition IA.1.** *If the inverse function $H^{-1} : (0,\infty) \to (0,1)$ exists and is strictly decreasing, and $\frac{\alpha - I(\rho,\alpha)}{\rho}\,\beta$ is strictly decreasing in $\rho$, then:*

$$
\begin{aligned}
&\rho \;=\; H^{-1}\left(\frac{C_0}{m}\right), \quad \textit{with} \quad m \;=\; \sqrt{\frac{2\sigma_v^2}{\ell(1+\gamma^2)N_{\bar{u}}}},\\
&\Delta \;=\; m\,\sqrt{1-\rho}, \quad \lambda \;=\; \frac{\ell N_{\bar{u}}}{1-\rho}, \quad N_0 \;=\; N_{\bar{u}}\,\frac{\rho}{1-\rho}, \quad N_0 + N_{\bar{u}} \;=\; \frac{N_{\bar{u}}}{1-\rho},\\
&V \;=\; \frac{\beta m\sqrt{1-\rho}}{\rho}, \qquad S \;=\; \frac{\alpha - I(\rho,\alpha)}{\rho}\,\beta\, m\sqrt{1-\rho}.
\end{aligned}
\tag{IA.21}
$$

*Also, the informed share $\rho$ is strictly decreasing in the information cost $C_0$. The variables $\Delta$, $V$ and $S$ are strictly decreasing in $\rho$ (and increasing in $C_0$), while $\lambda$ and $N_0$ are strictly increasing in $\rho$ (and decreasing in $C_0$).*

Thus, an exogenous decrease in the cost of acquiring information $C_0$ increases the informed share $\rho$ and the total trading activity $\lambda$, which induces a decrease in public volatility $V$, and bid–ask spread $S$.

The surprising result is that the price impact coefficient $\Delta$ decreases as well. Recall that in equilibrium in the benchmark model, price impact does not change when the informed share decreases (see Proposition 1). But this is because in the benchmark model the total trading activity $\lambda$ is constant. In the current setup, the trading activity increases when there is a decrease in the information cost, and this increase in liquidity causes a decrease in adverse selection (formally, this follows from the fact that $\Delta$ is proportional to $\sigma_I = \frac{\sigma_v}{\sqrt{\lambda}}$).

I now discuss welfare implications of lowering the information cost $C_0$, or equivalently of raising the informed share $\rho$. I define the measure of "aggregate trader welfare" $W$ to be equal to the total mass or traders $(N_0 + N_{\bar{u}})$ multiplied by the average trader expected utility ($\bar{\mathcal{U}}$):

$$
W \;=\; (N_0 + N_{\bar{u}})\,\bar{\mathcal{U}}
\tag{IA.22}
$$

To compute $W$, it is necessary to compute the average trader expected utility $\bar{\mathcal{U}}$. The next result describes the equilibrium expected utility of the different types of traders. In equilibrium, the natural buyers and sellers remain uninformed, and by symmetry their expected utility is the same. Thus, let $\mathcal{U}^{UP}$ be the expected utility of the uninformed patient traders, and $\mathcal{U}^{UI}$ the expected utility of the uninformed impatient traders. Let also $\mathcal{U}^{I}$ be the expected utility of the informed traders (who are the patient speculators). The next result computes all trader expected utilities, as well as the aggregate welfare.

**Proposition IA.2.** *The expected utilities of the different types of traders are:*

$$\mathcal{U}^{I} \;=\; 0, \qquad \mathcal{U}^{UP} \;=\; \frac{S}{2} - \Delta + \bar{u}, \qquad \mathcal{U}^{UI} \;=\; -\frac{S}{2} + \bar{u}. \tag{IA.23}$$

*The average trader expected utility is:*

$$\bar{\mathcal{U}} \;=\; (1-\rho)\bar{u} - \frac{1-\rho}{2}\,\Delta \;=\; (1-\rho)\bar{u} - \sqrt{\frac{\sigma_v^2(1-\rho)^3}{2\ell(1+\gamma^2)N_{\bar{u}}}}. \tag{IA.24}$$

*The aggregate trader welfare is:*

$$W \;=\; N_{\bar{u}}\bar{u} - \frac{N_{\bar{u}}}{2}\,\Delta \;=\; N_{\bar{u}}\bar{u} - \sqrt{\frac{\sigma_v^2 N_{\bar{u}}(1-\rho)}{2\ell(1+\gamma^2)}}, \tag{IA.25}$$

*and is strictly increasing in the informed share $\rho$.*

Equation (IA.25) shows that an exogenous decrease in the information cost $C_0$ (or equivalently an increase in the informed share $\rho$) leads to an increase in the aggregate trader welfare $W$. This is partially due to the fact that a decrease in information costs increases the total number of informed traders. Indeed, Proposition IA.1 implies that the number of informed traders is $N_0 \;=\; N_{\bar{u}}\,\rho/(1-\rho)$, while the number $N_{\bar{u}}$ of natural buyers and sellers is considered fixed. But the average trader welfare $\bar{\mathcal{U}}$ usually decreases when the informed share increases.[12]

Therefore, to understand better why welfare is increasing in the informed share, I now analyze the disaggregated welfare numbers. First, note that the informed traders always break even (Corollary IA.1), and hence their expected utility does not change. The uninformed patient traders submit limit orders in equilibrium, and thus benefit from providing liquidity (they gain the half-spread $S/2$), but lose from adverse selection to the informed

[12]Whether $\bar{\mathcal{U}}$ is increasing or decreasing in $\rho$ depends on the value of the parameter $z = \frac{\sigma_v}{\bar{u}}\sqrt{\frac{1}{2\ell(1+\gamma^2)N_{\bar{u}}}}$. Indeed, using the equation $\frac{\bar{\mathcal{U}}}{\bar{u}} = (1-\rho)-z(1-\rho)^{3/2}$, simple calculus shows that there are two cases, depending on the value of $z$. If $z < 2/3$, $\bar{\mathcal{U}}$ is decreasing in $\rho$ everywhere. If $z > 2/3$, $\bar{\mathcal{U}}$ has a reverse U-shape, and attains its maximum in $(0,1)$ at a point $\rho^*$ which is increasing in the parameter $z$ (and approaches 1 as $z$ becomes large). Note that $z$ is small when (i) the liquidity parameter $\ell$ is large, (ii) the number of natural buyers and sellers $N_{\bar{u}}$ is large, (iii) the natural buyer's private valuation $u$ is large, or (iv) the fundamental volatility $\sigma_v$ is small. Hence, intuitively, the case when $z$ is small can be thought as the case when the market is naturally more liquid.

traders (they lose the price impact coefficient $\Delta$). Since both $S$ and $\Delta$ are decreasing in the informed share (Proposition IA.1), an increase in $\rho$ has an ambiguous on the welfare of uninformed patient traders. Numerically, one can check that $\mathcal{U}^{UP}$ has a U-shape with respect to the informed share $\rho$. When $\rho$ is small, an increase in $\rho$ lowers the bid–ask spread more than it lowers the adverse selection, and hence the utility $\mathcal{U}^{UP}$ decreases. When $\rho$ is large, an increase in $\rho$ strongly lowers the adverse selection (the latter approaches 0 as $\rho$ approaches 1), and hence the utility $\mathcal{U}^{UP}$ increases.

The uninformed impatient traders submit market orders in equilibrium, and thus lose from taking liquidity (they lose the half-spread $S/2$). Since $S$ is decreasing in the informed share (Proposition IA.1), an increase in $\rho$ has the unambiguous effect to increase the welfare of the uninformed impatient traders.

If one aggregates the welfare of the uninformed traders (patient and impatient), the effect of the bid–ask spread cancels out, as the patient gain half the bid–ask spread, while the impatient lose half the bid–ask spread. In the aggregate, the uninformed traders lose from the adverse selection coefficient $\Delta$. And when $\rho$ is large, the adverse selection is small, which unambiguously increases the welfare of the uninformed traders. Since the informed traders break even (have zero aggregate welfare), the previous discussion about the uninformed trader welfare translates to the aggregate trader welfare, and therefore an increase in the informed share leads to an increase in the aggregate trader welfare.

In conclusion, an exogenous decrease in information costs always leads to an increase in aggregate trader welfare, for two reasons: (i) there is an increase in the total number of traders, as more informed traders enter the market when the information cost is low, and (ii) the uninformed traders benefit from a decrease in adverse selection due to the increase in liquidity (the increase in trading activity from the informed traders).

### 3.4 Proofs of Results

**Proof of Theorem IA.2**. Conditional on the information acquisition decision and on the behavior of the impatient traders, the existence of an MPE is proved in Theorem 1 in the paper, and the behavior of impatient traders is proved in Theorem IA.1 (see Section 2 in this Internet Appendix).

Because the traders have now the option of acquiring information, I extend the game assessment from Section 2.3 in this Internet Appendix, and I prove that the corresponding strategies must be optimal. Thus, I must show that $\mathcal{S}(l)$ is correct, which implies that the patient speculators acquire information, while all other types of traders remain uninformed. Also, I must show that the out-of-equilibrium behavior described in $\mathcal{S}$(m)–$\mathcal{S}$(o) is optimal.

**The Game Assessment**

The strategy profile $\mathcal{S}$ is given by the set of rules $\mathcal{S}$(a)–$\mathcal{S}$(h) in Section 1, $\mathcal{S}$(i)–$\mathcal{S}$(k) in Section 2, plus the following rules:

(l) The PS acquires information, while all the other types of traders (PNB, PNS, INB, INS, IS) remain uninformed.

(m) (Out-of-equilibrium behavior) If the PNB acquires information, then he subsequently pursues the following strategy. If a solution exists, denote by $a^*$ be the solution of $a^* - I(\rho, -a^*) = \frac{s+2\bar{u}}{V}$, and by $a_0$ the solution of $I(\rho, a_0) - I(\rho, -a_0) = \frac{2\bar{u}}{V}$; if a solution to one of these equations does not exist, set the corresponding value to $+\infty$.[13] Let $w_t = \frac{v_t - \mu_t}{V}$. There are two cases:

- If $a^* > \alpha$, the PNB submits $\mathcal{O} \in \{\mathrm{SMO}, \mathrm{BLO}, \mathrm{BMO}\}$ whenever $w_t$ lies in the corresponding interval in $\{(-\infty, -a^*), (-a^*, \alpha), (\alpha, \infty)\}$.
- If $a^* < \alpha$, the PNB submits $\mathcal{O} \in \{\mathrm{SMO}, \mathrm{SLO}, \mathrm{BLO}, \mathrm{BMO}\}$ whenever $w_t$ lies in the corresponding interval in $\{(-\infty, -\alpha), (-\alpha, -a^0), (-a^0, \alpha), (\alpha, \infty)\}$.

The PNS has a symmetric strategy to that of the PNB.

(n) (Out-of-equilibrium behavior) If an impatient trader acquires information, then he subsequently pursues the following strategy. Denote by $a^\pm = \frac{S/2 - \omega \pm u}{V}$, respectively, where $u \in \{-\bar{u}, 0, \bar{u}\}$ is the trader's private valuation. Then, the trader submits $\mathcal{O} \in \{\mathrm{BMO}, \mathrm{NO}, \mathrm{SMO}\}$ (where NO means that no order is being submitted) whenever $v_t - \mu_t$ lies in the corresponding interval in $\{(-\infty, -a^-), (-a^-, a^+), (a^+, \infty)\}$.

(o) (Out-of-equilibrium behavior) If the PS decides not to acquire information, then she submits no order and exits the model.

**Patient Natural Buyers**

I study the decision of the PNB to acquire information, and show that the optimal strategy conditional on acquiring information is the one specified by $\mathcal{S}$(m). Suppose the PNB pays

[13] Recall that according to Result 1 in the paper, the functions $w - I(\rho, w)$ and $I(\rho, w) - I(\rho, -w)$ are strictly increasing in $w$; therefore their sum, $w - I(\rho, -w)$, is also increasing in $w$. Thus, the solutions are unique if they exist.

$C$ and observes an asset value $v_t$, or equivalently a signal $w_t = \frac{v_t - \mu_t}{V}$. Denote by $\hat{u} = \frac{\bar{u}}{V}$ his normalized private valuation, and by $\hat{S} = \frac{s}{V}$ the normalized bid–ask spread. Let $\hat{\mathcal{U}}_{\mathcal{O}} = \frac{\mathcal{U}_{\mathcal{O}}}{V}$ be the normalized payoff of this trader from submitting $\mathcal{O}$ and subsequently following $\mathcal{S}$. As for the PS, one uses equation (A-19) in the Appendix of the paper—modified to include the private valuation $\bar{u}$—to compute $\hat{\mathcal{U}}_{\text{BMO}} - \hat{\mathcal{U}}_{\text{BLO}} = A(w_t) - \hat{S}$, $\hat{\mathcal{U}}_{\text{BMO}} - \hat{\mathcal{U}}_{\text{SLO}} = B(w_t) - \hat{S} + 2\hat{u}$, $\hat{\mathcal{U}}_{\text{BLO}} - \hat{\mathcal{U}}_{\text{SLO}} = D(w_t) + 2\hat{u}$, $\hat{\mathcal{U}}_{\text{BLO}} - \hat{\mathcal{U}}_{\text{SMO}} = \hat{S} - B(-w_t) + 2\hat{u}$, $\hat{\mathcal{U}}_{\text{SLO}} - \hat{\mathcal{U}}_{\text{SMO}} = \hat{S} - A(-w_t)$.

If a solution exists, denote by by $a^*$ the solution of $B(a^*) = \hat{S} + 2\hat{u}$, and by $a_0$ the solution of $D(a_0) = 2\hat{u}$; if a solution to one of these equations does not exist, set the corresponding value to $+\infty$. If the order preference of the informed PNB is denoted by "$>$", one gets BMO $>$ BLO $\iff w_t > \alpha$; BLO $>$ SLO $\iff w_t > -a_0$; BLO $>$ SMO $\iff w_t > -a^*$; SLO $>$ SMO $\iff w_t > -\alpha$.

One has $A(a^*) + D(a^*) = B(a^*) = \hat{S} + 2\hat{u} = A(\alpha) + D(a_0)$, which implies $A(a^*) - A(\alpha) = D(a_0) - D(a^*)$. Therefore, $a^* - \alpha$ and $a_0 - a^*$ have the same sign, which implies that either $\alpha < a^* < a_0$ or $a_0 < a^* < \alpha$.

**Case 1:** $\alpha < a^* < a_0$.

In this case, it is easy to verify that the informed PNB submits $\mathcal{O} \in \{\text{SMO}, \text{BLO}, \text{BMO}\}$ whenever $w_t$ lies in the corresponding interval in $\{(-\infty, -a^*), (-a^*, \alpha), (\alpha, \infty)\}$, as specified by $\mathcal{S}(\text{m})$. So far, I have excluded NO (no order) from this analysis. Now, I show that NO is ruled out by the penalty $\omega$ for non-trading.

Recall that $\omega = \gamma\Delta$, or $\hat{\omega} = \frac{\omega}{V} = \gamma\frac{\rho}{\beta}$. Since $\mathcal{U}_{\mathcal{O}}$ is increasing for $w_t > -a^*$ and decreasing for $w_t < -a^*$, it is necessary to compare $\hat{\mathcal{U}}_{\text{SMO}}(w_t = -a^*)$ with $-\hat{\omega}$. However,

$$\text{(IA.26)} \qquad \hat{\mathcal{U}}_{\text{SMO}}(w_t = -a^*) \;=\; -\frac{\hat{S}}{2} + a^* - \hat{u} \;=\; a^* - \frac{B(a^*)}{2} \;=\; \frac{a^* + I(\rho, -a^*)}{2}.$$

(The second equality follows from the definition of $a^*$: $B(a^*) = \hat{S} + 2\hat{u}$.) Because $B$ is an increasing function, and $a^* > \alpha$, one gets $\hat{S} + 2\hat{u} = B(a^*) > B(\alpha)$. From $\hat{u} > \hat{S}/2$, one gets $\hat{S} + 2\hat{u} > 2\hat{S}$. The inequalities above imply $\hat{S} + 2\hat{u} > \max\{B(\alpha), 2\hat{S}\}$. Denote by $b = \max\{B(\alpha), 2\hat{S}\}$, and by $a_*$ the solution of $B(a_*) = b$. One has $B(a^*) > B(a_*)$, therefore $a^* > a_*$. According to condition (IA.10), the function $A(\rho, -w) = -w - I(\rho, -w)$ is strictly decreasing in $w$, therefore $w - B(w)/2 = \big(w + I(\rho, -w)\big)/2$ is strictly increasing in $w$. Thus, $\mathcal{U}_{\text{SMO}}(w_t = -a^*) = a^* - B(a^*)/2 > a_* - B(a_*)/2 = a^* - b/2$. I now check that $a_* - b/2 \geq -\hat{\omega}$, which is equivalent to $a_* \geq b/2 - \hat{\omega}$, or $B(a_*) \geq B(b/2 - \hat{\omega})$. By the definition of $B$ and $a^*$, this is the same as $b \geq b/2 - \hat{\omega} - I\big(\rho, -(b/2 - \hat{\omega})\big)$, or $b/2 + \hat{\omega} + I\big(\rho, -(b/2 - \hat{\omega})\big) \geq 0$. But this follows from condition (IA.12).

Next, denote by $\mathsf{E}_{w_t}\hat{\mathcal{U}}^{\text{PNB}}$ the normalized expected payoff of the informed PNB before observing $w_t$, but after paying the normalized information cost $\hat{C} = C/V$. Using the formulas above, one computes $\mathsf{E}_{w_t}\hat{\mathcal{U}}^{\text{PNB}} = \int_{-\infty}^{-a^*}\big(-\hat{S}/2 - w - \hat{u}\big)\phi(w)\,\mathrm{d}w + \int_{-a^*}^{\alpha}\big(\hat{S}/2 + I(\rho,w) + \hat{u}\big)\phi(w)\,\mathrm{d}w + \int_{\alpha}^{\infty}\big(w - \hat{S}/2 + \hat{u}\big)\phi(w)\,\mathrm{d}w$. It is necessary to compare $\mathsf{E}_{w_t}\hat{\mathcal{U}}^{\text{PNB}} - \hat{C}$ with the normalized payoff of the uninformed PNB from equation (A-16) in the Appendix of the paper, $\hat{\mathcal{U}}^{\text{PNB}}_{\text{BLO}} = \hat{u} + \hat{S}/2 - \Delta/V$. From (IA.9), one gets $\hat{C} = \hat{C}_0 + \hat{u} = 2\big(\phi(\alpha) + \int_0^{\alpha} I(\rho,w)\phi(w)\,\mathrm{d}w\big) + \hat{u}$, and from the definition of $a^*$ one gets $\hat{u} = \big(B(a^*) - \hat{S}\big)/2$. Then, one computes:

$$\text{(IA.27)}\qquad \begin{aligned} \mathsf{E}_{w_t}\hat{\mathcal{U}}^{\text{PNB}} - \hat{C} - \hat{\mathcal{U}}^{\text{PNB}}_{\text{BLO}} \;=\; & \frac{\Delta}{V} + \frac{\hat{S}}{4} - \frac{B(a^*)}{2} - \Phi(-a^*)B(a^*) - \big(\phi(\alpha) - \phi(a^*)\big) \\ & + \int_{\alpha}^{a^*} I(\rho,-w)\phi(w)\,\mathrm{d}w - \int_0^{\alpha} D(w)\phi(w)\,\mathrm{d}w. \end{aligned}$$

I check that the PNB does not acquire information, which is equivalent to $\mathsf{E}_{w_t}\hat{\mathcal{U}}^{\text{PNB}} - \hat{C} - \hat{\mathcal{U}}^{\text{PNB}}_{\text{BLO}} < 0$. The derivative of (IA.27) with respect to $a^*$ equals $-B'(a^*)/2 - \Phi(-a^*)B'(a^*) < 0$, since $B$ is strictly increasing. Thus, it is sufficient to verify the inequality when $a^* = a_*$ and $B(a^*) = \max\{B(\alpha), 2\hat{S}\} = b$. This follows from condition (IA.14), which states that $\frac{\rho}{\beta} + \frac{\hat{S}}{4} - \frac{b}{2} - \Phi(-a_*)b - \big(\phi(\alpha) - \phi(a_*)\big) + F_1 - D_1 < 0$, with $F_1 = \int_{\alpha}^{a_*} I(\rho,-w)\phi(w)\,\mathrm{d}w$, and $D_1 = \int_0^{\alpha} D(w)\phi(w)\,\mathrm{d}w$.

**Case 2:** $a_0 < a^* < \alpha$.

In this case, one can check that the informed PNB submits $\mathcal{O} \in \{\text{SMO}, \text{SLO}, \text{BLO}, \text{BMO}\}$ whenever $w_t$ lies in the corresponding interval in $\{(-\infty,-\alpha), (-\alpha,-a_0), (-a_0,\alpha), (\alpha,\infty)\}$, as specified by $\mathcal{S}(\text{m})$. Again, NO is excluded from this analysis. I now show that NO is ruled out by the penalty $\omega$ for not trading.

Since $\mathcal{U}_{\mathcal{O}}$ is increasing for $w_t > -a_0$ and decreasing for $w_t < -a_0$, one needs to compare $\hat{\mathcal{U}}_{\text{BLO}}(w_t = -a_0)$ with $-\hat{\omega}$. However,

$$\text{(IA.28)}\qquad \hat{\mathcal{U}}_{\text{BLO}}(w_t = -a_0) \;=\; \frac{\hat{S}}{2} + I(\rho,-a_0) + \hat{u} \;=\; \frac{\hat{S}}{2} + \frac{I(\rho,a_0) + I(\rho,-a_0)}{2}.$$

(The second equality follows from the definition of $a_0$: $D(a_0) = I(\rho,a_0) - I(\rho,-a_0) = 2\hat{u}$.) Because $D$ is an increasing function, and $a_0 < \alpha$, one gets $2\hat{u} = D(a_0) < D(\alpha)$. From $\hat{u} > \hat{S}/2$, it follows that $2\hat{u} > \hat{S}$. Denote by $a_1$ the solution of $D(a_1) = \hat{S}$. The inequalities above imply $\hat{S} < D(a_0) < D(\alpha)$, which also implies $a_1 < a_0 < \alpha$. According to condition (IA.11), the function $I(\rho, w) + I(\rho, -w)$ is strictly decreasing in $w$ for $w > 0$. Thus, $\mathcal{U}_{\text{BLO}}(w_t = -a_0) > \hat{S}/2 + \big(I(\rho, \alpha) + I(\rho, -\alpha)\big)/2$. It remains to check that $\hat{S}/2 + \big(I(\rho, \alpha) + I(\rho, -\alpha)\big)/2 > -\hat{\omega}$, but only if $D(\alpha) > \hat{S}$. But this follows from condition (IA.13).

