Equivariant K-theory and equivariant cohomology

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Abstract. For T an abelian compact Lie group, we give a description of
T-equivariant K-theory with complex coefficients in terms of equivariant
cohomology. In the appendix we give applications of this by extending
results of Chang-Skjelbred and Goresky-Kottwitz-MacPherson from equivar-
iant cohomology to equivariant K-theory.

1. Introduction

Let T be an abelian compact Lie group, not necessarily connected. Let X
be a compact T-equivariant manifold, or more generally a finite T-CW
complex. We denote by $H_T^*(X)$ the T-equivariant (Borel) cohomology of
X, as described in Atiyah and Bott [1], and by $K_T^*(X)$ the T-equivariant
K-theory of X, as described in Segal [16]. All the cohomology theories in
this paper have complex coefficients, unless otherwise noted. For example,
$K_T^*(X) = K_T^*(X, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C}$. Also, when X is a point, we write K_T^* instead
of $K_T^*(X)$, and similarly for H_T^*.

The goal of this paper is to describe $K_T^*(X)$ in terms of $H_T^*(X)$. When T
is the trivial group, this is easy: it is a classical result that the Chern character
$ch : K^*(X) \to H^*(X)$ is an isomorphism (in this case one only needs to
tensor with $\mathbb{Q}$). In general however it is not true that the equivariant version
of the Chern character

$$ch_T : K_T^*(X) \to H_T^{**}(X)$$

is an isomorphism. (For the definition of ch_T see Lemma 3.1 and the dis-
cussion before it.)

The good news is that there is still a way in which one can describe
$K_T^*(X)$ in terms of $H_T^*(X)$. Details will be given later, but for now let

us outline the main steps of this description. Let $\mathcal{C}_T = \operatorname{Spec}_m K_T^*$ be the complex algebraic group of the maximal ideals of K_T^*. The construction of $\mathcal{C}_T$ is functorial in T, and if $H \hookrightarrow T$ is a compact subgroup of T, we can identify $\mathcal{C}_H$ as a subgroup of $\mathcal{C}_T$ via the map $\mathcal{C}_H \to \mathcal{C}_T$. If α is a point of $\mathcal{C}_T$, denote by $H(\alpha)$ the smallest compact subgroup H of T such that $\mathcal{C}_H$ contains α. Denote by $X^\alpha = X^{H(\alpha)}$, the subspace of X fixed by all elements of $H(\alpha)$. Denote by $\mathcal{O}$ the sheaf of algebraic functions on $\mathcal{C}_T$, and by $\mathcal{O}^h$ the sheaf of holomorphic functions. Then we will define a sheaf, denoted by $\mathcal{K}_T^*(X)$, whose stalk at a point $\alpha \in \mathcal{C}_T$ is

$$\mathcal{K}_T^*(X)_\alpha = \mathbf{H}_T^*(X^\alpha),$$

where $\mathbf{H}_T^*(-)$ is the extension of $H_T^*(-)$ by the ring of holomorphic germs at zero on H_T^*. Moreover, the transition functions of $\mathcal{K}_T^*(X)$ will be also defined entirely using the equivariant cohomology of X. Now, if Γ denotes the global sections functor, we will show that there exists an isomorphism

$$\Phi_T : K_T^*(X) \otimes_{\Gamma \mathcal{O}} \Gamma \mathcal{O}^h \cong \Gamma \mathcal{K}_T^*(X).$$

This is the sense in which equivariant K-theory with complex coefficients can be described in terms of equivariant cohomology.

We should say a few words about the isomorphism Φ_T. If we denote by ch_T the equivariant Chern character (to be defined below), we will see that Φ_T is esentially a sheaf version of ch_T. However, ch_T has to be twisted (translated) in an appropriate sense, to take into account the point α over which the germ of ch_T is taken.

One may call this a de Rham model for equivariant K-theory, but it would be a slight misnomer, since we do not describe $K_T^*(X)$ at the level of cocycles. As a matter of fact, we do more: we describe the classes themselves, as sections in a sheaf build from ordinary equivariant cohomology. But, if one were really intent on giving a de Rham model, one could use the sheaf model to define a cocycle in K-theory as a collection of germs of ordinary closed differential forms, and use a similar definition for coboundaries. We do not purse this avenue because it would only obscure the purely topological nature of our description of $K_T^*(X)$.