Next, denote by $\mathsf{E}_{w_t} \hat{\mathcal{U}}^{\text{PNB}}$ the normalized expected payoff of the informed PNB before observing $w_t$, but after paying the information cost $C$. Using the formulas above, one computes $\mathsf{E}_{w_t} \hat{\mathcal{U}}^{\text{PNB}} = \int_{-\infty}^{-\alpha} \big(-\hat{S}/2 - w - \hat{u}\big)\phi(w)\,\mathrm{d}w + \int_{-\alpha}^{-a_0} \big(\hat{S}/2 + I(\rho, -w) - \hat{u}\big)\phi(w)\,\mathrm{d}w + \int_{-a_0}^{\alpha} \big(\hat{S}/2 + I(\rho, w) + \hat{u}\big)\phi(w)\,\mathrm{d}w + \int_{\alpha}^{\infty} \big(w - \hat{S}/2 + \hat{u}\big)\phi(w)\,\mathrm{d}w$. It is necessary to compare $\mathsf{E}_{w_t} \hat{\mathcal{U}}^{\text{PNB}} - \hat{C}$ with the normalized payoff of the uninformed PNB from equation (A-16) in the Appendix of the paper, $\hat{\mathcal{U}}_{\text{BLO}}^{\text{PNB}} = \hat{u} + \hat{S}/2 - \Delta/V$. Using (IA.9), write $\hat{C} = \hat{C}_0 + \hat{u} = 2\big(\phi(\alpha) + \int_0^{\alpha} I(\rho, w)\phi(w)\,\mathrm{d}w\big) + \hat{u}$. Then, since $\hat{u} = D(a_0)/2$ (which comes from the definition of $a_0$), one computes:

$$\text{(IA.29)}\quad \mathsf{E}_{w_t} \hat{\mathcal{U}}^{\text{PNB}} - \hat{C} - \hat{\mathcal{U}}_{\text{BLO}}^{\text{PNB}} \;=\; \frac{\Delta}{V} - \frac{\hat{S}}{4} - \frac{D(a_0)}{2} - \Phi(-a_0)D(a_0) - \int_0^{a_0} D(w)\phi(w)\,\mathrm{d}w.$$

I check that the PNB does not acquire information, which is equivalent to $\mathsf{E}_{w_t} \hat{\mathcal{U}}^{\text{PNB}} - \hat{C} - \hat{\mathcal{U}}_{\text{BLO}}^{\text{PNB}} < 0$. The derivative of (IA.29) with respect to $a_0$ equals $-D'(a_0)/2 - \Phi(-a_0)D'(a_0) < 0$, since $D$ is strictly increasing. Thus, it is sufficient to verify the inequality when $a_0 = a_1$, which is defined by $D(a_1) = \hat{S}$. The last term in (IA.29) is negative, and $a_0 < \alpha$ implies $\Phi(-a_0) > \Phi(-\alpha) = 1/4$. Hence, since $D(a_0) = \hat{S}$, it is sufficient to verify that $\Delta/V - \hat{S}/4 - D(a_0)/2 - D(a_0)/4 = \Delta/V - \hat{S} < 0$, which is equivalent to $\Delta < s$. But this follows from condition (IA.10), which implies that $\Delta(1 + \gamma) < s$.

**Patient Speculators**

I show that, as specified by $\mathcal{S}(1)$, the PS prefers to acquire information. For this, I compute the expected payoff of the PS before observing $w_t$, by integrating (A-19) over $w_t$. If one denotes $H(x) = \int_0^x I(\rho, w)\phi(w)\mathrm{d}w$, the ex-ante normalized expected payoff equals $\mathsf{E}_{w_t} \hat{\mathcal{U}}^{\text{PS}} = 2\Big(\int_0^{\alpha} \big(w - \hat{S}/2\big)\phi(w)\,\mathrm{d}w + \int_{\alpha}^{\infty} \big(\hat{S}/2 + I(\rho, w)\big)\phi(w)\,\mathrm{d}w\Big) = 2\big(\phi(\alpha) + H(\alpha)\big)$; I have used the fact that $\int_0^{\alpha} \phi(w)\,\mathrm{d}w = \int_{\alpha}^{\infty} \phi(w)\,\mathrm{d}w = 1/4$. Since by (IA.9) the normalized information cost is by definition $\hat{C}_0 = 2\big(\phi(\alpha) + H(\alpha)\big)$, it follows that the PS's ex-ante expected payoff is always at least equal to $C_0$. This implies that the PS weakly prefers to acquire information.

Suppose that the PS does not acquire information, which is out-of-equilibrium behavior. If the PS arrives at $t$, she believes the asset value to be distributed according to the public density $\mathcal{N}\big(\mu_t, V^2\big)$. By submitting the BLO, the public mean moves up by $\gamma\Delta$ (the price impact of a BLO). The PS, however, knows that she is in fact uninformed, hence she maintains the belief that on average the asset value is the previous public mean ($\mu_t$), while all the other uninformed traders believe the asset value is on average $\mu_{t+1} = \mu_t + \gamma\Delta$. This difference of beliefs makes the PS now essentially an informed trader. Moreover, the asset value also changes by $v_{t+1} = v_t + \sigma_I \varepsilon_{t+1}$, where $\varepsilon_{t+1} \sim \mathcal{N}(0,1)$. Therefore, the PS's belief at $t+1$ is given by the density $\mathcal{N}(\mu_{t+1} - \gamma\Delta, V^2 + \sigma_I^2)$, which corresponds to a normalized density for the signal $w_{t+1} = \frac{v_{t+1} - \mu_{t+1}}{V}$ equal to:

$$g_1 = \mathcal{N}\left(-\gamma\frac{\Delta}{V}, \frac{V^2 + \sigma_I^2}{V^2}\right). \tag{IA.30}$$

By Lemma A2 in the Appendix of the paper applied to the density $g_1$, it follows that the normalized expected payoff of the PS from a BLO is $\hat{\mathcal{U}}^{\mathrm{PS}}_{\mathrm{BLO}} = \frac{\hat{S}}{2} J(\rho, g_1, 1) + I(\rho, g_1, 1)$. But, by condition (IA.10), $J = 1$, hence:

$$\hat{\mathcal{U}}^{\mathrm{PS}}_{\mathrm{BLO}} = \frac{\hat{S}}{2} + I(\rho, g_1, 1). \tag{IA.31}$$

But by condition (IA.16), this payoff is negative. By symmetry, the PS's expected payoff from submitting an SLO is also negative. Moreover, the expected payoff from submitting BMO or SMO is $-S/2 < 0$. Therefore, conditional on not acquiring information, the PS prefers to submit NO (no order), which has zero expected payoff. This proves that $\mathcal{S}(\mathrm{o})$ is optimal.

**Impatient Traders**

I study the decision of an impatient trader (INB or IS, or by symmetry INS) whether to acquire information, and I show that the optimal strategy conditional on acquiring information is the one specified by $\mathcal{S}(\mathrm{n})$. Suppose the impatient trader pays $C$ and observes an asset value $v_t$, or equivalently a signal $w_t = \frac{v_t - \mu_t}{V}$. Denote by $\hat{u} = \frac{u}{V}$ his normalized private valuation, where $u \in \{0, \bar{u}\}$; by $\hat{S} = \frac{s}{V}$ the normalized bid–ask spread; by $\hat{r} = \frac{\bar{r}}{V}$ the normalized waiting cost; and by $\hat{\omega} = \frac{\omega}{V} = \frac{\gamma\Delta}{V}$ the normalized commitment parameter (which is a penalty for not trading).

Let $\hat{\mathcal{U}}_{\mathcal{O}} = \frac{\mathcal{U}_{\mathcal{O}}}{V}$ be the normalized payoff of this trader from submitting $\mathcal{O}$ and subsequently following $\mathcal{S}$. As in the proof of Theorem IA.1 for the INB, let $\mathcal{U}_{\mathrm{BLO}}$ be the expected payoff from a one-stage deviation by submitting a BLO at $t$ and switching to BMO at $t+1$. An informed trader who arrives at $t$ and observes the signal $w_t$, has at $t+1$ a belief about the signal $w_{t+1}$ given by $g_1 = \mathcal{N}\left(\nu_1, \tau_1^2\right)$, where $\nu_1 = w_t - \gamma\frac{\rho}{\beta}$ and $\tau_1 = \frac{\sigma_I}{V} = \rho\sqrt{\frac{1+\gamma^2}{2\beta^2}}$ (see Definition 1 in the paper). I use the notation from Lemma A1 in the Appendix of the paper. By equation (A-2), the informed trader computes a probability of SMO at $t+1$ equal to:

$$\text{(IA.32)} \qquad P_{\text{SMO}} \;=\; \frac{1-\rho}{4} + \rho\int_{-\infty}^{\alpha} g_1(z)\,\mathrm{d}z \;=\; \frac{1-\rho}{4} + \rho\,\Phi\Big(\frac{\alpha-\nu_1}{\tau_1}\Big).$$

The only time the informed trader benefits from the BLO is when his order is executed at $t+1$ by an SMO, in which case his normalized expected payoff is $\hat{S}/2 + \nu_1 + \delta_{2,\text{SMO}}$, where $\delta_{2,\text{SMO}}$ is the adverse selection coming from the SMO (it usually has a negative sign). If the order is of the other 3 types, $\mathcal{O} \in \{\text{BMO}, \text{BLO}, \text{SLO}\}$, the normalized expected payoff from switching to a BMO immediately after the order $\mathcal{O}$ is $-\hat{S}/2 + \nu_{2,\mathcal{O}} = -\hat{S}/2 + \nu_1 - \delta_{\mathcal{O}} + \delta_{2,\mathcal{O}}$, where the last equality follows from equation (A-5) in the Appendix of the paper. Since the average adverse selection $\delta_{2,\mathcal{O}}$ is 0 over all 4 types of orders (see Lemma A1), the average normalized payoff net of waiting costs is:

$$\text{(IA.33)} \qquad \hat{\mathcal{U}}_{\text{BLO}} \;=\; \nu_1 - \frac{\hat{S}}{2} + P_{\text{SMO}}\,\hat{S} - \sum_{\mathcal{O}\in\{\text{BMO},\text{BLO},\text{SLO}\}} P_{\mathcal{O}}\delta_{\mathcal{O}}.$$

But the informed trader incurs a waiting cost equal to $\frac{\bar{r}}{\lambda}$, because between $t$ and $t+1$ he waits a clock time equal to $\frac{1}{\lambda}$. Since $\nu_1 = w_t - \delta_{\text{BLO}}$, if $\delta = \delta_{\text{BMO}} = \frac{\rho}{\beta}$, one computes:

(IA.34)

$$\hat{\mathcal{U}}_{\text{BMO}} \;=\; w_t - \frac{\hat{S}}{2}, \quad \hat{\mathcal{U}}_{\text{BLO}} \;=\; w_t - \gamma\delta - \frac{\hat{S}}{2} + P_{\text{SMO}}\hat{S} - P_{\text{BMO}}\delta - P_{\text{BLO}}\gamma\delta + P_{\text{SLO}}\gamma\delta - \frac{\bar{r}}{\lambda V}.$$

The (non-normalized) difference in payoff between BMO and BLO is:

$$\text{(IA.35)} \qquad \mathcal{U}_{\text{BMO}} - \mathcal{U}_{\text{BLO}} \;=\; -P_{\text{SMO}}s + P_{\text{BMO}}\Delta + P_{\text{BLO}}\gamma\Delta + (1-P_{\text{SLO}})\gamma\Delta + \frac{\bar{r}}{\lambda}.$$

Hence, to have $\mathcal{U}_{\text{BMO}} > \mathcal{U}_{\text{BLO}}$ is sufficient to have $\frac{\bar{r}}{\lambda} > s$, which is true since $\bar{r} > \lambda S$.

It follows that BMO is always preferred to BLO, and by a symmetric argument SMO is always preferred to SLO. Therefore, I need to compare only BMO, SMO, and NO. Denote by $a^{\pm} = \hat{S}/2 - \hat{\omega} \pm \hat{u}$. One has $a^+ + a^- = \hat{S}/2 - \hat{\omega} > 0$, since $S/2 - \omega = S/2 - \gamma\Delta > 0$ by condition (IA.10). (Indeed, $S > \Delta(1+\gamma) > 2\gamma\Delta$, since $\gamma \approx 0.2554 < 1$). This implies $-a^- < a^+$. Note that at $w_t = -a^-$ the trader is indifferent between SMO and NO, while at $w_t = a^+$ the trader is indifferent between BMO and NO. This proves that the strategy specified by $\mathcal{S}(\mathrm{n})$ is indeed optimal.

Next, denote by $\mathsf{E}_{w_t}\hat{\mathcal{U}}$ the normalized expected payoff of the impatient trader before observing $w_t$, but after paying the information cost $C = C_0 + u$, where $u$ is the trader's private valuation. Denote by $f(x) = \phi(x) - x\Phi(-x)$, and note that $f'(x) = -\Phi(-x)$. Using the formulas above, one computes $\mathsf{E}_{w_t}\hat{\mathcal{U}} = \int_{-\infty}^{-a^-}\big(-\hat{S}/2 - w - \hat{u}\big)\phi(w)\mathrm{d}w + \int_{-a^-}^{a^+}(-\hat{\omega})\phi(w)\mathrm{d}w + \int_{a^+}^{\infty}\big(w - \hat{S}/2 + \hat{u}\big)\phi(w)\,\mathrm{d}w = -\hat{\omega} + f(a^+) + f(a^-)$. Then, the expected payoff net of information cost is $\mathsf{E}_{w_t}\hat{\mathcal{U}} - \hat{C}_0 - \hat{u} = -\hat{\omega} - \hat{C}_0 + f(a^+) + f(a^-) - \hat{u}$. I now show that $\mathsf{E}_{w_t}\hat{\mathcal{U}}$ is decreasing in $\hat{u}$. Indeed, one verifies that the derivative of $\mathsf{E}_{w_t}\hat{\mathcal{U}} - \hat{C}_0 - \hat{u}$ with respect to $\hat{u}$ equals $\Phi(-a^+) - \Phi(-a^-) - 1 = -\Phi(-a^-) - \Phi(a^+) < 0$. Thus, $\mathsf{E}_{w_t}\hat{\mathcal{U}} - \hat{C}_0 - \hat{u}$ attains its maximum at $\hat{u} = 0$. When $\hat{u} = 0$, denote by $a = a^+ = a^- = \hat{S}/2 - \hat{\omega}$. Then, $\mathsf{E}_{w_t}\hat{\mathcal{U}} - \hat{C}_0 - \hat{u} = 2f(a) - \hat{\omega} - \hat{C}_0$. By condition (IA.15), $2f(a) - \hat{\omega} - \hat{C}_0 < 0$. It follows that the impatient trader has a negative expected payoff conditional on acquiring information. Thus, the impatient trader does not acquire information in equilibrium.

The proof of Theorem IA.2 is now complete. □

**Proof of Corollary IA.1**. Ex ante, an informed trader who arrives at $t$ believes that his signal $w_t = \frac{v_t - \mu_t}{V}$ is distributed according to the standard normal density $\mathcal{N}(0,1)$. From equation (10), the informed trader's expected payoff when $w$ is in the interval $\mathcal{O} \in \{\mathrm{BMO}, \mathrm{BLO}, \mathrm{SLO}, \mathrm{SMO}\}$ is, respectively, $\big\{-\frac{S}{2} + wV, \frac{S}{2} + I(\rho, w)V, \frac{S}{2} + I(\rho, -w)V, -\frac{S}{2} - wV\big\}$. By integrating this payoff over the standard normal density, one obtains the formula for $C_0$ in equation (IA.9). Hence, the ex-ante expected utility of the informed trader is 0. □

**Proof of Proposition IA.1**. Combining equations (IA.7) and (IA.8), one obtains:

$$\text{(IA.36)} \qquad \lambda \;=\; \ell(N_0 + N_{\bar{u}}) \;=\; \frac{\ell N_{\bar{u}}}{1-\rho}, \qquad \Delta \;=\; \sqrt{\tfrac{2}{1+\gamma^2}}\,\frac{\sigma_v}{\sqrt{\lambda}} \;=\; m\,\sqrt{1-\rho}.$$

From the definition of $F(\rho)$, one obtains:

$$\text{(IA.37)} \qquad C_0 \;=\; F(\rho)\Delta \;=\; mF(\rho)\sqrt{1-\rho} \;=\; m\,H(\rho) \quad\Longrightarrow\quad \rho \;=\; H^{-1}\left(\frac{C_0}{m}\right).$$

The other formulas follow from simple algebraic manipulation, and from the equations for $V$ and $S$ in (IA.8).

It is clear that the informed share $\rho$ is strictly decreasing in the information cost $C_0$, because both $H$ and its inverse are strictly decreasing functions. Moreover, it is clear that $\Delta$, $V$ are strictly decreasing in $\rho$, while $\lambda$ and $N_0$ are strictly increasing in $\rho$. For the bid–ask spread parameter, Result 2 implies that $\frac{S}{\Delta}$ is strictly decreasing in $\rho$ (see also Figure 3). Since $\Delta = m\sqrt{1-\rho}$ is also strictly decreasing in $\rho$, it follows that $S$ is also strictly decreasing in $\rho$. $\square$

**Proof of Proposition IA.2**. The fact that $\mathcal{U}^I = 0$ is essentially Corollary IA.1. The formula $\mathcal{U}^{UP} = \frac{S}{2} - \Delta + \bar{u}$ follows from Lemma A3 in the Appendix of the paper. The formula $\mathcal{U}^{UI} = -\frac{S}{2} + \bar{u}$ is straightforward, since in the equilibrium the uninformed impatient traders submit market orders which execute at the ask or at the bid, which are a half bid–ask spread away from the public mean. The equation $\bar{\mathcal{U}} = (1-\rho)\bar{u} - \frac{1-\rho}{2}\Delta$ follows from the formula:

$$\bar{\mathcal{U}} \; = \; \rho\mathcal{U}^I + \frac{1-\rho}{2}\mathcal{U}^{UP} + \frac{1-\rho}{2}\mathcal{U}^{UI}. \tag{IA.38}$$

To obtain the second part of equation (IA.24), I use the formulas from Proposition IA.1. To obtain equation (IA.25), note that from Proposition IA.1, $N_0 + N_{\bar{u}} = \frac{N_{\bar{u}}}{1-\rho}$. $\square$

# 4 Verification of Numerical Results

In this section, I verify the various properties of the information functions $I(\rho, w, j)$ and $J(\rho, w, j)$ from Definition 1 in the paper. The definition of these functions is completely formal, but each variable has an interpretation in the model. The definitions are:

$$\text{(IA.39)} \qquad I(\rho, w, j) = \sum_{Q \in \mathcal{Q}} P(Q)\nu(Q), \qquad J(\rho, w, j) = \sum_{Q \in \mathcal{Q}} P(Q),$$

where the different variables are interpreted as follows: First, the input $w$ represents the initial signal $w_0 = \frac{v_0 - \mu_0}{V}$ of an informed trader, before she submits a BLO at $t = 0$. The input $\rho$ is the informed share. The input $j$ represents the rank $j_0 = 1, 2, \ldots$ of the initial BLO in the bid queue. Let $g_t$ be the posterior density of the signal $w_t = \frac{v_t - \mu_t}{V}$ before trading at $t$ (after observing the sequence of orders $\mathcal{O}_0 = \text{BLO}$, $\mathcal{O}_1$, …, $\mathcal{O}_{t-1}$), and its mean is $\nu_t = \mathsf{E}(g_t)$. I define an execution sequence to be a sequence of orders $Q = (\mathcal{O}_0 = \text{BLO}, \mathcal{O}_1, \ldots, \mathcal{O}_T = \text{SMO})$, where $\mathcal{O}_t \in \{\text{BMO}, \text{BLO}, \text{SLO}, \text{SMO}\}$, such that the last order (SMO) executes the initial BLO. (This translates into the final rank of the BLO being 0, after trading at $T$.) Let $P(Q)$ be the ex-ante probability of a particular execution sequence $Q = (\mathcal{O}_0, \mathcal{O}_1, \ldots, \mathcal{O}_T)$. Also, let $\nu(Q) = \nu_{T+1} - \frac{\rho}{\beta}$ be the expected signal $w_T$ after the execution at $T$.[14] Then, $I(\rho, w, j)$ is the expected signal immediately after execution, where the expectation is take at $t = 0$ over all possible execution sequences. Finally, the function $J(\rho, w, j)$ is simply the probability that an initial BLO is eventually executed, when the BLO starts from rank $j = j_0$ in the bid queue. If $j = 1$, I write $I(\rho, w)$ instead of $I(\rho, w, 1)$, and $J(\rho, w)$ instead of $J(\rho, w, 1)$.

Since there are infinitely many such execution sequences, I do not attempt to compute the information function in closed form. The definition, however, suggests a simple Monte Carlo procedure, described in Section 4.1, and refined in Section 4.2. The verification of the main numerical results is done in Section 4.3.

## 4.1 A Monte Carlo Procedure

In this section, I describe a Monte Carlo procedure to compute the information function with a good approximation. I choose two main parameters:

[14] The expected signal $w_T$ after $T$ is similar to the expected signal $w_{T+1}$ before $T+1$ (which is simply $\nu_{T+1}$), except that the public means at $T$ and $T+1$ differ by the price impact of an SMO, which is $-\Delta$. Therefore, $\nu(Q) = \nu_{T+1} - \frac{\Delta}{V} = \nu_{T+1} - \frac{\rho}{\beta}$. For more details, see the proof of Lemma A2 in the Appendix of the paper.

- $M$ = the number of execution sequences $Q$ for which one computes $\nu(Q)$, and

- $L$ = the maximum length of the execution sequence $Q$.

I now describe the procedure. Fix $w \in \mathbb{R}$ and $\rho \in (0,1)$. For each $m = 1, \ldots, M$, the procedure yields a random order sequence $Q^{(m)}$, along with two numbers, $\nu^{(m)}$ and $i^{(m)}$. The number $\nu^{(m)}$ is interpreted the average signal at execution of the initial BLO along the sequence $Q^{(m)}$, while $i^{(m)}$ is equal to 1 if the sequence $Q^{(m)}$ executes the initial BLO until time $L$, or is equal to 0 otherwise.

As suggested by Definition 1 in the paper, one starts a sequence $Q^{(m)}$ by specifying the initial density, $g_1 = \mathcal{N}\left(w - \gamma\frac{\rho}{\beta}, \rho^2\frac{1+\gamma^2}{2\beta^2}\right)$, and the initial bid rank (rank in the bid queue), $j_0 = 1$. Then, one computes $P_{\mathcal{O}}$, the probability of each type of order $\mathcal{O} \in \{\text{BMO}, \text{BLO}, \text{SLO}, \text{SMO}\}$, by using the formula $P_{\mathcal{O}} = \pi_{g_1,\mathcal{O}}$, where, as in equation (5) in the paper,

$$\text{(IA.40)} \quad \pi_{g,\mathcal{O}} = \frac{1-\rho}{4} + \rho\int_{z\in i_{\mathcal{O}}} g(z)\mathrm{d}z, \quad \text{with} \quad i_{\mathcal{O}} \in \{(\alpha,\infty),(0,\alpha),(-\alpha,0),(-\infty,-\alpha)\}.$$

Using the 4 probabilities $P_{\mathcal{O}}$, I choose $\mathcal{O}_1$ randomly (with probability $P_{\mathcal{O}}$) among the 4 types of orders. I then update $g_2$ to include the information contained in the order $\mathcal{O}_1$, by using the formula $g_2 = f_{g_1,\mathcal{O}_1}$, where, as in equation (5) in the Appendix of the paper,

$$\text{(IA.41)} \qquad f_{g,\mathcal{O}}(x) = \frac{\int\left(\frac{1-\rho}{4} + \rho\mathbf{1}_{z\in i_{\mathcal{O}}}\right) g(z)\phi\left(x; z-\delta_{\mathcal{O}}, \rho\sqrt{\frac{1+\gamma^2}{2\beta^2}}\right)\mathrm{d}z}{\pi_{g,\mathcal{O}}},$$

where $\delta_{\mathcal{O}} \in \left\{\frac{\rho}{\beta}, \gamma\frac{\rho}{\beta}, -\gamma\frac{\rho}{\beta}, -\frac{\rho}{\beta}\right\}$, respectively, and $\phi(\,\cdot\,; m, s)$ is the normal density with mean $m$ and standard deviation $s$.

Also, if the order $\mathcal{O}_1 \in \{\text{BMO}, \text{BLO}, \text{SLO}, \text{SMO}\}$, set the increment in the bid rank to be, respectively, $j_{\mathcal{O}_1} \in \{0, +1, 0, -1\}$. Hence, one updates the bid rank to $j_1 = j_0 + j_{\mathcal{O}_1}$. One continues to choose randomly $\mathcal{O}_t$ and update $g_{t+1}$, its mean $\nu_{t+1}$, and the bid rank $j_t$ in the same way as described thus far, until one of the following two scenarios occurs:

- The bid rank $j_T = 0$ for some $T \leq L$. This means that the initial BLO is executed at $T$ (the bid rank is 0). In this case, one sets $\nu^{(m)} = \nu_{T+1} - \frac{\rho}{\beta}$, and $i^{(m)} = 1$.

- The bid rank $j_t > 0$ for all $t = 1, \ldots, L$. This means that the initial BLO has not been executed until $L$. In this case, one sets $\nu^{(m)} = \nu_{L+1} - \frac{\rho}{\beta}$, and $i^{(m)} = 0$.

I finally define the numerical estimate of the information function as the average value of the numbers $\nu^{(m)}$, i.e.,

$$\hat{I}(\rho, w) = \frac{1}{M} \sum_{m=1}^{M} \nu^{(m)}. \tag{IA.42}$$

Note that $\hat{g}$ estimates the informed trader's average signal $w_T$ after the random sequence $Q^{(m)}$ executes the initial BLO. (Proposition 2 in the paper shows that this interpretation is correct.)

The function $J$ from Definition 1 in the paper can be numerically estimated by the average of the numbers $i^{(m)}$, i.e.,

$$\hat{J}(\rho, w) = \frac{1}{M} \sum_{m=1}^{M} i^{(m)}. \tag{IA.43}$$

Note that $\hat{J}$ estimates what percentage of the time the random sequence $Q^{(m)}$ leads to the execution of the initial BLO.

## 4.2 Practical Issues

The default values used in practice are:

- Informed share, $\rho \in [0.05, 0.95]$, step $= 0.05$ (19 values);
- Initial signal: $w \in [-5\alpha, 5\alpha] \approx [-3.3724, 3.3724]$, step $= \frac{\alpha}{30} \approx 0.0225$ (301 values);
- Number of iterations: $M = 5,000$;
- Maximum order flow length: $L = 100$;
- Span for the moving average along the values of $w$: $Span = 25$.

The baseline procedure described in Section 4.1 in this Internet Appendix computes the exact density $g_t$ at each step, and is therefore extremely memory-intensive. A close alternative is to approximate the density $g_t$ with a normal density $\mathcal{N}(\nu_t, \tau_t^2)$. This approach is consistent with the principle stated in Section II in the paper, that information processing is difficult, and traders (including informed traders) may not be able to compute all the moments of the density. Thus, implicitly here I assume that informed traders compute their beliefs using only the first two moments of the density.

Thus, I modify the procedure in Section 4.1 by changing only the way the density $g_t$ is updated, in such a way that it stays normal. Note that the initial density $g_1$ is normal. By induction, assuming that $g_t$ is normal, equation (A-8) from Lemma A1 in the Appendix of the paper provides simple formulas to compute the mean $\nu_{t+1}$ and standard deviation $\tau_{t+1}$ of the correct density $g_{t+1,c\mathcal{O}_t}$ after the informed trader observes the order $\mathcal{O}_t$. Then, approximate $g_{t+1,\mathcal{O}_t}$ with the normal density $\mathcal{N}(\nu_{t+1,\mathcal{O}_t}, \tau^2_{t+1,\mathcal{O}_t})$, and continue the procedure as before.[15]

With this procedure, if one sets $M = 15,000$ and $L = 40,000$, the function $I$ is computed with a very good approximation. To improve the precision, I also consider a smoothed-out version of $\hat{g}$ which takes a moving average over the initial signal $w$, with *Span* $= 25$. I perform the following tests to verify that the procedure produces good approximations.

First, I check that estimate of the function $\hat{J}$ is approximately equal to 1. I find that the minimum over all the values of $\rho$ and $w$ considered is 0.9587, and the mean is 0.9857. These results indicate that for most sequences $Q^{(m)}$ the initial BLO is executed, which reduces the potential bias that might occur because in a small percentage of cases the BLO is not executed.

Second, I compare the raw estimate ($I_{\text{raw}}$) with a smoothed-out version ($I_{\text{smooth}}$) which takes a moving average over the values of $w$, with *Span* $= 25$. Figure IA.1 shows that the two estimates are approximately equal, which means that taking $M = 15,000$ and $L = 40,000$ provides very good estimates. Numerically, I also verify the magnitude of the error when $w = 1$, and the maximum error is $\max_\rho |I_{\text{raw}}(\rho, 1) - I_{\text{smooth}}(\rho, 1)| \approx 0.0100$, which is relatively small.

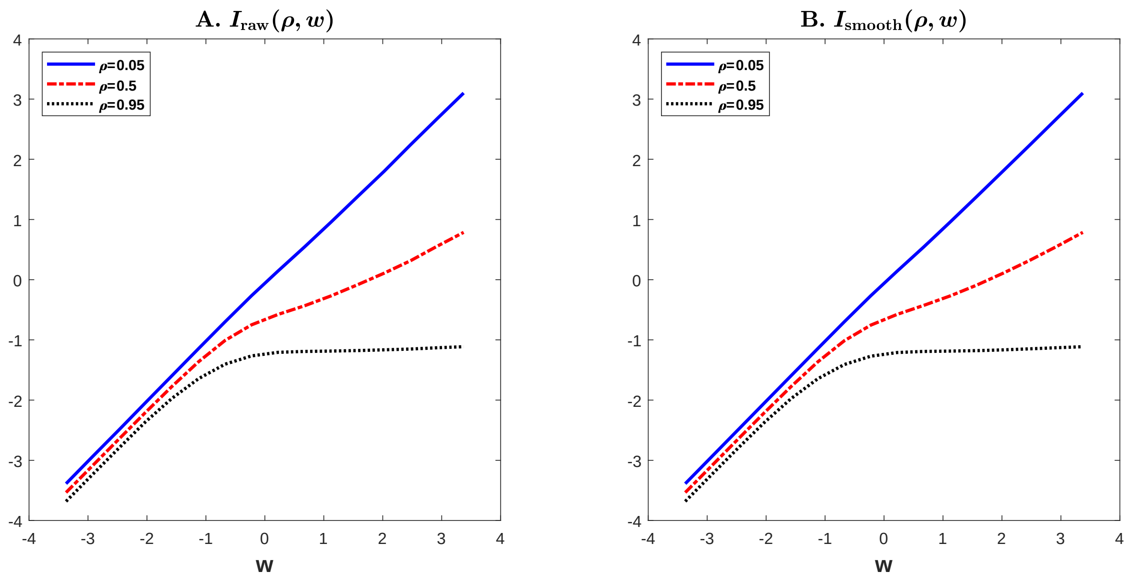
Third, I compare the smoothed-out version for $M = 15,000$ and $L = 40,000$ with the smoothed-out version for $M = 5,000$ and $L = 100$. The goal is to verify that smoothing out the estimates increases precision enough that one does not need to consider large values of $M$ and $L$ (which are very time consuming). As before, the maximum error is $\max_\rho |I_{M=15,000,\, L=40,000}(\rho, 1) - I_{M=1,000,\, L=100}(\rho, 1)| \approx 0.0102$, which again is relatively small.

[15]This approximate procedure improves the execution time by a factor larger than $10^4$ when $\rho$ is large, and larger than $10^6$ when $\rho$ is small. To verify that the approximation is good, I verify that average absolute difference in $\nu_t$ (for the 19 values of $\rho$) is $0.0111 \times \Delta$, with a standard deviation of $0.0098 \times \Delta$, and a maximum of $0.0321 \times \Delta$. These values are small compared for instance with the price impact of a limit order, which is $\gamma\Delta \approx 0.2554 \times \Delta$. Also, the average absolute difference in $\tau_t$ is $0.0336 \times \Delta$, the maximum $0.0746 \times \Delta$, and the standard deviation $0.0207 \times \Delta$.

FIGURE IA.1

**Raw versus Smoothed-Out Estimates of the Information Function**

Figure IA.1 shows estimates of the information function $I(\rho, w)$, where $\rho$ is the informed share, and $w$ is the initial signal observed by the informed trader who submits a BLO. Graph A shows $I_{\text{raw}}(\rho, w)$, which is the raw output from the Monte Carlo procedure. Graph B shows $I_{\text{smooth}}(\rho, w)$, which is the smoothed-out version of $I_{\text{raw}}(\rho, w)$ that takes a moving average over $w$ with $Span = 25$. Each graph considers the function for the following values of the informed share: $\rho = 0.05$ (solid line), $\rho = 0.5$ (dashed-dotted line), and $\rho = 0.95$ (dotted line). The Monte-Carlo parameters are $M = 15,000$ and $L = 40,000$.




## 4.3 Numerical Properties of the Information Function

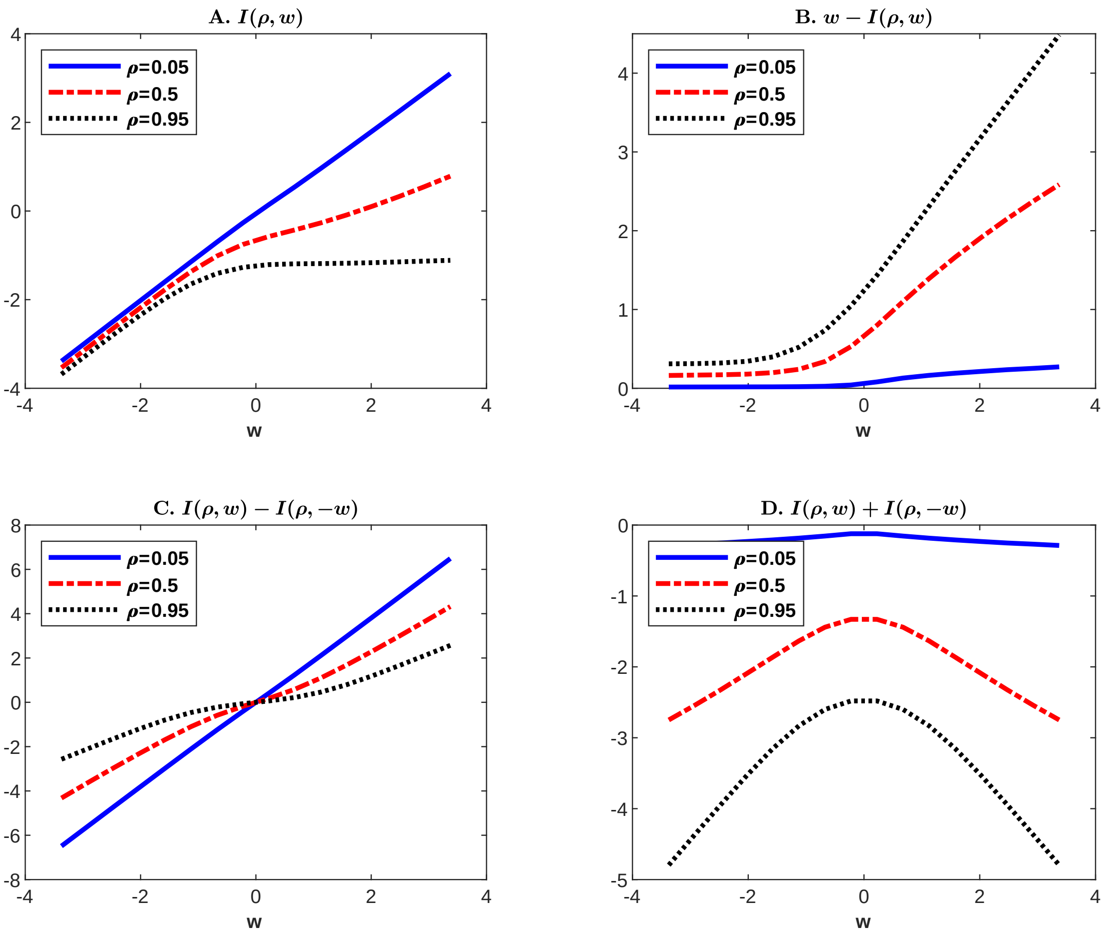
**Verification of Result 1**. I verify numerically the conditions of Result 1, which are stated in the equations (IA.10) in this Internet Appendix. I use the Monte Carlo procedure described above, with normal approximation of densities, and moving-average smoothing. In general, I use the values of $\rho$ and $w$ described in Section 4.2, and the Monte-Carlo parameters $M = 5,000$, and $L = 100$ mentioned above.

I begin with the part of the condition (IA.10) which states that the functions $I(\rho, w)$, $A(\rho, w) = w - I(\rho, w)$, $D(\rho, w) = I(\rho, w) - I(\rho, -w)$ are strictly increasing in $w$. As a consequence, the function $B(\rho, w) = A(\rho, w) + D(\rho, w)$ is also strictly increasing in $w$. I only display the results for $\rho \in \{0.05, 0.50, 0.95\}$. Graphs A, B, and C in Figure IA.2 show that the functions $I(\rho, w)$, $w - I(\rho, w)$, and $I(\rho, w) - I(\rho, -w)$ are indeed strictly increasing in $w$. Moreover, Graph B shows the inequality $w - I(\rho, w) > 0$.

FIGURE IA.2

**Functions of $\rho$ and $w$**

Figure IA.2 shows estimates of various functions of the informed share $\rho$ and the signal $w$. If $I(\rho, w)$ is the information function, Graph A shows the function $I(\rho, w)$, Graph B shows the function $w - I(\rho, w)$, Graph C shows the function $I(\rho, w) - I(\rho, -w)$, and Graph D shows the function $I(\rho, w) + I(\rho, -w)$. Each graph shows the function for the following values of the informed share: $\rho = 0.05$ (solid line), $\rho = 0.5$ (dashed-dotted line), and $\rho = 0.95$ (dotted line). The Monte-Carlo parameters are $M = 5,000$ and $L = 100$.

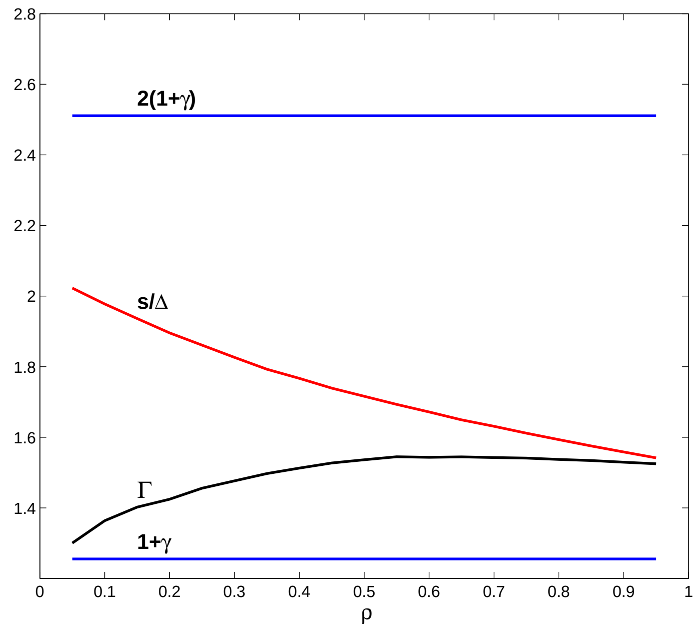



I now verify the inequality $\max\left(\frac{\rho(1+\gamma)}{\beta}, -2I(\rho,0) - 2\frac{\rho\gamma}{\beta}\right) < \alpha - I(\rho,\alpha)$ from (IA.10). In addition, I also prove the inequality $\alpha - I(\rho,\alpha) < \frac{2\rho(1+\gamma)}{\beta}$. Using the formulas $S = \big(\alpha - I(\rho,\alpha)\big)V$, and $V = \beta\rho^{-1}\Delta$, these inequalities are equivalent to: (i) $\frac{S}{\Delta} \in [1+\gamma, 2(1+\gamma)]$, and (ii) $\frac{S}{\Delta} > \Gamma(\rho) = -\frac{2\beta}{\rho}I(\rho,0) - 2\gamma$. Figure IA.3 shows that indeed $\frac{S}{\Delta}$ is larger than $1+\gamma$ and $\Gamma(\rho) = -\frac{2\beta}{\rho}I(\rho,0) - 2\gamma$, and smaller than $2(1+\gamma)$. This completes the verification of the inequalities in (IA.10). Note that this figure is related to Figure 3 in the paper, which shows $\frac{S}{\Delta}$, as well as its components.

FIGURE IA.3

**Spread Inequalities**

Figure IA.3 shows various functions of the informed share $\rho$: the ratio $\frac{S}{\Delta}$, where $S$ is the bid–ask spread parameter and $\Delta$ is the price impact parameter $\Delta$; the function $\Gamma(\rho) = -\frac{2\beta}{\rho} I(\rho, 0) - 2\gamma$; the two constant function $1+\gamma$; and the constant function $2(1+\gamma)$. The Monte Carlo parameters are $M = 5,000$ and $L = 100$.




In condition (IA.10), I state that $I$ increases if the initial density $g_1$ has a positive shift in mean. In the computation of $I$, the mean of $g_1$ is $w - \gamma\frac{\delta}{\rho}$, therefore a positive shift in mean of $g_1$ implies that one considers a larger value of $w$. But, as shown before, $I$ is strictly increasing in $w$. The result is nevertheless needed in more generality, therefore I also verified it for initial standard deviations other than $\rho\sqrt{(1+\gamma^2)/(2\beta^2)}$.[16]

In condition (IA.10), I state that $I = I(\rho, w, j)$ decreases in $j$ if $w > 0$. Thus, one computes $I_j(\rho, w) = I(\rho, w, j)$ by initializing $j = 1, 2, 3, 10$ in the regular Monte Carlo procedure. I verify that this property holds, by checking that $I_1 - I_{10} > I_1 - I_3 > I_1 - I_2 > 0$. The results are shown in Figure IA.4. One sees that, except when $\rho = 0.95$ (in which case the three functions are too close to each other), the desired inequalities are true when $w > 0$.

The last part of condition (IA.10) is that $J(\rho, w) = 1$. But this is discussed at the end of Section 4.2, where this identity is shown to hold with a good approximation. $\square$

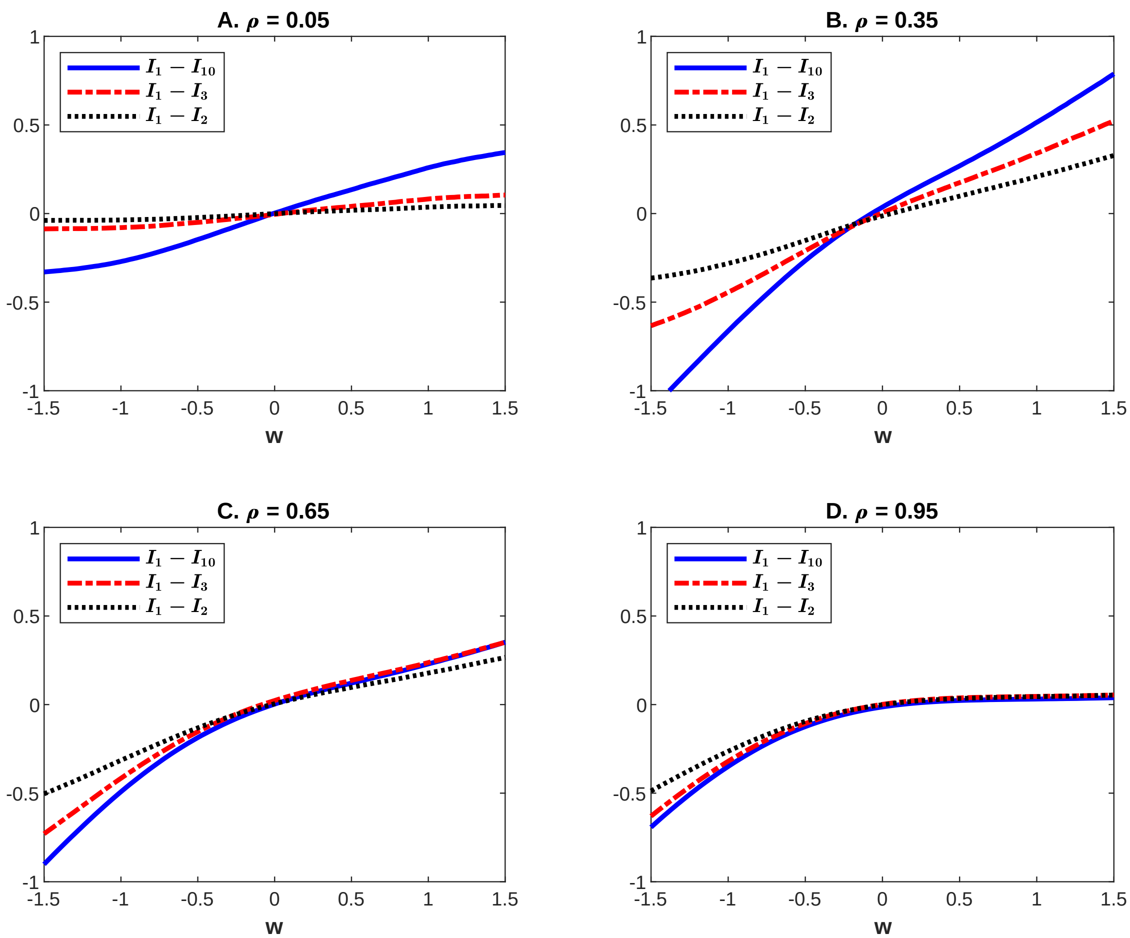
**Verification of Result 2**. The verification is essentially illustrated in Figure 3 in the pa-

[16] I did not try non-normal densities, as all of the densities are approximated by normal ones.

FIGURE IA.4

**Dependence of $I(\rho, w, j)$ on the Bid Rank $j$**

Figure IA.4 shows the differences $I_1 - I_{10}$, $I_1 - I_3$, and $I_1 - I_2$, for the information function $I_j(\rho, w) = I(\rho, w, j)$ corresponding to a general bid rank $j$. Graphs A, B, C, and D correspond, respectively, to an informed share $\rho \in \{0.05, 0.35, 0.65, 0.95\}$. The Monte Carlo parameters are $M = 5,000$ and $L = 100$.




per. The slippage function $I^s(\rho, w)$ is estimated with the same procedure as described above, except that for each $m = 1, 2, \ldots, M$ I consider $\nu^{(m)} = \nu_T$ instead of $\nu^{(m)} = \nu_{T+1} - \frac{\rho}{\beta}$. $\square$

**Verification of Result IA.1**. I verify numerically the conditions (IA.11)–(IA.17) from Section 3.2 in this Internet Appendix.

The condition (IA.11) is that $I(\rho, w) + I(\rho, -w)$ is strictly decreasing in $w$ whenever $w > 0$. But this is shown in Graph D of Figure IA.2, which shows that the function $I(\rho, w) + I(\rho, -w)$ is strictly increasing in $w$ when $w > 0$.

Recall the following notation (see equation (IA.18)):

(IA.44)

$$
\begin{aligned}
A(\rho, w) &= w - I(\rho, w), \qquad B(\rho, w) = w - I(\rho, -w), \qquad D(\rho, w) = I(\rho, w) - I(\rho, -w), \\
F(\rho) &= \frac{2\beta}{\rho}\left(\phi(\alpha) + \int_0^\alpha I(\rho, w)\phi(w)\,\mathrm{d}w\right), \qquad \hat{\omega} = \frac{\omega}{V} = \frac{\gamma\rho}{\beta}, \qquad \hat{S} = \alpha - I(\rho, \alpha), \\
\hat{C}_0 &= 2\left(\phi(\alpha) + \int_0^\alpha I(\rho, w)\phi(w)\,\mathrm{d}w\right), \qquad b = \max\big(B(\alpha), 2\hat{S}\big), \qquad a_* = B^{-1}(b), \\
F_1 &= \int_\alpha^{a_*} I(\rho, -w)\phi(w)\,\mathrm{d}w, \qquad D_1 = \int_0^\alpha D(w)\phi(w)\,\mathrm{d}w.
\end{aligned}
$$

Condition (IA.12) is: $\frac{b}{2} + \hat{\omega} + I\big(\rho, -(\frac{b}{2} - \hat{\omega})\big) > 0$. I use the numerical procedure to compute $I(\rho, w)$ for $w \in [-5\alpha, 5\alpha]$. Using a spline interpolation, one computes $I(\rho, -x)$ for $x = \frac{b}{2} - \hat{\omega}$. Graph A of Figure IA.5 displays $x + 2\hat{\omega} + I(\rho, -x)$ as a function of $\rho$, and shows that condition (IA.12) is satisfied.

Condition (IA.13) is: $\frac{\hat{S}}{2} + \frac{I(\rho,\alpha)+I(\rho,-\alpha)}{2} + \hat{\omega} > 0$ whenever $D(\alpha) > \hat{S}$. Graph B of Figure IA.5 displays $\big(\frac{\hat{S}}{2} + \frac{I(\rho,\alpha)+I(\rho,-\alpha)}{2} + \hat{\omega}\big) \cdot \mathbf{1}_{D(\alpha)>\hat{S}}$ as a function of $\rho$, and shows that condition (IA.13) is satisfied.

Condition (IA.14) is: $\frac{\rho}{\beta} + \frac{\hat{S}}{4} - \frac{b}{2} - \Phi(-a_*)b - \big(\phi(\alpha) - \phi(a_*)\big) + F_1 - D_1 < 0$. Graph C of Figure IA.5 displays $\frac{\rho(1+\gamma)}{\beta} + \frac{\hat{S}}{4} - \frac{b}{2} - \Phi(-a_*)b - \big(\phi(\alpha) - \phi(a_*)\big) + F_1 - D_1$ as a function of $\rho$, and shows that condition (IA.14) is satisfied.

Condition (IA.15) is: $2f\big(\frac{\hat{S}}{2} - \hat{\omega}\big) - \hat{\omega} - \hat{C}_0 < 0$, where $f(x) = \phi(x) - x\Phi(-x)$. Graph D of Figure IA.5 displays $2f\big(\frac{\hat{S}}{2} - \hat{\omega}\big) - \hat{\omega} - \hat{C}_0$ as a function of $\rho$, and shows that condition (IA.15) is satisfied.

Condition (IA.16) is: $\frac{\hat{S}}{2} + I(\rho, g_1, 1) < 0$, where $g_1 = \mathcal{N}\left(-\gamma\frac{\Delta}{V}, \frac{V^2+\sigma_I^2}{V^2}\right)$. Graph E of Figure IA.5 displays $\frac{\hat{S}}{2} + I(\rho, g_1, 1)$ as a function of $\rho$, and shows that condition (IA.16) is satisfied.

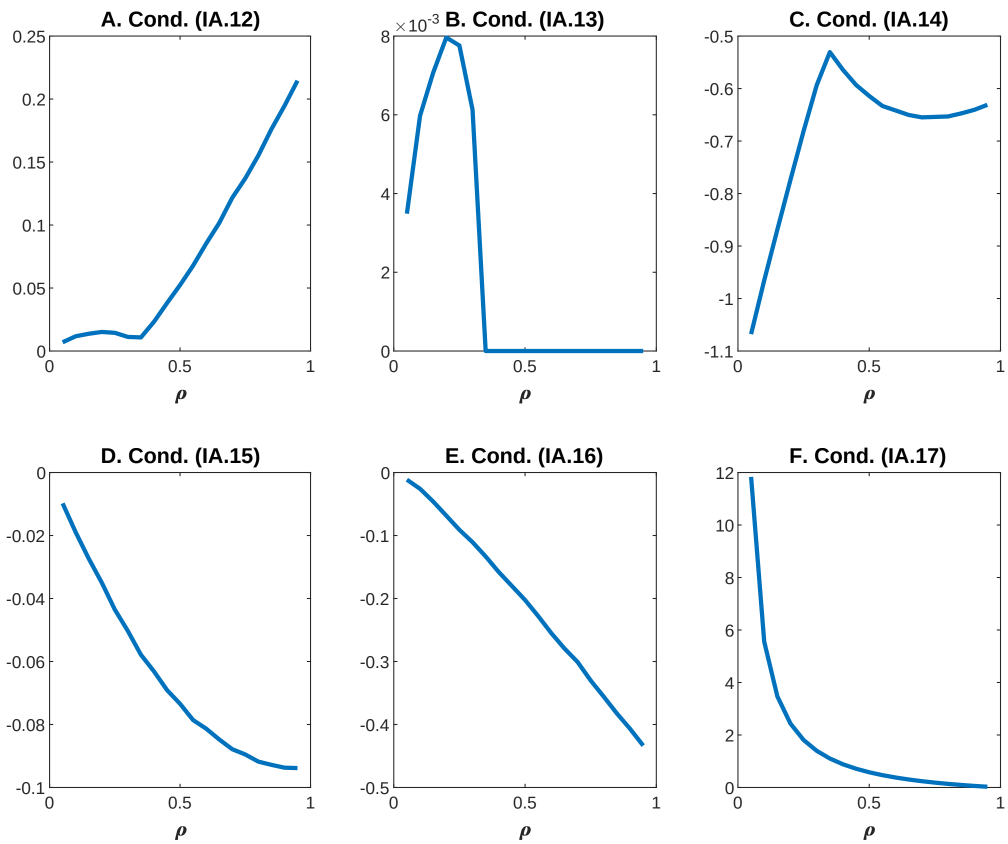
Condition (IA.17) is: $F(\rho) = \frac{2\beta}{\rho}\left(\phi(\alpha) + \int_0^\alpha I(\rho, w)\phi(w)\,\mathrm{d}w\right) : (0,1) \to (0,\infty)$ is one-to-one and strictly decreasing in $\rho$. Graph F of Figure IA.5 displays $F$ as a function of $\rho$, and shows that condition (IA.17) is satisfied. $\square$

**Verification of Result IA.2**. From condition (IA.17), the function $F(\rho) : (0,1) \to (0,\infty)$ is one-to-one and strictly decreasing in $\rho$ (see Graph F of Figure IA.5). But this implies that the function $H(\rho) = F(\rho)\sqrt{1-\rho}$ is also strictly decreasing in $\rho$. Graph A of Figure IA.6 shows that this is the case, and that the function $H : (0,1) \to (0,\infty)$ is also one-to-one.

FIGURE IA.5

## Conditions for Information Acquisition

Figure IA.5 shows various functions of the informed share $\rho$ that correspond to the conditions (IA.12)–(IA.17) from Section 3.2 in this Internet Appendix. With the notation from equation (IA.44), Graph A shows $x+2\hat{\omega}+I(\rho,-x)$, where $x=\frac{b}{2}-\hat{\omega}$ (condition (IA.12)); Graph B shows $\left(\frac{\hat{S}}{2}+\frac{I(\rho,\alpha)+I(\rho,-\alpha)}{2}+\hat{\omega}\right)\cdot\mathbf{1}_{D(\alpha)>\hat{S}}$ (condition (IA.13)); Graph C shows $\frac{\rho}{\beta}+\frac{\hat{S}}{4}-\frac{b}{2}-\Phi(-a_*)b-\big(\phi(\alpha)-\phi(a_*)\big)+F_1-D_1$ (condition (IA.14)); Graph D shows $2f\left(\frac{\hat{S}}{2}-\hat{\omega}\right)-\hat{\omega}-\hat{C}_0$, where $f(x)=\phi(x)-x\Phi(-x)$ (condition (IA.15)); Graph E shows $\frac{\hat{S}}{2}+I(\rho,g_1,1)$, where $g_1=\mathcal{N}\left(-\gamma\frac{\Delta}{V},\frac{V^2+\sigma_I^2}{V^2}\right)$ (condition (IA.16)); and Graph F shows $F(\rho)=\frac{2\beta}{\rho}\left(\phi(\alpha)+\int_0^\alpha I(\rho,w)\phi(w)\,\mathrm{d}w\right)$ (condition (IA.17)). The Monte Carlo parameters are $M=5,000$ and $L=100$.

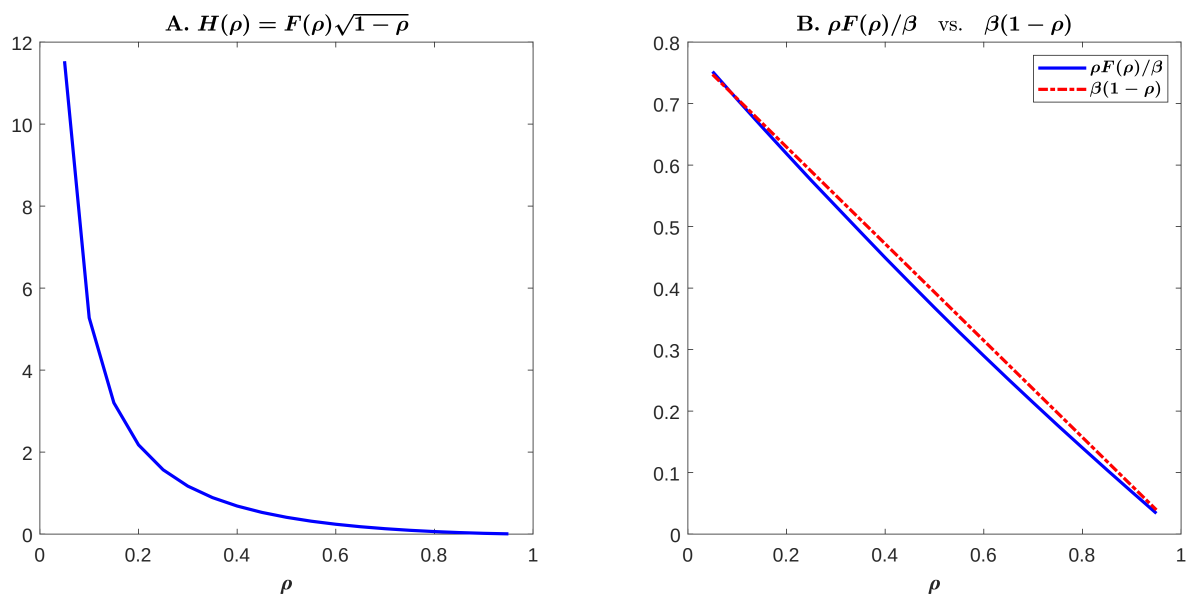



Finally, Graph B of Figure IA.6 shows that the approximation $\frac{\rho}{\beta}F(\rho)\approx\beta(1-\rho)$ is good. □

FIGURE IA.6

**Approximation for the Information Cost**

Figure IA.6 shows various functions of the informed share $\rho$. Let $F(\rho) = \frac{2\beta}{\rho}\left(\phi(\alpha) + \int_0^\alpha I(\rho, w)\phi(w)\,\mathrm{d}w\right)$, as in equation (IA.44). Graph A shows the function $H(\rho) = F(\rho)\sqrt{1-\rho}$. Graph B shows the function $\frac{\rho}{\beta}F(\rho) = 2\left(\phi(\alpha) + \int_0^\alpha I(\rho, w)\phi(w)\,\mathrm{d}w\right)$ (solid line) in comparison with the linear function $\beta(1-\rho)$ (dashed line). The Monte Carlo parameters are $M = 5,000$ and $L = 100$.



# 5 Extensions – Theory

## 5.1 Public Information Processing

In the benchmark model in the paper, the uninformed traders can compute the posterior belief of the asset value only partially: they compute correctly (i) the first moment of the posterior belief conditional on the order type, and (ii) the second moment of the posterior belief conditional on the order arrival (not conditional on the order type), but they cannot compute any higher moments. In this section, I assume that the uninformed traders can compute the entire posterior belief.

As in the previous sections, I define the public density as the uninformed traders' belief about the fundamental value just before trading at $t$. Let $\mu_t$ and $\sigma_t$ be respectively the mean and the volatility of the public density. Because the uninformed traders can compute all moments of the distribution, the shape of the public density is no longer constant and normal as in the benchmark model, but it keeps changing after each order. Nevertheless, at the initial date $t = 0$ the public density is assumed to be $\mathcal{N}(0, V^2)$, which is normal with standard deviation equal to the public volatility parameter $V$ from equation (7) in the paper. Equivalently, if I define the normalized public density $g_t$ as the density of the signal $w_t = \frac{v_t - \mu_t}{V}$, then the initial normalized public density is standard normal: $g_0 = \mathcal{N}(0, 1)$.

The other assumptions are as in the benchmark model. In particular, the informed trader can compute only the average signal at the time of her limit order's execution. In the benchmark model, this expectation is equal to the information function $I$ from Definition 1 in the paper. I assume that in the current setup, the informed traders use the same information function when computing their expected payoff from a limit order.[17]

Proposition IA.3 shows that there exists an MPE of the model if the conditions in Result IA.5 are satisfied. These conditions are verified in Section 6 below. The next result also describes several properties of the equilibrium.

**Proposition IA.3.** *If the conditions in Result IA.5 are satisfied, there exists an MPE of the game. In equilibrium, if an order* $\mathcal{O} \in \{\text{BMO}, \text{BLO}, \text{SLO}, \text{SMO}\}$ *arrives at* $t$*, the public mean changes from* $\mu_t$ *to* $\mu_t + \Delta_{g_t,\mathcal{O}}$*, and the public density changes from* $g_t$ *to* $g_{t+1} = f_{g_t,\mathcal{O}}$*, where* $\delta_{g,\mathcal{O}} = \frac{\Delta_{g,\mathcal{O}}}{V}$ *and* $f_{g,\mathcal{O}}$ *are defined in equation* (IA.64) *in the Appendix. At date* $t$*, the*

[17] This is consistent with the principle that information processing is difficult. The computation of the exact information function in the current setup would be much more difficult, because the information function now depends on an additional parameter: the public density $\psi$, which keeps changing shape with each order.

*ask price is* $\mu_t + H_t^a$ *and the bid price is* $\mu_t - H_t^b$*, where* $H_t^a = S/2 + \Delta_{g_t,\mathrm{BMO}} - \Delta$ *and* $H_t^b = S/2 - \Delta_{g_t,\mathrm{SMO}} - \Delta$.

I call $H_t^a$ the "ask half-spread" and $H_t^b$ the "bid half-spread." These quantities are not equal to each other, because in the current context the public density is now precisely computed by the uninformed traders, and is therefore no longer normal. In fact, Proposition IA.3 describes the exact shape of the posterior density and its evolution.

The last statement in Proposition IA.3 is the indifference condition for the uninformed traders.[18] Consider an uninformed trader who is first in the ask queue at date $t$. If a buy market order arrives, then his expected payoff (net of his private valuation) is given by the ask half-spread ($H_t^a$), minus the adverse selection of the buy market order ($\Delta_{g_t,\mathrm{BMO}}$). The expected payoff is the same regardless of the date $t$, because otherwise the uninformed traders would have an incentive to modify their position in the bid queue. The discussion thus far explains why the expected payoff $H_t^a/2 - \Delta_{g_t,\mathrm{BMO}}$ is constant. I set this constant equal to $S/2-\Delta$, because I want the equilibrium to be on average the same as the stationary equilibrium of Section III in the paper. More directly, this equality holds if the initial normalized public density $g_0$ is the standard normal density $\mathcal{N}(0,1)$.

In equilibrium the informed trader has the same threshold strategy as in Corollary 2 in the paper, because she uses the same information function $I$ as in the stationary equilibrium. Depending on her signal $w_t = \frac{v_t - \mu_t}{V}$, she submits a BMO if $w_t \in (\alpha, \infty)$, BLO if $w_t \in (0, \alpha)$, SLO if $w_t \in (-\alpha, 0)$, or SMO if $w_t \in (-\infty, -\alpha)$. The magnitude of the price impact thus depends on how the normalized public density $g_t$ is averaged out over the intervals that define the informed trader's strategy.

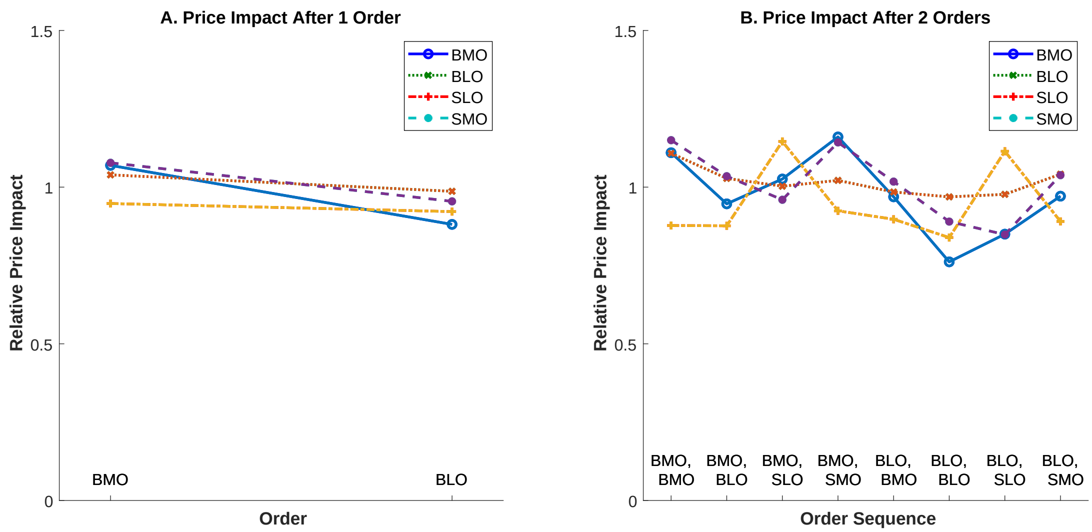
Figure IA.7 shows the relative price impact coefficient after observing a particular order flow sequence of length 1 and 2. The relative price impact of a buy market order is the ratio $\Delta_{g_t,\mathrm{BMO}}/\Delta$, where $\Delta$ is the price impact of a BMO in the benchmark model, and $g_t$ is the public density after the particular order flow sequence, assuming that the initial density is standard normal. Graph A of Figure IA.7 shows the relative price impact for the 4 types of order after a BMO is observed (the 4 points on the left) or after a BLO is observed (the 4 points on the right). There is no need to analyze the relative price impact for an SLO or an SMO, because the results are symmetric with respect to the line $y = 1$. In all cases, the price impact coefficients are close to the benchmark price impact, indicating that the standard normal approximation is good.

[18]This result is essentially Corollary 8 from the paper translated into the current context.

FIGURE IA.7

**Price Impact with Exact Densities**

Figure IA.7 shows the relative price impact of an order $\mathcal{O} \in \{\text{BMO}, \text{BLO}, \text{SLO}, \text{SMO}\}$, conditional on a sequence of orders being observed. The relative price impact of an order $\mathcal{O}$ is the price impact coefficient of that order, divided by the price impact of the same type of order in the benchmark model. For all graphs, the initial normalized public density (before the sequence of orders) is the standard normal density. Graph A shows the relative price impact coefficient of the order $\mathcal{O}$ after an order $\mathcal{O}_1$ is observed, where $\mathcal{O}_1$ is either BMO or BLO. Graph B shows the relative price impact coefficient of the order $\mathcal{O}$ after two orders $\mathcal{O}_1$ and $\mathcal{O}_2$ are observed, where $\mathcal{O}_1 \in \{\text{BMO}, \text{BLO}\}$ and $\mathcal{O}_2 \in \{\text{BMO}, \text{BLO}, \text{SLO}, \text{SMO}\}$. The informed share is $\rho = 0.10$.




Graph B of Figure IA.7 shows the relative price impact for the 4 types of order after observing a sequence of 2 orders, $\mathcal{O}_1 \in \{\text{BMO}, \text{BLO}\}$ and $\mathcal{O}_2 \in \{\text{BMO}, \text{BLO}, \text{SLO}, \text{SMO}\}$. The figure suggests testable implications of the model. To give an example, after an order flow sequence containing a BLO, the price impact of a BMO is generally lower than in the benchmark model. For instance, after observing 2 BLOs in a row, the relative price impact of a BMO is significantly smaller than 1. The intuition is that the occurrence of a BLO indicates to the uninformed traders that the asset mispricing is relatively smaller (i.e., its standard deviation is lower) than after observing a market order. Therefore, a subsequent BMO is less likely to be informed, and therefore has a smaller price impact.

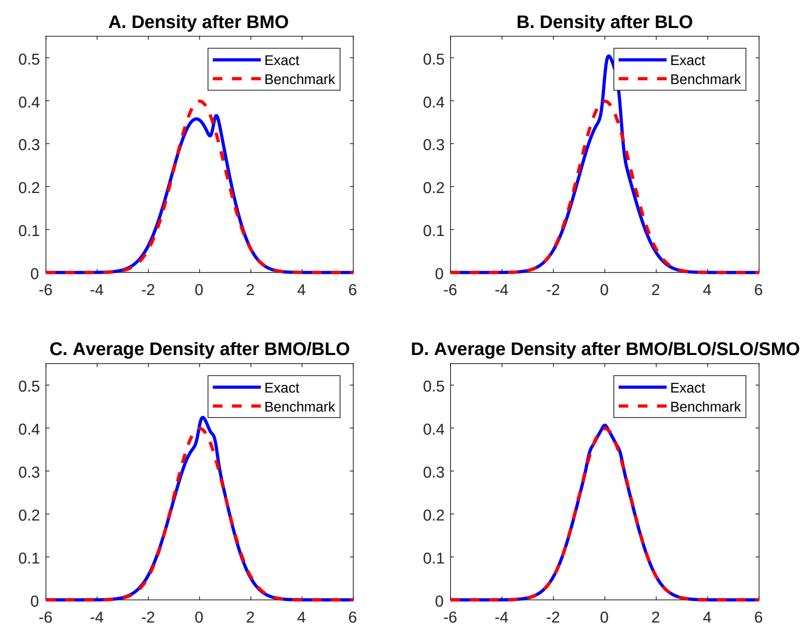
Overall, I conclude that the price impact of various types of orders stays close to the price impact in the benchmark model. To provide further evidence, consider Figure IA.8. Graphs A and B of Figure IA.8 show the normalized public density after a BMO and after a BLO.[19] In both cases, the shape of the posterior density is not normal, although the

[19]The normalized public densities after an SMO or an SLO are, respectively, symmetric around the $y$-axis

FIGURE IA.8

**Average Public Density**

Figure IA.8 shows the normalized public density after various types of orders. Suppose at $t$ the signal $w_t = \frac{v_t - \mu_t}{V}$ has a normal density $\mathcal{N}(0,1)$. Graphs A and B show the normalized public density at $t+1$ after, respectively, observing a BMO or a BLO. Graph C shows the average normalized public density at $t+1$ after observing either a BMO or BLO (with equal probability). Graph D shows the average normalized public density at $t+1$ after observing either a BMO, BLO, SLO, or SMO (with equal probability). In all graphs, the public density is displayed as a solid line, while the standard normal density is displayed as a dashed line. The informed share is $\rho = 0.10$.




difference from the standard normal density is not large. If I take the average of these shapes (Graphs C and D of Figure IA.8), the average public density is quite close to the benchmark public density, which is standard normal. Very similar results are obtained if instead of the average public density after 1 order, one considers the average public density after several orders. All these results suggest that the assumption that the uninformed traders process information by using a normal approximation to the public density is plausible.

## 5.2 No-Order Region

In the benchmark model, each informed trader receives a penalty $\omega$ (called the "commitment parameter") if after observing the fundamental value she chooses not to trade. In this section, I set $\omega = 0$. Because in this case the informed trader might choose not to submit any order, I assume that whenever this event occurs, an uninformed trader is drawn instantly

to the densities after a BMO or a BLO.

from the pool.[20]

In this section, I use the same methodology as in the benchmark model, except that now the optimal strategy of the informed trader includes a "no-order region" when the informed share $\rho$ is above a threshold. Thus, instead of the parameter $\alpha$ (the threshold signal between BLO and BMO), I introduce two parameters that are functions of $\rho$. The first parameter, $\alpha^0$, is the threshold signal between not trading and BLO, with $\alpha^0 = 0$ if the no-order region is empty. The second parameter, $\alpha^1$, is the threshold signal between BLO and BMO, with $\alpha^1 = \alpha$ if the no-order region is empty.

I start by defining parameters similar to those in Section III.B in the paper, except that all parameters now depend on the informed share $\rho$. To indicate the different values, I add a superscript "1" to the parameters, and write $\alpha^1$, $\beta^1$, $\gamma^1$, $I^1$, $\Delta^1$, $V^1$, and $S^1$. Also, the new information function $I^1$ cannot be defined independent of the other parameters, as in Definition 1 in the paper. Therefore, in Definition IA.2 below, I define all parameters at the same time. Recall that $\phi(\,\cdot\,)$ is the standard normal density, and $\Phi(\,\cdot\,)$ is its cumulative density. In the next definition, I also use the parameters $\alpha$, $\beta$, $\gamma$, $I$, $V$, $\Delta$, and $S$ from Section III.B of the paper.

**Definition IA.2.** *Let $\rho > 0$ and $w \in \mathbb{R}$. Define the function $I^1(\rho, w)$ as in Definition 1 in the paper, except that instead of the numeric parameters $\alpha$, $\beta$, and $\gamma$, one uses the functions $\alpha^1 = \alpha^1(\rho)$, $\beta^1 = \beta^1(\rho)$, $\gamma^1 = \gamma^1(\rho)$ defined below, and equation (5) is replaced with:*

$$\text{(IA.45)} \qquad \begin{aligned} \pi_{g,\mathcal{O}} &= \int \left( \frac{1-\rho}{4} + \frac{\rho}{4}\mathbf{1}_{z \in i_{\mathrm{NO}}} + \rho \mathbf{1}_{z \in i_{\mathcal{O}}} \right) g(z) \mathrm{d}z, \\ f_{g,\mathcal{O}}(x) &= \frac{\int \left( \frac{1-\rho}{4} + \frac{\rho}{4}\mathbf{1}_{z \in i_{\mathrm{NO}}} + \rho \mathbf{1}_{z \in i_{\mathcal{O}}} \right) g(z) \phi\left( x; z - \delta_{\mathcal{O}}, \rho \sqrt{\frac{1+(\gamma^1)^2}{2(\beta^1)^2}} \right) \mathrm{d}z}{\pi_{g,\mathcal{O}}}, \end{aligned}$$

*where $i_{\mathcal{O}} \in \{(\alpha^1, \infty), (\alpha^0, \alpha^0), (-\alpha^0, \alpha^0), (-\alpha^1, -\alpha^0), (-\infty, -\alpha^1)\}$, respectively, for $\mathcal{O} \in \{\mathrm{BMO}, \mathrm{BLO}, \mathrm{NO}, \mathrm{SLO}, \mathrm{SMO}\}$. The functions $\alpha^0 = \alpha^0(\rho)$, $\alpha^1 = \alpha^1(\rho)$, $\beta^1 = \beta^1(\rho)$, and $\gamma^1 = \gamma^1(\rho)$ are defined by the implicit equations:*

$$\text{(IA.46)} \qquad \begin{aligned} \alpha^1 - I^1(\rho, \alpha^1) + 2I^1(\rho, \alpha^0) &= 0, \qquad 1 - \Phi(\alpha^1) = \Phi(\alpha^1) - \Phi(\alpha^0), \\ \beta^1 &= \frac{1}{4\phi(\alpha^1)}, \qquad \gamma^1 = \frac{\phi(\alpha^0) - \phi(\alpha^1)}{\phi(\alpha^1)}. \end{aligned}$$

[20] This assumption is consistent with the principle of working in event time rather than calendar time. Indeed, if an informed trader decides not to trade at $t$, the clock only gets restarted when an uninformed trader arrives to the market and trades, in which case this event occurs at $t + 1$. If one worked in calendar time instead, the model would be more complicated, because the time elapsed between order arrivals would become informative about the fundamental value.

*If there is no solution of the equations above with $\alpha^0 > 0$, define $\alpha^0 = 0$, $\alpha^1 = \alpha$, $\beta^1 = \beta$, $\gamma^1 = \gamma$, and $I^1 = I$. Finally, define:*

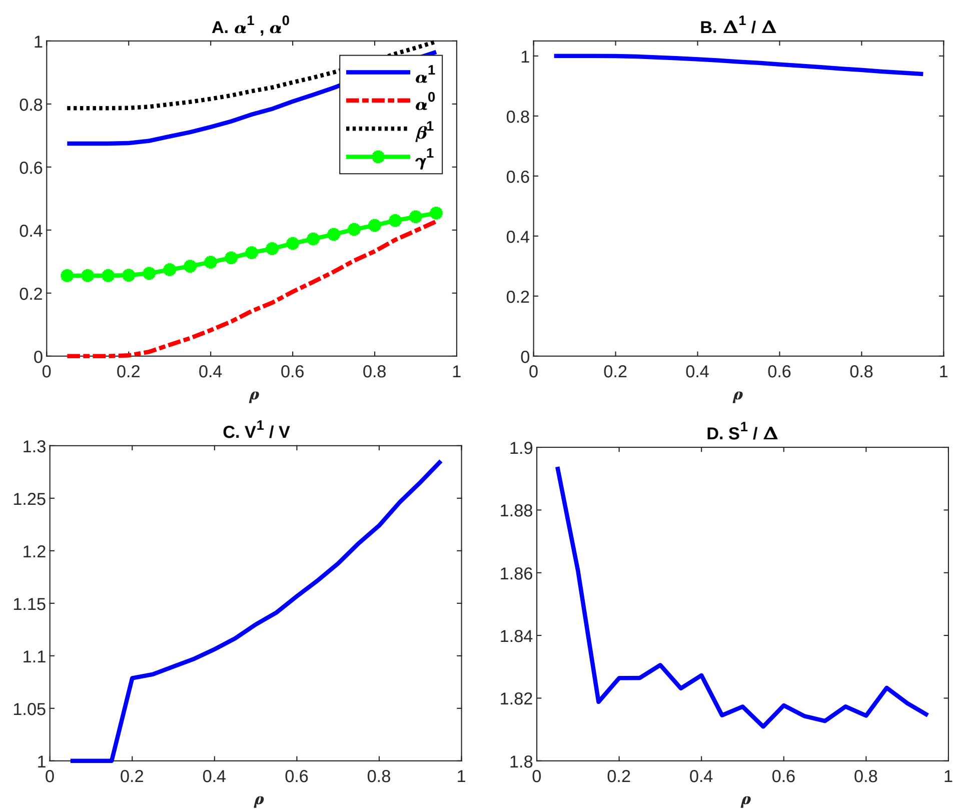
$$\text{(IA.47)} \qquad \Delta^1 \;=\; \sqrt{\frac{2}{1+(\gamma^1)^2}}\,\frac{\sigma_v}{\sqrt{\lambda}}, \qquad V^1 \;=\; \beta^1 \rho^{-1} \Delta^1, \qquad S^1 \;=\; \left(\alpha^1 - I^1(\rho, \alpha^1)\right) V^1.$$

Note that, as in Corollary 3 in the paper, the condition $1 - \Phi(\alpha^1) = \Phi(\alpha^1) - \Phi(\alpha^0)$ from Definition IA.2 ensures that all orders are equally likely, and that the equilibrium is stationary.

FIGURE IA.9

**Equilibrium with No-Order Region**

Figure IA.9 shows several functions of the informed share $\rho$ that correspond to certain variables in the equilibrium with no-order region. Graph A shows the threshold signal $\alpha^1$ between BLO and BMO, and the threshold $\alpha^0$ of the no-order region (between BLO and NO = no order). Graph B shows the price impact coefficient $\Delta^1$, divided by the benchmark value $\Delta$. Graph C shows the public volatility $V^1$, divided by the benchmark value $V$. Graph D shows the bid–ask spread $S^1$, divided by the benchmark price impact coefficient $\Delta$.




Proposition IA.4 shows that there exists an MPE of the model if the conditions in Result IA.6 are satisfied. I verify these conditions numerically in Section 6 below. Proposition IA.4 also describes several properties of the equilibrium.

**Proposition IA.4.** *If the conditions in Result IA.6 are satisfied, there exists an MPE of the game. In equilibrium, if an order $\mathcal{O} \in \{\mathrm{BMO}, \mathrm{BLO}, \mathrm{SLO}, \mathrm{SMO}\}$ arrives at $t$, the public mean changes from $\mu_t$ to $\mu_t + \Delta^1_{\mathcal{O}}$, where, respectively, $\Delta^1_{\mathcal{O}} \in \{\Delta^1, \gamma^1\Delta^1, -\gamma^1\Delta^1, -\Delta^1\}$. An informed trader who arrives at $t \geq 0$ and observes $w_t = \frac{v_t - \mu_t}{V^1}$ in the interval $i_{\mathcal{O}} \in \{(\alpha^1, \infty), (\alpha^0, \alpha^0), (-\alpha^0, \alpha^0), (-\alpha^1, -\alpha^0), (-\infty, -\alpha^1)\}$ optimally submits an order $\mathcal{O} \in \{\mathrm{BMO}, \mathrm{BLO}, \mathrm{NO}, \mathrm{SLO}, \mathrm{S}$ respectively. At date $t$, the ask price is $\mu_t + S^1/2$ and the bid price is $\mu_t - S^1/2$.*

In equilibrium, the size of the no-order region $(-\alpha^0, \alpha^0)$ depends on the informed share $\rho$. Result IA.3 below shows that there exists a threshold informed share ($\rho_0 \approx 0.1560$, such that when the informed share is below this threshold the no-order region is empty. I call this region the "low-information regime," as opposed to the case when $\rho > \rho_0$ which is called the "high-information regime." Result IA.3 also shows that the size of the no-order region is increasing in the informed share in the high-information regime. The intuition is that competition among informed traders makes it less profitable to trade when the informed trader's signal is moderate. Differently said, competition among informed traders raises the bar for their incentive to trade on information.

**Result IA.3.** There exists a threshold informed share $\rho_0 \approx 0.1560$, such that when $\rho < \rho_0$ (the "low-information regime") the no-order region is empty and the equilibrium coincides with the benchmark equilibrium. When $\rho < \rho_0$ (the "high-information regime"), (*i*) the no-order region $(-\alpha^0, \alpha^0)$ is nonempty and increasing in $\rho$, (*ii*) $\alpha^1$, $\beta^1$ and $\gamma^1$ are increasing in $\rho$, (*iii*) the relative public volatility $V^1/V$ is increasing in $\rho$, (*iv*) the price impact coefficient $\Delta^1$ is decreasing in $\rho$.

Figure IA.9 helps illustrate graphically some stylized facts from Result IA.3. In the high-information regime, adverse selection as measured by the price impact coefficient $\Delta^1$ is decreasing in $\rho$. Indeed, when competition among informed traders increases, the threshold ($\alpha^0$) for trading on information increases, and because of the decrease in informed trading activity, the overall adverse selection decreases, although not strongly (by at most 7%). (Recall that the overall adverse selection parameter $\Delta$ is constant in the benchmark model.) The effect on the bid–ask spread is ambiguous. In the benchmark model, the increase in competition among the informed traders makes the market more dynamically efficient and therefore lowers the bid–ask spread (although not very strongly). In the high-information regime, the existence of a no-order region decreases dynamic efficiency relative to the benchmark model. Indeed, according to Result IA.3, the relative public volatility $V^1/V$ is increasing with $\rho$. Nevertheless, the public volatility itself $V^1$ is still decreasing in $\rho$. Overall, the effect of $\rho$ in the high-information regime is very weak, and the bid–ask spread appears approximately constant in this region, although the result is less conclusive because the estimation error is relatively large in this case.

## 6 Extensions – Proofs and Numerical Results

In this section, I provide proofs and numerical verifications for the results in Section V in the paper and in Section 5 in this Internet Appendix. I also provide further discussion.

### 6.1 Non-Stationary Equilibria

**Proof of Proposition 5**. The proof follows closely the proof of Theorem 1 in the paper. The difference consists in the fact that the public volatility is no longer constant. Recall that the normalized public volatility at $t$ is defined by:

$$\theta_t = \frac{\sigma_t}{V}, \tag{IA.48}$$

where $\sigma_t$ is the public volatility at $t$, and $V$ is the constant public volatility parameter from equation (7) in the paper. In the benchmark (or stationary) model, $\theta_t$ is constant and equal to 1. In the non-stationary case, however, $\theta_t$ changes over time, and along with it the other parameters of the model. For instance, denote by $\tilde{\alpha}_t$ the threshold signal between BLO and BMO at $t$. Because the problem is symmetric with respect to the public mean $\mu_t$, it follows that an informed trader who observes a signal $w_t = \frac{v_t - \mu_t}{V}$ at $t$ submits an order $\mathcal{O} \in \{\text{BMO}, \text{BLO}, \text{SLO}, \text{SMO}\}$ whenever the signal belongs, respectively, to the interval:

$$i_{\mathcal{O}} \in \{(\tilde{\alpha}_t, \infty), (0, \tilde{\alpha}_t), (-\tilde{\alpha}_t, 0), (-\infty, -\tilde{\alpha}_t)\}. \tag{IA.49}$$

Then $\tilde{\alpha}_t$ also evolves over time as well, and is determined in equilibrium jointly with $\theta_t$ and with the other parameters, including the information function $\tilde{I}(\rho, w, \theta)$ (see Definition A1 in the paper).

By the assumptions made in Section II in the paper, the uninformed traders update the public volatility by calculating the average variance over the 4 types of orders, which leads to a deterministic evolution of the public volatility.[21]

The update of $\theta_t$ follows equation (A-7) from Lemma A1 in the Appendix of the paper, which implies that the average normalized public variance evolves by the formula ($\bar{\tau}_t$ in the lemma coincides with $\theta_t$ in this context):

$$\theta_{t+1}^2 = \theta_t^2 + \hat{\sigma_I}^2 - \mathsf{E}_{\mathcal{O}}\, \delta_{t+1,\mathcal{O}}^2, \tag{IA.50}$$

[21] Otherwise, the public volatility would be stochastic, as it would change depending on the type of order submitted at each date.

where $\delta_{t+1,\mathcal{O}}$ is the normalized price impact after observing $\mathcal{O}$ at $t$, and $\hat{\sigma_I}$ is the normalized inter-arrival volatility, which is constant and satisfies:

$$\text{(IA.51)} \qquad \hat{\sigma_I} \;=\; \frac{\sigma_I}{V} = \rho\sqrt{\frac{1+\gamma^2}{2\beta^2}}.$$

But the normalized public density at $t$ is of the form:

$$\text{(IA.52)} \qquad g_t \;=\; \mathcal{N}\big(0,\theta_t^2\big).$$

Using part (A-8) of Lemma A1 in the Appendix of the paper, one computes:

$$\text{(IA.53)} \quad \theta_{t+1}^2 \;=\; \rho^2\,\frac{1+\gamma^2}{2\beta^2} + \theta_t^2 - 2\rho^2\theta_t^2\left(\frac{\Big(\phi\big(\frac{\tilde{\alpha}_t}{\theta_t}\big)\Big)^2}{\frac{1-\rho}{4} + \rho\big(1-\Phi\big(\frac{\tilde{\alpha}}{\theta_t}\big)\big)} + \frac{\Big(\phi\big(\frac{0}{\theta_t}\big) - \phi\big(\frac{\tilde{\alpha}_t}{\theta_t}\big)\Big)^2}{\frac{1-\rho}{4} + \rho\big(\Phi\big(\frac{\tilde{\alpha}}{\theta_t}\big) - \Phi(0)\big)}\right),$$

which proves equation (28) in the paper.

I compute the normalized price impact $\delta_{t+1,\mathcal{O}}$ of the order $\mathcal{O}$ by using equation (A-8) in the Appendix of the paper. The formula for $\mathcal{O} \in \{\mathrm{BMO}, \mathrm{BLO}, \mathrm{SLO}, \mathrm{SMO}\}$ is, respectively,

$$\text{(IA.54)} \qquad \delta_{t+1,\mathcal{O}} \;=\; \left\{\, \frac{\rho}{\tilde{\beta}}\,\theta_t\;,\; \tilde{\gamma}\frac{\rho}{\tilde{\beta}}\,\theta_t\;,\; -\tilde{\gamma}\frac{\rho}{\tilde{\beta}}\,\theta_t\;,\; \frac{\rho}{\tilde{\beta}}\,\theta_t \,\right\},$$

where:

$$\text{(IA.55)} \qquad \tilde{\beta} \;=\; \frac{\frac{1-\rho}{4} + \rho\big(1-\Phi\big(\frac{\tilde{\alpha}}{\theta}\big)\big)}{\phi\big(\frac{\tilde{\alpha}}{\theta}\big)}, \qquad \tilde{\gamma} \;=\; \frac{\phi(0) - \phi\big(\frac{\tilde{\alpha}}{\theta}\big)}{\phi\big(\frac{\tilde{\alpha}}{\theta}\big)}\;\frac{\frac{1-\rho}{4} + \rho\big(1-\Phi\big(\frac{\tilde{\alpha}}{\theta}\big)\big)}{\frac{1-\rho}{4} + \rho\big(\Phi\big(\frac{\tilde{\alpha}}{\theta}\big) - \Phi(0)\big)},$$

which is part of equation (A-21) in the paper.

The other formulas are justified in the same way as in the benchmark case. In particular, one has:

$$\text{(IA.56)} \qquad \tilde{S}_t \;=\; \big(\tilde{\alpha}_t - \tilde{I}(\rho,\tilde{\alpha}_t,\theta_t)\big)\,V, \qquad \tilde{\Delta}_t \;=\; \delta_{t+1,\mathcal{O}}\,V.$$

The only exception is equation (29) from Corollary 8 in the paper:

$$\text{(IA.57)} \qquad \frac{\tilde{S}_t}{2} - \tilde{\Delta}_t \;=\; \frac{S}{2} - \Delta.$$

This equation represents the indifference condition for the uninformed traders, meaning that they have the same expected utility whether their order is executed now or is executed later (see Lemma A3 in the Appendix of the paper). If equation (IA.57) is divided by $V$, one obtains:

$$\text{(IA.58)} \qquad \tilde{\alpha} - \tilde{I}(\rho, \tilde{\alpha}, \theta) - 2\, \frac{\rho\, \theta\, \phi\big(\frac{\tilde{\alpha}}{\theta}\big)}{\frac{1-\rho}{4} + \rho\big(1 - \Phi\big(\frac{\tilde{\alpha}}{\theta}\big)\big)} \;=\; \alpha - I(\rho, \alpha) - 2\, \frac{\rho}{\beta}.$$

This justifies the first equality in equation (A-21) in the paper.

The arguments presented thus far show that the notation made in Definition A1 in the paper mirrors the notation in the benchmark case. Hence, the proof of Theorem 1 in the paper can be adapted to show that the usual strategies of the informed and uninformed traders are optimal in this case, as well.

It only remains to state the conditions for the new information function $\tilde{I}$ that mirror the conditions in Result 1 in the paper. Thus, if the conditions in Result IA.4 are true, the proof of the theorem is now complete. □

**Result IA.4.** For all $\rho \in (0, 1)$ and $\theta > 0$, the functions $\tilde{I}(\rho, w, \theta)$, $w - \tilde{I}(\rho, w, \theta)$ and $\tilde{I}(\rho, w, \theta) - \tilde{I}(\rho, -w, \theta)$ are strictly increasing in $w$, and:

$$\text{(IA.59)} \qquad \max\Big( \frac{\rho(1 + \tilde{\gamma})}{\tilde{\beta}}\,,\, -2\tilde{I}(\rho, 0, \theta) - 2\frac{\rho\tilde{\gamma}}{\beta} \Big) \;<\; \tilde{\alpha} - \tilde{I}(\rho, \tilde{\alpha}, \theta).$$

Let $\tilde{I}(\rho, g_1, \theta, j)$ and $\tilde{J}(\rho, g_1, \theta, j)$ be as in Definition A1 in the paper, but for general density $g_1$ and rank $j \in \mathbb{N}_+$, and denote by $\tilde{I}(\rho, w, \theta, j)$ and $\tilde{J}(\rho, w, \theta, j)$ the same functions for $g_1 = \mathcal{N}\left(w - \gamma\frac{\rho}{\beta}, \rho^2\frac{1+\gamma^2}{2\beta^2}\right)$. When $j = 1$, the argument $j$ can be omitted. Then, $(i)$ $\tilde{I}(\rho, g_1, \theta, j)$ increases if $g_1$ has a positive shift in mean, $(ii)$ $\tilde{I} = \tilde{I}(\rho, w, \theta, j)$ decreases in $j$ if $w > 0$, and $(iii)$ $\tilde{J} = 1$.

I also include here the conditions on the preference parameters, $\bar{u} > S/2$ and $\omega > \gamma\Delta$.

**Proof of Corollary 8**. This is discussed in the course of proving Proposition 5. □

**Proof of Proposition 6**. The proof mirrors the proof of Proposition 4 in the paper. □

**Proof of Proposition 7**. This is a simple application of the methods used to prove Proposition 5. Indeed, if one applies equation (A-2) in the Appendix of the paper to the normal density $g_t = \phi\big(0, \theta_t^2\big)$, one computes:

$$\text{(IA.60)} \qquad P_{\mathrm{MO}} \;=\; \frac{1-\rho}{4} + \rho\left(1 - \Phi\Big(\frac{\tilde{\alpha}}{\theta}\Big)\right), \qquad P_{\mathrm{LO}} \;=\; \frac{1-\rho}{4} + \rho\left(\Phi\Big(\frac{\tilde{\alpha}}{\theta}\Big) - \Phi(0)\right).$$

Taking their ratio, one proves equation (32) in the paper. $\square$

**Verification of Results 4, 5, and IA.4**. To verify Result 4, I need to show that the non-stationary equilibrium that starts with normalized public density $\theta_0 = \theta$ converges to the stationary equilibrium, which corresponds to $\theta = 1$. But $\theta$ evolves deterministically according to equation (28) in the paper, and can numerically be seen to converge to 1 regardless of the initial value. This is done in Figure 4 in the paper.

To verify Results 5 and IA.4, I need to solve numerically for the non-stationary equilibrium that begins with $\theta_0 = \theta$. It is thus necessary to compute the parameters $\tilde{\alpha}$, $\tilde{\beta}$, $\tilde{\gamma}$, and $\tilde{I}$ from Definition A1 in the paper. These parameters are functions of $\rho$ and $\theta$, while $\tilde{I}$ also depends on the initial signal $w$. Numerically, I consider the following values of $\rho$ and $\theta$:

- Informed share, $\rho \in [0.05, 0.95]$, step $= 0.05$ (19 values);
- Normalized public volatility, $\theta \in [0.50, 1.50]$, step 0.10 (11 values);

Note that I choose values of $\theta$ around $\theta = 1$, because in that case the equilibrium is stationary.

The computation of the parameters $\tilde{\alpha}$, $\tilde{\beta}$, $\tilde{\gamma}$, and $\tilde{I}$ amounts to finding a fixed point of the equations in Definition A1 in the paper. A natural starting point is to use the parameters in the benchmark model ($\theta = 1$), and take $\tilde{\alpha}^{(0)}(\rho, \theta) = \alpha$, $\tilde{\beta}^{(0)}(\rho, \theta) = \beta$, $\tilde{\gamma}^{(0)}(\rho, \theta) = \gamma$, and $\tilde{I}^{(0)}(\rho, w, \theta) = I(\rho, w)$. I then update $\tilde{\alpha}$, by solving numerically for each value of $\rho$ and $\theta$ the implicit equation:

$$\tilde{\alpha} - \tilde{I}(\rho, \tilde{\alpha}, \theta) - 2\, \frac{\rho\, \theta\, \phi\big(\frac{\tilde{\alpha}}{\theta}\big)}{\frac{1-\rho}{4} + \rho\big(1 - \Phi\big(\frac{\tilde{\alpha}}{\theta}\big)\big)} \;=\; \alpha - I(\rho, \alpha) - 2\, \frac{\rho}{\beta}. \tag{IA.61}$$

(see equation (A-21) in the paper). This produces the function $\tilde{\alpha}^{(1)}(\rho, \theta)$. Using (IA.55), one computes also $\tilde{\beta}^{(1)}(\rho, \theta)$ and $\tilde{\gamma}^{(1)}(\rho, \theta)$. The function $\tilde{I}^{(1)}(\rho, w, \theta)$ is obtained as explained in Definition A1 in the paper, using the current values for the parameters $\tilde{\alpha}$, $\tilde{\beta}$, and $\tilde{\gamma}$. I iterate the procedure until $|\tilde{\alpha}^{(m+1)} - \tilde{\alpha}^{(m)}|$ is below some tolerance level, say $10^{-4}$.

This process, however, is very computationally intensive, and I stop after only 3 iterations. Nevertheless, I impose the stringent condition that the solution $\tilde{\alpha}$ of the implicit equation (IA.61) must be unique at each step of the iteration. With this condition, for $\theta \geq 1$ there is always an approximate solution, but for $\theta < 1$ there is no longer a unique solution for sufficiently large values of $\rho$ (in Figure 5 in the paper I omit the data points for which the numerical procedure fails to produce a solution).[22] Now, visual inspection of Figure 5 in the paper shows that the price impact coefficient $\tilde{\Delta}$ is indeed increasing in $\theta$, as stated in Result 5. The same is true about the bid–ask spread $\tilde{S}$, since $\tilde{\Delta}$ and $\tilde{S}$ are related by a linear relation that does not depend on $\theta$ (see equation (IA.57)).

When the procedure yields a solution, the numerical verification of the conditions in Result IA.4 for $\tilde{I}(\rho, w, \theta)$ is then done as for Result 1. Even when the numerical procedure does not yield a solution, there is reason to believe that the conditions in Result IA.4 still hold. Indeed, for $\theta = 1$ the function $\tilde{I}(\rho, w, 1)$ coincides with the benchmark information function $I(\rho, w)$, therefore by continuity the conditions in Result IA.4 must still hold in a neighborhood of $\theta = 1$. In order for the conditions to hold for a general $\theta$, I suspect that a larger value of the commitment parameter $\omega$ is necessary (larger than the current value $\gamma\Delta$), such that the inequality condition in (IA.59) becomes less stringent. □

## 6.2 Public Information Processing

As in the case of Theorem 1 in the paper, the results depend on certain conditions being true. I verify them numerically in the following result.

**Result IA.5.** Besides the conditions (IA.10) from Result 1 in the paper, the following condition is true:

$$S - \Delta \; > \; \max\Big(-H_t^b \, , \; \frac{S}{2} - H_t^a\Big). \tag{IA.62}$$

I also include here the conditions on the preference parameters, $\bar{u} > S/2$ and $\omega > \gamma\Delta$.

**Verification of Result IA.5**. The stringent condition is $S - \Delta > \frac{S}{2} - H_t^a$. But, as shown in Figure IA.7, the approximation of the normalized public density by the standard normal density is very good, and so $\frac{S}{2} - H_t^a$ is close to 0, while $S - \Delta$ is large. This condition is therefore easily satisfied. □

**Proof of Proposition IA.3**. Let $\psi_t$ be the public density at $t$, and let $\mu_t$ and $\sigma_t$ be respectively its mean and standard deviation. Let also $g_t$ be the normalized public density (i.e., the density of the signal $w_t = \frac{v_t - \mu_t}{V}$). By assumption, the initial density $g_0$ is the standard normal density (which is the normalized public density in the benchmark model).

[22] In that case, I conjecture that economically the model still has an equilibrium, and therefore by properly choosing the initial parameter values (different than the benchmark values), one can solve for the equilibrium using the same numerical procedure. Because of the computational constraints, however, I choose to not pursue this path.

The proof is simplified by the assumption that the informed trader uses the same information function as in the benchmark model. Therefore, for the informed traders I use the same proof as in Theorem 1 in the paper. It follows that in equilibrium the informed trader has the same threshold strategy as in Corollary 2 in the paper: depending on her signal $w_t = \frac{v_t - \mu_t}{V}$, she submits an order $\mathcal{O} \in \{\mathrm{BMO}, \mathrm{BLO}, \mathrm{SLO}, \mathrm{SMO}\}$ if the signal lies in the interval $i_{\mathcal{O}}$, where, respectively,

$$i_{\mathcal{O}} \in \{ (\alpha, \infty), (0, \alpha), (-\alpha, 0), (-\infty, -\alpha) \}. \tag{IA.63}$$

The magnitude of the price impact thus depends on how the normalized public density $g_t$ is averaged out over the intervals that define the informed trader's strategy. Then, for an order $\mathcal{O} \in \{\mathrm{BMO}, \mathrm{BLO}, \mathrm{SLO}, \mathrm{SMO}\}$, define:

$$\begin{aligned} \delta_{g,\mathcal{O}} &= \frac{\rho \int_{i_{\mathcal{O}}} z g(z) \mathrm{d}z}{\frac{1-\rho}{4} + \rho \int_{i_{\mathcal{O}}} g(z) \mathrm{d}z}, \\ f_{g,\mathcal{O}}(x) &= \frac{\int \left( \frac{1-\rho}{4} + \rho \mathbf{1}_{z \in i_{\mathcal{O}}} \right) g(z) \phi \left( x; z - \delta_{g,\mathcal{O}}, \rho \sqrt{\frac{1+\gamma^2}{2\beta^2}} \right) \mathrm{d}z}{\frac{1-\rho}{4} + \rho \int_{i_{\mathcal{O}}} g(z) \mathrm{d}z}, \\ \Delta_{g,\mathcal{O}} &= \delta_{g,\mathcal{O}} V. \end{aligned} \tag{IA.64}$$

By Lemma A1 in the Appendix of the paper, it follows that the normalized public density is updated by the rule:

$$g_{t+1} = f_{g_t,\mathcal{O}}, \tag{IA.65}$$

and the public mean is updated by the rule:

$$\mu_{t+1} = \mu_t + \Delta_{g_t,\mathcal{O}}. \tag{IA.66}$$

I now analyze the optimal strategy of a (patient) uninformed buyer UB, with private valuation $\bar{u}$. If he submits a BLO, Lemma A2 in the Appendix of the paper implies that his expected utility is:

$$\mathcal{U}_{\mathrm{BLO}}^{\mathrm{UB}} = H_t^b + \Delta_{g_t,\mathrm{SMO}} + \bar{u}, \tag{IA.67}$$

where by definition $H_t^b = S/2 - \Delta_{g_t,\mathrm{SMO}} - \Delta$ is the bid half-spread. Hence, one obtains:

$$\mathcal{U}_{\mathrm{BLO}}^{\mathrm{UB}} \;=\; \frac{S}{2} - \Delta + \bar{u}, \tag{IA.68}$$

which is simply the indifference condition of the uninformed traders. If the uninformed buyer submits a BMO instead, his expected utility is:

$$\mathcal{U}_{\mathrm{BMO}}^{\mathrm{UB}} \;=\; -H_t^a + \bar{u}, \tag{IA.69}$$

since the ask price is $\mu_t + H_t^a$. Similarly, the expected utility from SMO is:

$$\mathcal{U}_{\mathrm{SMO}}^{\mathrm{UB}} \;=\; -H_t^b - \bar{u}. \tag{IA.70}$$

Since $\frac{S}{2} - \Delta + \bar{u} > 0$ when $\bar{u} > \frac{S}{2}$, the uninformed buyer prefers to submit a BLO to doing nothing (the inequality $S > \Delta$ follows from condition (IA.10) from Result 1 in the paper). He also prefers to submit BLO to SLO, since the latter order makes him lose his private valuation $\bar{u}$. Using the results above, if $\bar{u} = \frac{S}{2}$, it follows that BLO is optimal if the following inequality holds:

$$S - \Delta \;>\; \max\Big(-H_t^b\ ,\ \frac{S}{2} - H_t^a\Big). \tag{IA.71}$$

But this condition is ensured by Result IA.5. □

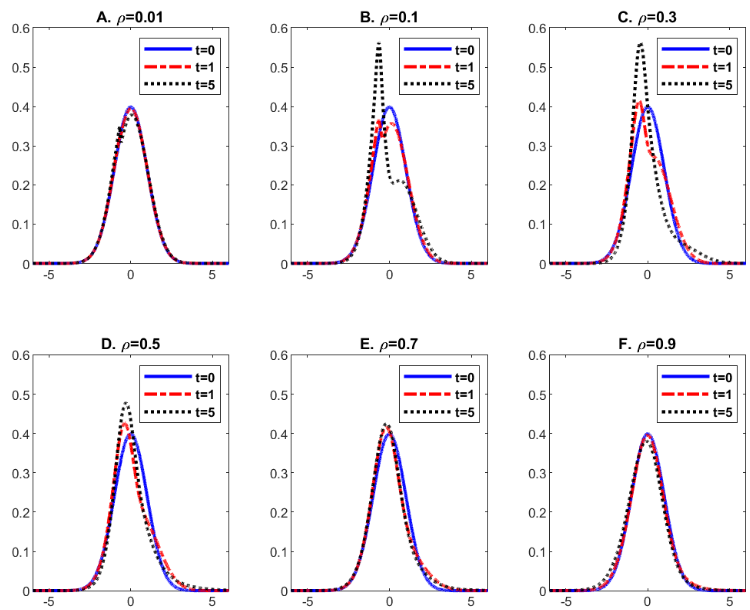
In the remainder of this section, I complement the results of Section 5.1 by giving a few more example to illustrate the evolution of the normalized public density. In all cases, start at $t = 0$ with the stationary public density $\mathcal{N}(\mu_0, V^2)$, which after normalization becomes the standard normal density $\mathcal{N}(0, 1)$.

Figure IA.10 shows the public density after $t = 0$, $t = 1$, and $t = 5$ BMOs for various values of the informed share $\rho$. By visual inspection, the public density appears close to the standard normal density even after a sequence of 5 BMOs (this sequence happens with probability $4^{-5}$, which is less than 1 in 1000). Interestingly, the deviation of the public densities from the standard normal density is at its smallest level when the informed share is either small or large, and it peaks for an intermediate value $\rho$ near 0.2. When $\rho$ is small, the order flow is uninformative, hence the posterior is not far from the prior. When $\rho$ is large, the information decays very quickly, and the informed trader becomes essentially

FIGURE IA.10

**Exact Public Density after Series of BMOs**

Figure IA.10 shows the evolution of the normalized public density $g_t$ after $t = 0$, $t = 1$ and $t = 5$ BMOs. In all graphs, the initial normalized public density (at $t = 0$) is the standard normal density $\mathcal{N}(0, 1)$. Graphs A, B, C, D, E, and F correspond, respectively, to an informed share $\rho \in \{0.01, 0.1, 0.3, 0.5, 0.7, 0.9\}$.



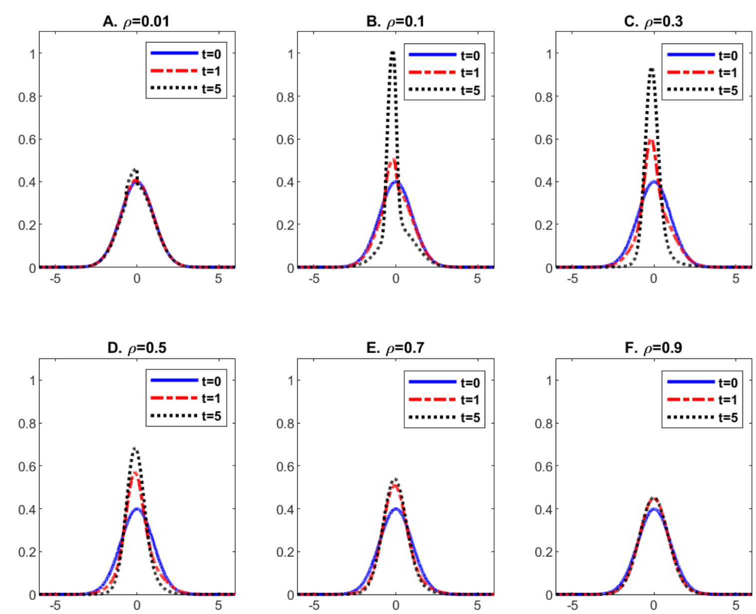

uninformed. Similarly, Figure IA.11 shows the public density after $t = 0$, $t = 1$, and $t = 5$ BLOs for various values of the informed share $\rho$. Here one can observe visually a slightly larger difference between the posterior density after a series of BLOs and the standard normal density than after a series of BMOs. Nevertheless, as explained below, what matters for risk-neutral traders is the difference in means, not the difference in higher moments.

A more direct measure for the quality of the approximation is to compare the exact posterior mean of the public density with 0, which is the posterior mean under the approximate density. Since in the model traders are risk-neutral, it is important to compute posterior means, for instance one wants to compute the exact price impact of order flow. Consider an informed share $\rho = 0.2$, for which the deviation of the posterior density from normality is close to the largest. Then, I am interested in the standard deviation of the posterior mean, when the posterior mean is estimated after 200 random series of $t = 5$ orders. I estimate a standard deviation of 0.062, which is reasonably small, compared with 1, the standard deviation of the (normalized) public density.

FIGURE IA.11

**Exact Public Density after Series of** BLOs

Figure IA.11 shows the evolution of the normalized public density $g_t$ after $t = 0$, $t = 1$ and $t = 5$ BLOs. In all graphs, the initial normalized public density (at $t = 0$) is the standard normal density $\mathcal{N}(0, 1)$. Graphs A, B, C, D, E, and F correspond, respectively, to an informed share $\rho \in \{0.01, 0.1, 0.3, 0.5, 0.7, 0.9\}$.




Additionally, I compute the average density over 200 random series of length $t = 5$. The result for all $\rho$ is almost indistinguishable from the standard normal density, giving another indication that traders are reasonable in making the standard normal approximation. In fact, an alternative justification of this approximation is that in practice there is significant uncertainty about the actual shape of the public density. Thus, it is reasonable to expect that the uninformed traders are not fully Bayesian, and only do approximate inferences about the public density in order to preserve tractability.

## 6.3 No-Order Region

Let $0 \leq \alpha^0 \leq \alpha^1$. For $\mathcal{O} \in \{\text{BMO}, \text{BLO}, \text{NO}, \text{SLO}, \text{SMO}\}$, define respectively $i_{\mathcal{O}} \in \{(\alpha^1, \infty), (\alpha^0, \alpha^0), (-\alpha^0, \alpha^0), (-\alpha^1, -\alpha^0), (-\infty, -\alpha^1)\}$, which are the intervals defining the strategy of the informed trader. For $\mathcal{O} \in \{\text{BMO}, \text{BLO}, \text{SLO}, \text{SMO}\}$, define respectively $\delta_{\mathcal{O}} \in \left\{\frac{\rho}{\beta^1}, \gamma^1 \frac{\rho}{\beta^1}, -\gamma^1 \frac{\rho}{\beta^1}, -\frac{\rho}{\beta^1}\right\}$. Let $\phi(\cdot; M, S)$ be the normal density with mean $M$ and standard deviation $S$, $\phi(\cdot)$ the standard normal density ($M = 0$, $S = 1$), and $\Phi(\cdot)$ the cumulative normal density. Denote the normalized inter-arrival volatility by $\hat{\sigma_I}^1 = \frac{\sigma_I}{V^1} = \rho\sqrt{\frac{1+(\gamma^1)^2}{2(\beta^1)^2}}$.

**Lemma IA.1.** *In the context of Proposition IA.4, consider a trader who just before trading at $t$ believes that the signal $w_t = \frac{v_t - \mu_t}{V} = z$ has probability density function $g_t(z)$. Then, the following are true:*

*(a) The probability of observing $\mathcal{O} \in \{\mathrm{BMO}, \mathrm{BLO}, \mathrm{SLO}, \mathrm{SMO}\}$ at $t$ is:*

$$P_{\mathcal{O}} \;=\; \frac{1-\rho}{4} + \frac{\rho}{4}\int_{z \in i_{\mathrm{NO}}} g_t(z)\mathrm{d}z + \rho \int_{z \in i_{\mathcal{O}}} g_t(z)\mathrm{d}z. \tag{IA.72}$$

*After seing the order $\mathcal{O}$ at $t$, the posterior density of $w_{t+1} = \frac{v_{t+1} - \mu_{t+1}}{V}$ is:*

$$g_{t+1,\mathcal{O}}(x) \;=\; \frac{\int \big(\frac{1-\rho}{4} + \frac{\rho}{4}\mathbf{1}_{z \in i_{\mathrm{NO}}} + \rho \mathbf{1}_{z \in i_{\mathcal{O}}}\big) g_t(z) \phi\big(x; z - \delta_{\mathcal{O}}, \hat{\sigma_I}^1\big)\mathrm{d}z}{P_{\mathcal{O}}}. \tag{IA.73}$$

*(b) Suppose $g_t(\cdot)$ is not necessarily normal, and has mean $\nu_t$ and standard deviation $\tau_t$. Define the "normalized price impact" $\delta_{t+1,\mathcal{O}} = \mathsf{E}(w_t \mid g_t, \mathcal{O}) - \mathsf{E}(w_t \mid g_t)$ as the change in the expectation of $w_t$ after observing $\mathcal{O} \in \{\mathrm{BMO}, \mathrm{BLO}, \mathrm{SLO}, \mathrm{SMO}\}$ at $t$. Then,*

$$\delta_{t+1,\mathcal{O}} \;=\; \frac{\frac{\rho}{4}\int_{i_{\mathrm{NO}}} g_t(z)(z - \nu_t)\mathrm{d}z + \rho \int_{i_{\mathcal{O}}} g_t(z)(z - \nu_t)\mathrm{d}z}{P_{\mathcal{O}}}. \tag{IA.74}$$

*Denote by $\nu_{t+1,\mathcal{O}}$ and $\tau_{t+1,\mathcal{O}}$ the mean and standard deviation, respectively, of the posterior density $g_{t+1,\mathcal{O}}(x)$. Define:*

$$V_{t+1,\mathcal{O}} \;=\; \frac{1}{P_{\mathcal{O}}}\int g_t(z)\Big(\frac{1-\rho}{4} + \frac{\rho}{4}\mathbf{1}_{z \in i_{\mathrm{NO}}} + \rho \mathbf{1}_{z \in i_{\mathcal{O}}}\Big)\bigg(\Big(\frac{z - \nu_t}{\tau_t}\Big)^2 - 1\bigg)\mathrm{d}z. \tag{IA.75}$$

*Then,*

$$\nu_{t+1,\mathcal{O}} \;=\; \nu_t - \delta_{\mathcal{O}} + \delta_{t+1,\mathcal{O}}, \qquad \tau^2_{t+1,\mathcal{O}} \;=\; \tau_t^2(1 + V_{t+1,\mathcal{O}}) + (\hat{\sigma_I}^1)^2 - \delta^2_{t+1,\mathcal{O}}, \tag{IA.76}$$

$$\mathsf{E}(w_t \mid g_t, \mathcal{O}) \;=\; \nu_{t+1,\mathcal{O}} + \delta_{\mathcal{O}} \;=\; \nu_t + \delta_{t+1,\mathcal{O}}. \tag{IA.77}$$

*Let $\bar{\nu}_{t+1} = \mathsf{E}_{\mathcal{O}}\big(\nu_{t+1,\mathcal{O}}\big)$ and $\bar{\tau}^2_{t+1} = \mathsf{E}_{\mathcal{O}}\big(\tau^2_{t+1,\mathcal{O}}\big)$, where $\mathsf{E}_{\mathcal{O}}$ represents the average over $\mathcal{O} \in \{\mathrm{BMO}, \mathrm{BLO}, \mathrm{SLO}, \mathrm{SMO}\}$, with weights $P_{\mathcal{O}}$. Then, $\mathsf{E}_{\mathcal{O}}(\delta_{t+1,\mathcal{O}}) = \mathsf{E}_{\mathcal{O}}(V_{t+1,\mathcal{O}}) = 0$, and*

$$\bar{\nu}_{t+1} \;=\; \nu_t - \mathsf{E}_{\mathcal{O}}\,\delta_{\mathcal{O}}, \qquad \bar{\tau}^2_{t+1} \;=\; \tau_t^2 + (\hat{\sigma_I}^1)^2 - \mathsf{E}_{\mathcal{O}}\,\delta^2_{t+1,\mathcal{O}}. \tag{IA.78}$$

*(c) If $g_t(\cdot) = \mathcal{N}(\nu_t, \tau_t^2)$ is normal, let $i_{\mathcal{O}} = (L_{\mathcal{O}}, H_{\mathcal{O}})$, $\ell_{\mathcal{O}} = \frac{L_{\mathcal{O}} - \nu_t}{\tau_t}$, $h_{\mathcal{O}} = \frac{H_{\mathcal{O}} - \nu_t}{\tau_t}$. Then,*

$$
\begin{aligned}
P_{\mathcal{O}} &= \frac{1-\rho}{4} + \frac{\rho}{4}\big(\Phi(h_{\mathrm{NO}}) - \Phi(\ell_{\mathrm{NO}})\big) + \rho\big(\Phi(h_{\mathcal{O}}) - \Phi(\ell_{\mathcal{O}})\big), \\
\nu_{t+1,\mathcal{O}} &= \nu_t - \delta_{\mathcal{O}} + \delta_{t+1,\mathcal{O}}, \quad \tau_{t+1,\mathcal{O}}^2 \;=\; \tau_t^2(1 + V_{t+1,\mathcal{O}}) + (\hat{\sigma_I}^1)^2 - \delta_{t+1,\mathcal{O}}^2, \\
\delta_{t+1,\mathcal{O}} &= \frac{\frac{\rho}{4}\tau_t\big(\phi(\ell_{\mathrm{NO}}) - \phi(h_{\mathrm{NO}})\big) + \rho\tau_t\big(\phi(\ell_{\mathcal{O}}) - \phi(h_{\mathcal{O}})\big)}{P_{\mathcal{O}}}, \\
V_{t+1,\mathcal{O}} &= \frac{\frac{\rho}{4}\big(\ell_{\mathrm{NO}}\phi(\ell_{\mathrm{NO}}) - h_{\mathrm{NO}}\phi(h_{\mathrm{NO}})\big) + \rho\big(\ell_{\mathcal{O}}\phi(\ell_{\mathcal{O}}) - h_{\mathcal{O}}\phi(h_{\mathcal{O}})\big)}{P_{\mathcal{O}}}.
\end{aligned}
\tag{IA.79}
$$

*If one writes $\bar{\nu}_{t+1} = f(\nu_t)$, then*

(IA.80)

$$
f'(\nu_t) \;=\; 1 - \frac{\rho^2}{\tau_t \beta^1} \left( \gamma^1\Big(\phi\big(\tfrac{\alpha^0+\nu_t}{\tau_t}\big) + \phi\big(\tfrac{\alpha^0-\nu_t}{\tau_t}\big)\Big) \;+\; (1-\gamma^1)\Big(\phi\big(\tfrac{\alpha^1+\nu_t}{\tau_t}\big) + \phi\big(\tfrac{\alpha^1-\nu_t}{\tau_t}\big)\Big) \right).
$$

*(d) If $g_t(\cdot)$ is the standard normal density, with $\nu_t = 0$ and $\tau_t = 1$, then for all $\mathcal{O}$ at $t$,*

$$
P_{\mathcal{O}} \;=\; \frac{1}{4}, \qquad \delta_{t+1,\mathcal{O}} \;=\; \delta_{\mathcal{O}}, \qquad \nu_{t+1,\mathcal{O}} \;=\; 0, \qquad \bar{\tau}_{t+1} \;=\; 1. \tag{IA.81}
$$

*Hence, the normalized density $g_t$ has constant volatility.*

***Proof***. Conditional on observing $w_t = \frac{v_t - \mu_t}{V} = z$, the probability of an order $\mathcal{O} \in \{\mathrm{BMO}, \mathrm{BLO}, \mathrm{SLO}, \mathrm{SMO}\}$ at $t$ is

$$
\mathsf{P}\big(\mathcal{O}_t \;=\; \mathcal{O} \mid w_t = z\big) = \frac{1-\rho}{4} + \frac{\rho}{4}\,\mathbf{1}_{z \in i_{\mathrm{NO}}} + \rho\,\mathbf{1}_{z \in i_{\mathcal{O}}}.
$$

Indeed, if the trader at $t$ is uninformed (with probability $1-\rho$), he submits an order $\mathcal{O}$ with equal probability $\frac{1}{4}$. If the trader at $t$ is informed (with probability $\rho$), she submits an order $\mathcal{O}$ if and only if $z \in i_{\mathcal{O}}$; but if $z \in i_{\mathrm{NO}}$, she does not trade and is immediately replaced by an uniformed trader who submits $\mathcal{O}$ with equal probability $\frac{1}{4}$. Integrating over $z$, one obtains $P_{\mathcal{O}} = \frac{1-\rho}{4} + \frac{\rho}{4}\int_{z \in i_{\mathrm{NO}}} g_t(z)\mathrm{d}z + \rho\int_{z \in i_{\mathcal{O}}} g_t(z)\mathrm{d}z$, which proves (IA.72).

I now compute the density of the normalized asset value at $t+1$ after observing an order $\mathcal{O} \in \{\mathrm{BMO}, \mathrm{BLO}, \mathrm{SLO}, \mathrm{SMO}\}$ at $t$. Immediately after $t$ the public mean moves to $\mu_{t+1} = \mu_t + \Delta_{\mathcal{O}}$, where $\Delta_{\mathcal{O}} \in \{\Delta^1, \gamma^1\Delta^1, -\gamma^1\Delta^1, -\Delta^1\}$. Since $\frac{\Delta^1}{V^1} = \frac{\rho}{\beta^1}$, note that $\delta_{\mathcal{O}} = \frac{\Delta_{\mathcal{O}}}{V^1} \in \Big\{\frac{\rho}{\beta^1}, \gamma^1\frac{\rho}{\beta^1}, -\gamma^1\frac{\rho}{\beta^1}, -\frac{\rho}{\beta^1}\Big\}$. If $z = w_t$ and $\delta_v = \frac{v_{t+1} - v_t}{V}$, write $x = w_{t+1} = \frac{v_{t+1} - (\mu_t + \Delta_{\mathcal{O}})}{V} = \delta_v + z - \delta_{\mathcal{O}}$. But $\delta_v$ has a normal distribution given by with mean 0 and standard deviation $\hat{\sigma_I}^1 = \frac{\sigma_I}{V^1}$, hence $\mathsf{P}\big(w_{t+1} = x \mid \mathcal{O}_t = \mathcal{O}, w_t = z\big) = \mathsf{P}\big(\delta_v = x - z + \delta_{\mathcal{O}}\big) =$

$\phi(x-z+\delta_{\mathcal{O}};0,\hat{\sigma_I}^1)=\phi(x;z-\delta_{\mathcal{O}},\hat{\sigma_I}^1)$. One computes also:

$$\begin{aligned}\mathsf{P}\big(w_{t+1}=x,\mathcal{O}_t=\mathcal{O}\mid w_t=z\big) &= \mathsf{P}\big(w_{t+1}=x\mid \mathcal{O}_t=\mathcal{O},w_t=z\big)\,\mathsf{P}\big(\mathcal{O}_t=\mathcal{O}\mid w_t=z\big)\\ &= \phi\big(x;z-\delta_{\mathcal{O}},\hat{\sigma_I}^1\big)\Big(\frac{1-\rho}{4}+\frac{\rho}{4}\mathbf{1}_{z\in i_{\mathrm{NO}}}+\rho\mathbf{1}_{z\in i_{\mathcal{O}}}\Big).\end{aligned}$$

Thus, the posterior density is:

$$\begin{aligned}g_{t+1,\mathcal{O}}(x) &= \mathsf{P}\big(w_{t+1}=x\mid w_t\sim g_t(z),\mathcal{O}_t=\mathcal{O}\big)\\ &= \frac{\int \mathsf{P}\big(w_{t+1}=x,\mathcal{O}_t=\mathcal{O}\mid w_t=z\big)g_t(z)\mathrm{d}z}{\int \mathsf{P}\big(\mathcal{O}_t=\mathcal{O}\mid w_t=z\big)g_t(z)\mathrm{d}z}\\ &= \frac{\int\big(\frac{1-\rho}{4}+\frac{\rho}{4}\mathbf{1}_{z\in i_{\mathrm{NO}}}+\rho\mathbf{1}_{z\in i_{\mathcal{O}}}\big)\phi\big(x;z-\delta_{\mathcal{O}},\hat{\sigma_I}^1\big)g_t(z)\mathrm{d}z}{P_{\mathcal{O}}}.\end{aligned}$$

This proves (IA.73).

To prove part (b), start with $\mathsf{P}\big(w_t=z\mid \mathcal{O}_t=\mathcal{O}\big)=\frac{\frac{1-\rho}{4}+\frac{\rho}{4}\mathbf{1}_{z\in i_{\mathrm{NO}}}+\rho\mathbf{1}_{z\in i_{\mathcal{O}}}}{P_{\mathcal{O}}}$. Multiplying by $z$ and integrating, one gets $\mathsf{E}\big(w_t\mid g_t,\mathcal{O}\big)=\frac{\int z\big(\frac{1-\rho}{4}+\frac{\rho}{4}\mathbf{1}_{z\in i_{\mathrm{NO}}}+\rho\mathbf{1}_{z\in i_{\mathcal{O}}}\big)g_t(z)\mathrm{d}z}{P_{\mathcal{O}}}$, and by subtracting $\nu_t=\mathsf{E}\big(w_t\mid g_t\big)$ one gets $\delta_{t+1,\mathcal{O}}=\frac{\int\big(\frac{1-\rho}{4}+\frac{\rho}{4}\mathbf{1}_{z\in i_{\mathrm{NO}}}+\rho\mathbf{1}_{z\in i_{\mathcal{O}}}\big)(z-\nu_t)g_t(z)\mathrm{d}z}{P_{\mathcal{O}}}$. But $\int(z-\nu_t)g_t(z)\mathrm{d}z=0$, hence $\delta_{t+1,\mathcal{O}}=\frac{\int\big(\frac{\rho}{4}\mathbf{1}_{z\in i_{\mathrm{NO}}}+\rho\mathbf{1}_{z\in i_{\mathcal{O}}}\big)(z-\nu_t)g_t(z)\mathrm{d}z}{P_{\mathcal{O}}}$, which proves (IA.74).

To compute the mean of $g_{t+1,\mathcal{O}}(x)$, integrate the formula (IA.73) over $x$, and obtain $\nu_{t+1,\mathcal{O}}=\frac{\int(\frac{1-\rho}{4}+\frac{\rho}{4}\mathbf{1}_{z\in i_{\mathrm{NO}}}+\rho\mathbf{1}_{z\in i_{\mathcal{O}}})(z-\delta_{\mathcal{O}})g_t(z)\mathrm{d}z}{P_{\mathcal{O}}}$. This is similar to the formula proved for $\delta_{t+1,\mathcal{O}}$, except that $\nu_t$ is replaced by $\delta_{\mathcal{O}}$. One gets $\nu_{t+1,\mathcal{O}}=\delta_{t+1,\mathcal{O}}+\nu_t-\delta_{\mathcal{O}}$, which proves the first part of (IA.76).

For the second part of (IA.76), note that for any (not necessarily normal) distribution $g$ with mean $\nu$ and variance $\sigma_I^2$, $\int(x+a)^2g(x)\mathrm{d}x=\sigma_I^2+(\nu+a)^2$. Then,

$$\text{(IA.82)}\quad \int(x-\nu_t+\delta_{\mathcal{O}})^2g_{t+1,\mathcal{O}}(x)\mathrm{d}x \;=\; \tau_{t+1,\mathcal{O}}^2+(\nu_{t+1,\mathcal{O}}-\nu_t+\delta_{\mathcal{O}})^2 \;=\; \tau_{t+1,\mathcal{O}}^2+\delta_{t+1,\mathcal{O}}^2.$$

One also integrates directly $\int(x-\nu_t+\delta_{\mathcal{O}})^2g_{t+1,\mathcal{O}}(x)\mathrm{d}x$ by replacing $g_{t+1,\mathcal{O}}(x)$ as in (IA.73). Using the formula $\int(x-\nu_t+\delta_{\mathcal{O}})^2\phi(x;z-\delta_{\mathcal{O}},\hat{\sigma_I}^1)\mathrm{d}x=(z-\nu_t)^2+(\hat{\sigma_I}^1)^2$, one obtains:

(IA.83)
$$\int(x-\nu_t+\delta_{\mathcal{O}})^2g_{t+1,\mathcal{O}}(x)\mathrm{d}x \;=\; (\hat{\sigma_I}^1)^2+\frac{\int g_t(z)\big(\frac{1-\rho}{4}+\frac{\rho}{4}\mathbf{1}_{z\in i_{\mathrm{NO}}}+\rho\mathbf{1}_{z\in i_{\mathcal{O}}}\big)(z-\nu_t)^2\mathrm{d}z}{P_{\mathcal{O}}}.$$

Putting together (IA.82) and (IA.83), one obtains the desired formula for $\tau_{t+1,\mathcal{O}}^2$. Equation (IA.77) follows directly from (IA.74) and (IA.76). Finally, proving $\mathsf{E}_{\mathcal{O}}(\delta_{t+1,\mathcal{O}})=0$ and $\mathsf{E}_{\mathcal{O}}(V_{t+1,\mathcal{O}})=0$ is straightforward, which also implies equation (IA.78).

To prove part (c), first use (IA.72) to compute $P_{\mathcal{O}} = \frac{1-\rho}{4} + \frac{\rho}{4}\big(\Phi(h_{\mathrm{NO}}) - \Phi(\ell_{\mathrm{NO}})\big) + \rho\big(\Phi(h_{\mathcal{O}}) - \Phi(\ell_{\mathcal{O}})\big)$. To prove the formula for $\delta_{t+1,\mathcal{O}}$, make the change of variable $z' = \frac{z-\nu_t}{\tau_t}$ and denote by $i'_{\mathcal{O}} = (\ell_{\mathcal{O}}, h_{\mathcal{O}})$. Then,

$$\begin{aligned}
\delta_{t+1,\mathcal{O}} &= \frac{\rho\tau_t \int_{i'_{\mathrm{NO}}} \phi(z')z'\mathrm{d}z + \rho\tau_t \int_{i'_{\mathcal{O}}} \phi(z')z'\mathrm{d}z}{P_{\mathcal{O}}} \\
&= \frac{\frac{\rho}{4}\tau_t\big(\phi(\ell_{\mathrm{NO}}) - \phi(h_{\mathrm{NO}})\big) + \rho\tau_t\big(\phi(\ell_{\mathcal{O}}) - \phi(h_{\mathcal{O}})\big)}{P_{\mathcal{O}}}.
\end{aligned}$$

A similar computation for $V_{t+1,\mathcal{O}}$ finishes the proof of (IA.79). Finally, by taking the average of $\nu_{t+1,\mathcal{O}}$ over $\mathcal{O} \in \{\mathrm{BMO}, \mathrm{BLO}, \mathrm{SLO}, \mathrm{SMO}\}$, one obtains:

(IA.84)

$$\begin{aligned}
\bar{\nu}_{t+1} &= f(\nu_t) = \nu_t - \sum_{\mathcal{O}} P_{\mathcal{O}}\delta_{\mathcal{O}} \\
&= \nu_t - \frac{1-\rho}{4}\sum_{\mathcal{O}}\delta_{\mathcal{O}} - \frac{\rho}{4}\sum_{\mathcal{O}}\big(\Phi(h_{\mathrm{NO}}) - \Phi(\ell_{\mathrm{NO}})\big)\delta_{\mathcal{O}} - \rho\sum_{\mathcal{O}}\big(\Phi(h_{\mathcal{O}}) - \Phi(\ell_{\mathcal{O}})\big)\delta_{\mathcal{O}} \\
&= \nu_t - \rho\sum_{\mathcal{O}}\big(\Phi(h_{\mathcal{O}}) - \Phi(\ell_{\mathcal{O}})\big)\delta_{\mathcal{O}},
\end{aligned}$$

where the last equality follows from $\sum_{\mathcal{O}}\delta_{\mathcal{O}} = 0$. If one differentiates the endpoints of $i'_{\mathcal{O}}$ with respect to $\nu_t$, one gets $-\frac{1}{\tau_t}$ in all cases, hence $f'(\nu_t) = 1 - \rho\sum_{\mathcal{O}}\big(\phi(h_{\mathcal{O}}) - \phi(\ell_{\mathcal{O}})\big)\big(-\frac{1}{\tau_t}\big)\delta_{\mathcal{O}}$. Using $\delta_{\mathcal{O}} \in \left\{\frac{\rho}{\beta^1}, \gamma^1\frac{\rho}{\beta^1}, -\gamma^1\frac{\rho}{\beta^1}, -\frac{\rho}{\beta^1}\right\}$, a straightforward calculation proves (IA.80).

To prove part (d), I substitute $\nu_t = 0$ and $\tau_t = 1$ in the formulas above. I only prove the results for $\mathcal{O} = \mathrm{BMO}$ and BLO, the proof for the other order types being symmetric. To compute the probability of a BMO, I use the equality $\int_{\alpha^0}^{\alpha^1}\phi(z)\mathrm{d}z = \int_{\alpha^1}^{\infty}\phi(z)\mathrm{d}z$. This implies $\int_{\alpha^1}^{\infty}\phi(z)\mathrm{d}z = \frac{1}{2}\int_{\alpha^0}^{\infty}\phi(z)\mathrm{d}z$, from which one computes:

(IA.85)
$$\begin{aligned}
P_{\mathrm{BMO}} &= \frac{1-\rho}{4} + \frac{\rho}{4}\int_{-\alpha^0}^{\alpha^0}\phi(z)\mathrm{d}z + \rho\int_{\alpha^1}^{\infty}\phi(z)\mathrm{d}z \\
&= \frac{1-\rho}{4} + \frac{\rho}{2}\int_{0}^{\alpha^0}\phi(z)\mathrm{d}z + \frac{\rho}{2}\int_{\alpha^0}^{\infty}\phi(z)\mathrm{d}z = \frac{1-\rho}{4} + \frac{\rho}{2}\frac{1}{2} = \frac{1}{4}.
\end{aligned}$$

The probability of a BLO is $P_{\mathrm{BLO}} = \frac{1-\rho}{4} + \frac{\rho}{4}\int_{-\alpha^0}^{\alpha^0}\phi(z)\mathrm{d}z + \rho\int_{\alpha^0}^{\alpha^1}\phi(z)\mathrm{d}z$. Using $\int_{\alpha^0}^{\alpha^1}\phi(z)\mathrm{d}z = \int_{\alpha^1}^{\infty}\phi(z)\mathrm{d}z$, note that $P_{\mathrm{BMO}} = P_{\mathrm{BLO}}$, hence $P_{\mathrm{BLO}} = \frac{1}{4}$.

The normalized price impact of a BMO is:

(IA.86)
$$\delta_{t+1,\mathrm{BMO}} = \frac{\frac{\rho}{4}\int_{-\alpha^0}^{\alpha^0}\phi(z)z\mathrm{d}z + \rho\int_{\alpha^1}^{\infty}\phi(z)z\mathrm{d}z}{P_{\mathrm{BMO}}} = \frac{\rho\phi(\alpha^1)}{1/4} = \frac{\rho}{\beta^1} = \delta_{\mathrm{BMO}}.$$

The normalized price impact of a BLO is:

$$\text{(IA.87)}\qquad \begin{aligned}\delta_{t+1,\mathrm{BLO}} &= \frac{\frac{\rho}{4}\int_{-\alpha^0}^{\alpha^0}\phi(z)z\mathrm{d}z+\rho\int_{\alpha^0}^{\alpha^1}\phi(z)z\mathrm{d}z}{P_{\mathrm{BLO}}} = \frac{\rho\big(\phi(\alpha^0)-\phi(\alpha^1)\big)}{1/4}\\ &= \frac{\phi(\alpha^0)-\phi(\alpha^1)}{\phi(\alpha^1)}\,\frac{\rho\phi(\alpha^1)}{1/4} = \gamma^1\frac{\rho}{\beta^1} = \delta_{\mathrm{BLO}}.\end{aligned}$$

By symmetry, it follows that $\delta_{t+1,\mathcal{O}}=\delta_{\mathcal{O}}$ for all orders $\mathcal{O}\in\{\mathrm{BMO},\mathrm{BLO},\mathrm{SLO},\mathrm{SMO}\}$.

I now compute $\nu_{t+1,\mathcal{O}}=\nu_t-\delta_{\mathcal{O}}+\delta_{t+1,\mathcal{O}}=\nu_t=0$. Also, $\bar{\tau}_{t+1}^2=\mathsf{E}_{\mathcal{O}}\big(\tau_{t+1,\mathcal{O}}^2\big)=\mathsf{E}_{\mathcal{O}}\big(\tau_t^2+(\hat{\sigma_I}^1)^2-\delta_{\mathcal{O}}^2\big)$. But $\mathsf{E}_{\mathcal{O}}(\delta_{\mathcal{O}}^2)=\frac{1}{4}\Big((\frac{\rho}{\beta^1})^2+(\gamma^1\frac{\rho}{\beta^1})^2+(-\gamma^1\frac{\rho}{\beta^1})^2+(-\frac{\rho}{\beta^1})^2\Big)=(\hat{\sigma_I}^1)^2$, hence $\bar{\tau}_{t+1}^2=\tau_t^2+(\hat{\sigma_I}^1)^2-(\hat{\sigma_I}^1)^2=\tau_t^2$, from which $\bar{\tau}_{t+1}=\tau_t=1$. Thus, the posterior mean is equal to 0 irrespective of the order $\mathcal{O}$ at $t$, while the posterior variance is equal to 1 on average. This means that the normalized density $\mathcal{N}(0,1)$ does not change. □

As for the benchmark model, the existence of the equilibrium with no-order region depends on certain properties of the information function $I^1$ being true. These are stated in Result IA.6 below. Conditional on these properties being true, this proves Proposition IA.4.

**Proof of Proposition IA.4**. This proof is essentially identical to that of Theorem 1. The only difference is that the informed trader no longer faces a penalty $\omega\geq\gamma\Delta$ from not trading. Instead, the commitment parameter is $\omega=0$. As a result, it is sometimes optimal for the informed trader to submit no order and exit the model, if the signal $w_t$ is the no-order region $(-\alpha^0,\alpha^0)$. Indeed, if one applies Lemma A2 in the Appendix of the paper, it follows that the normalized expected payoff of the informed trader from a BLO is $\hat{\mathcal{U}}_{\mathrm{BLO}}^I(w_t)=\frac{S^1}{2V}I^1(\rho,w_t)+J^1(\rho,w_t)$. But, by Result IA.6, $J^1=1$, hence:

$$\text{(IA.88)}\qquad 2\hat{\mathcal{U}}_{\mathrm{BLO}}^I(w_t) = \alpha^1-I^1(\rho,\alpha^1)+2I^1(\rho,w_t).$$

By Result IA.6, $I^1(\rho,w)$ is strictly increasing in $w$. The informed trader prefers BLO to NO (no order) if and only if $\alpha^1-I^1(\rho,\alpha^1)+2I^1(\rho,w_t)>0$. Hence, the worst-case scenario for the informed trader is to observe a positive signal $w_t\approx 0$.

From Result IA.6, there always exists a solution $\alpha_*^0$ to the system of equations (IA.46). If $\alpha_*^0>0$, then $\alpha^0=\alpha_*^0$ and $\hat{\mathcal{U}}_{\mathrm{BLO}}^I(\alpha^0)=\alpha^1-I^1(\rho,\alpha^1)+2I^1(\rho,\alpha^0)=0$, hence the informed trader is indifferent between BLO and NO at the threshold $w_t=\alpha^0$. This means that the informed trader has a no-order region. The rest of the proof in this case now continues as in Theorem 1 in the paper.

If $\alpha_*^0 < 0$, then $\alpha^0 = 0$ (by Definition IA.2), and $\hat{\mathcal{U}}_{\rm BLO}^I(0) = \alpha^1 - I^1(\rho, \alpha^1) + 2I^1(\rho, 0) > \alpha^1 - I^1(\rho, \alpha^1) + 2I^1(\rho, \alpha_*^0) = 0$, which means that even when $w_t = 0$ the trader still prefers BLO to NO. Thus, the informed trader does not have a no-order region in this case. In this case, by Definition IA.2, all the other parameters have the same values as in the benchmark model, and the rest of the proof in this case continues as in Theorem 1 in the paper. □

**Result IA.6.** For all $\rho \in (0,1)$, the functions $I^1(\rho, w)$, $w - I^1(\rho, w)$ and $I^1(\rho, w) - I^1(\rho, -w)$ are strictly increasing in $w$, and $S^1 > \Delta^1(1 + \gamma^1)$. The system of equations (IA.46) in the paper always has a real solution, although not necessarily with $\alpha^0 > 0$. Let $I^1(\rho, w, j)$ and $J^1(\rho, w, j)$ be as in Definition 1 in the paper. Then, $(i)$ $I^1 = I^1(\rho, w, j)$ decreases in $j$ if $w > 0$, and $(ii)$ $J^1(\rho, w, j) = 1$.

I also include here the conditions on the preference parameter, $\bar{u} > S/2$.

**Verification of Result IA.6**. The function $I^1$ is close to the function $I$, and hence its properties are very similar to the properties of the information function. It remains to verify that the system of equations (IA.46) always has a real solution, which is straightforward to check. This property is related to the verification below of Result IA.3. □

**Verification of Result IA.3**. The verification is essentially done by visual inspection of Figure IA.9. It remains just to show the existence of the threshold $\rho = \rho_0$ below which the no-order region is empty. The worst-case scenario for the informed trader is when she gets a positive signal $w_t$ which is very close to 0. In that case, a similar result to Lemma A2 in the Appendix of the paper implies that the informed trader gets a continuation payoff from BLO which is very close to $S/2 + I^1(\rho, 0)V$. The rest of the verification is straightforward.

Consider thus the function:

$$f(\rho) \;=\; \frac{\frac{S}{2} + I(\rho,0)\,V}{\Delta} \;=\; \Big(\alpha - I(\rho,\alpha) + 2I(\rho,0)\Big)\,\frac{\beta}{2\rho}. \tag{IA.89}$$

Figure IA.12 shows the function $f(\rho)$. The function $f$ is strictly decreasing in $\rho$, and takes both positive and negative values. Therefore, by continuity there must be some $\rho_0 \in (0,1)$ for which $f(\rho_0) = 0$. By spline interpolation, one estimates:

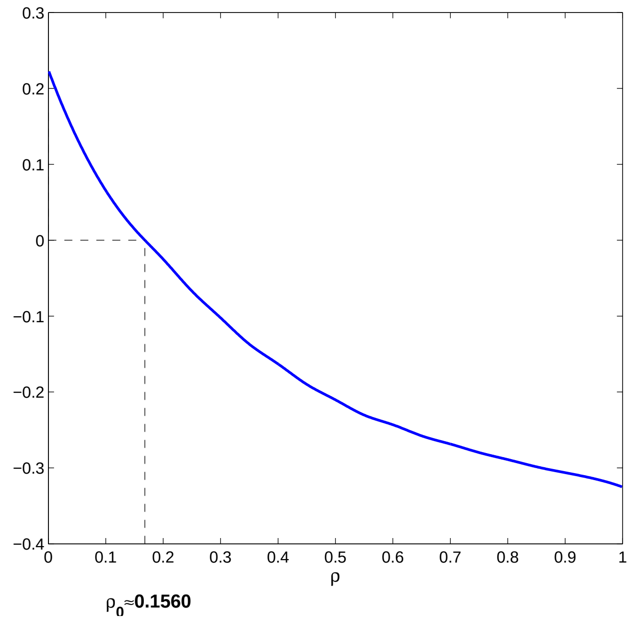
$$\rho_0 \;\approx\; 0.1560. \tag{IA.90}$$

□

FIGURE IA.12

**The Commitment Threshold**

Figure IA.12 shows $f(\rho) = \big(\alpha - I(\rho, \alpha) + 2I(\rho, 0)\big)\,\frac{\beta}{2\rho}$ as a function of the informed share $\rho$. The Monte Carlo parameters are $M = 5,000$ and $L = 100$.




## 6.4 Private Information Processing

In the benchmark model, if an informed trader arrives at date $t$ and submits a limit order, she can compute the average signal at the execution of her limit order, but she can do this computation only once, at $t$, after which she cannot condition on the subsequent order flow. Thus, the informed trader becomes essentially uninformed after one trading round, and, like the other patient uninformed traders in the order book, has no incentive to later cancel her order. In this section, I provide some heuristic discussion regarding what happens if informed traders can continue to process information correctly at least for the next few trading rounds.

To get intuition for this situation, consider the case of an informed trader who observes initially a moderately positive signal $w_t = \frac{v_t - \mu_t}{V} > 0$ and submits a BLO. Then, if she can correctly compute her posterior belief after observing the order flow, it may happen that after adverse order flow (a sequence of buy orders, market or limit), the posterior mean of her belief becomes negative. In that case, she can cancel the BLO, after which she has three options: (i) submit an SLO, (ii) submit an SMO, or (iii) exit the model. I conjecture that option (i) can be ruled out by out-of-equilibrium beliefs, as in the proof of Theorem 1 in the paper, because it provides information to the other limit order book traders. Option (ii) is expensive, because by submitting a market order the informed trader loses half the bid–ask spread. Moreover, it is unlikely that the informed trader's belief switches from moderately positive to extremely negative, to justify the submission of an SMO. Option (iii) is then the most likely outcome, provided the exchange does not charge cancellation cost, or provided that the informed trader does not worry about losing time priority in the book. In the model setup, time priority does not matter because the tick size is 0, but in practice tick size can be an important factor in the decision to cancel a limit order.

Another possibility is to leave the BLO in the book, but just modify the price to a lower price, away from the market. This move has the advantage of lowering the expected negative signal at execution, which, using the terms in the paper, is a form of "reverse slippage." Nevertheless, the informed trader now has to wait more. In the model, the waiting costs are 0, but with positive waiting costs the decision to step away from the market is less attractive.

Note also that signal shifts from moderately positive to extremely negative are rare. Such an adverse move could occur if for instance the informed trader observed a series of BMOs. Indeed, such order flow would increase the public mean $\mu$, and therefore decrease her private signal $w_{t+\tau}$. Consider, however, the case in which the informed share $\rho$ is significantly smaller than $\alpha\beta \approx 0.5306$; denote this situation by $\rho \ll \alpha\beta$. Then, the normalized price impact parameter satisfies:

$$\text{(IA.91)} \qquad \frac{\Delta}{V} = \frac{\rho}{\beta} \ll \alpha.$$

This implies that, compared to the informed trader's signal $w_t$ (which is between 0 and $\alpha$), the normalized change in public mean is significantly smaller, as long as the signal $w_t$ it is not very close to 0. Hence, it would take a large number of BMOs to decrease the signal $w_{t+\tau}$ below 0. The ex-ante probability of such an adverse order flow is small. Conversely, if $\rho$ is large, then the informed trader quickly becomes uninformed, and hence her signal does not change much after each order, either. One thus expects that adverse signal moves are more likely when the informed share is in an intermediate range.

A formal analysis along the lines discussed above is beyond the scope of the present paper. Yet, the arguments above suggest that allowing traders to continuously process information correctly does not fundamentally alter the robustness of the main results.

# 7 Stationary Filtering

In this section, I show that in a stationary equilibrium the variance of changes in the asset value is the same as the variance of changes in the public mean. Let $v_t$ be a diffusion process with no drift, and constant volatility $\sigma_v$. Suppose each period the market gets (public) information about $v_t$. Let $\mathcal{J}_t$ be the public information set available at time $t$. Denote by $\mu_t = \mathsf{E}(v_t|\mathcal{J}_t) = \mathsf{E}_t(v_t)$ the public mean at time $t$ (i.e., the expected asset value given all public information). (The subscript $t$ indicates conditioning on the information set, $\mathcal{J}_t$.) This filtration problem is called "stationary" if the public variance is constant over time: $\mathsf{Var}_t(v_t) = \mathsf{Var}_{t+1}(v_{t+1})$. This means that each period the public has the same prior variance about the asset value.

**Proposition IA.5.** *The filtration problem is stationary if and only if*

$$\mathsf{Var}(v_{t+1} - v_t) \;=\; \mathsf{Var}(\mu_{t+1} - \mu_t).$$

***Proof.*** The value $v_t$ can be decomposed into two orthogonal (uncorrelated) components: $v_t = \mu_t + \mu_t$. Moreover, $\mathsf{Var}(\mu_t) = \mathsf{Var}_t(v_t)$. Similarly, $v_{t+1} = \mu_{t+1} + \mu_{t+1}$, and $\mathsf{Var}(\mu_{t+1}) = \mathsf{Var}_{t+1}(v_{t+1})$. So the stationary condition reads $\mathsf{Var}(v_{t+1} - \mu_{t+1}) = \mathsf{Var}(v_t - \mu_t)$.

The difference $v_{t+1} - \mu_t$ can be decomposed in two ways:

$$\begin{aligned} v_{t+1} - \mu_t &= (v_{t+1} - \mu_{t+1}) + (\mu_{t+1} - \mu_t) \\ &= (v_{t+1} - v_t) + (v_t - \mu_t). \end{aligned}$$

First, I check that these are orthogonal decompositions. The first condition is that $\mathsf{cov}(v_{t+1} - \mu_{t+1}, \mu_{t+1} - \mu_t) = 0$, i.e., that $\mathsf{cov}(\mu_{t+1}, \mu_{t+1} - \mu_t) = 0$. As $\mathsf{cov}(\mu_{t+1}, \mu_{t+1}) = 0$, one can just check that $\mathsf{cov}(\mu_{t+1}, \mu_t) = 0$. Note that $\mu_{t+1}$ has zero conditional mean: $\mathsf{E}_{t+1}(\mu_{t+1}) = 0$. By the law of iterated expectations, one also has $\mathsf{E}_t(\mu_{t+1}) = \mathsf{E}(\mu_{t+1}) = 0$. Then, $\mathsf{cov}(\mu_{t+1}, \mu_t) = \mathsf{E}(\mu_{t+1}\mu_t) - \mathsf{E}(\mu_{t+1})\,\mathsf{E}(\mu_t) = \mathsf{E}(\mu_{t+1}\mu_t) = \mathsf{E}\,\mathsf{E}_t(\mu_{t+1}\mu_t) = \mathsf{E}(\mu_t\,\mathsf{E}_t(\mu_{t+1})) = 0$. The second condition is that $\mathsf{cov}(v_{t+1} - v_t, v_t - \mu_t) = 0$. But $v_t$ has independent increments, so $v_{t+1} - v_t$ is independent of $v_t$ and anything contained in the information set at time $t$. (This is true as long as the market does not get at $t$ information about the asset value at a future time.)

Now, I use the orthogonal decompositions to decompose the variance of $v_{t+1} - \mu_t$: $\mathsf{Var}(v_{t+1} - \mu_{t+1}) + \mathsf{Var}(\mu_{t+1} - \mu_t) = \mathsf{Var}(v_{t+1} - v_t) + \mathsf{Var}(v_t - \mu_t)$. But being stationary is equivalent to $\mathsf{Var}(v_{t+1} - \mu_{t+1}) = \mathsf{Var}(v_t - \mu_t)$, which is then equivalent to $\mathsf{Var}(v_{t+1} - v_t) = \mathsf{Var}(\mu_{t+1} - \mu_t)$. $\square$

## References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