There were previous attempts to give a de Rham type of model for $K_T^*(X)$. The earliest version appeared in Baum, Brylinski and MacPherson [4]. The ideas were further developed in Block and Getzler [5], and Duflo and Vergne [10]. In fact, both the idea of describing equivariant K-theory as a sheaf and twisting the equivariant Chern character are present in Duflo and Vergne. The problem in their paper is that they cannot prove the Mayer–Vietoris property for their cohomology theory, because they work with $\mathcal{C}^\infty$ functions. The advantage of our approach is that we work with coherent analytic sheaves over the affine (Stein) manifold $\mathcal{C}_T$, and in this case the global section functor is exact.

The present paper is inspired mainly by Grojnowski's preprint [12]. In this seminal work, he uses ideas from the papers mentioned above to define equivariant elliptic cohomology with complex coefficients. His model starts with an elliptic curve $\mathcal{E}$, and constructs for every torus T a complex variety $\mathcal{E}_T$ and a coherent analytic sheaf $\mathrm{Ell}_T^*(X)$ over $\mathcal{E}_T$, whose stalk at each point is defined in terms of equivariant cohomology. The sheaf $\mathrm{Ell}_T^*(X)$ is then defined by Grojnowski to be the ("delocalized") complex T-equivariant cohomology of X. Interestingly enough, equivariant K-theory is never explicitly mentioned in Grojnowski's preprint, although he was most likely aware that an analogous construction to that of $\mathrm{Ell}_T^*(X)$ should lead to equivariant K-theory, if the elliptic curve $\mathcal{E}$ is replaced by the multiplicative group $\mathcal{C} = \mathbb{C} \setminus \{0\}$. The main contribution of the present paper is to do exactly that: it starts with the multiplicative group $\mathcal{C}$, out of which it defines the base complex variety $\mathcal{C}_T$, and constructs a coherent analytic sheaf $\mathcal{K}_T^*(X)$ over $\mathcal{C}_T$. Then, the ring of global sections in $\mathcal{K}_T^*(X)$ turns out to be a faithfully flat extension of equivariant K-theory[1].

A simple exercise shows that if one starts instead with the additive group $\mathcal{A} = \mathbb{C}$, the resulting sheaf is nothing else but ordinary equivariant cohomology. An important conclusion is that, when working with complex coefficients, the difference between equivariant cohomology, K-theory and elliptic cohomology stems mainly from the fact that these theories are associated to different complex groups of dimension one: the additive, the multiplicative, and the elliptic groups, respectively.

The results of this paper can be extended in several directions. First, we can describe $K_T^*(X)$ directly, instead of describing its faithfully flat extension $K_T^*(X) \otimes_{\Gamma\mathcal{O}} \Gamma\mathcal{O}^h$. But in order to do that, one needs to define algebraic sheaves $\mathcal{F}_\alpha$ (see Definition 2.9) instead of holomorphic ones. And this can only be done using completions, because the logarithm map is not algebraic. The construction therefore becomes more complicated, and we decided to relegate it to another paper. Second, if G is a nonabelian connected compact Lie group, and T its maximal torus, then $K_G^*(-) = K_T^*(-)^W$, where W is the Weyl group, so one can also describe $K_G^*(-)$ using Borel equivariant cohomology. Third, we can prove a similar result whenever the coefficient ring R of the cohomology theories involved is an algebra over $\mathbb{Q}$ adjoined the roots of unity. We need R to be a $\mathbb{Q}$-algebra because the logarithm map is only defined over $\mathbb{Q}$, and we need to invert the roots of unity because we want to split $R[z]/\langle z^n - 1\rangle$ into a direct sum of n copies of R.

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[1] Besides its contribution to equivariant K-theory, one can regard the present paper as giving a rigorous definition for Grojnowski's equivariant elliptic cohomology: for this, it is enough to change the base manifold $\mathcal{C}_T$ to Grojnowsi's $\mathcal{E}_T$. See also Rosu [15] for a definition of equivariant elliptic cohomology when $T = S^1$.

While the details of the sheafifying process are somewhat technical, in principle the construction allows one to infer some results in equivariant K-theory from the corresponding ones in equivariant cohomology. In the Appendix we give examples of this, extending results of Chang–Skjelbred [9] and Goresky–Kottwitz–MacPherson [11] from equivariant cohomology to equivariant K-theory.

2. A sheaf-valued cohomology theory

The purpose of this section is to define a sheaf valued T-equivariant cohomology theory, which we denote by $\mathcal{K}_T^*(-)$. In the next sections we are going to show that global sections of this sheaf are essentially equivariant K-theory. We already knew that $K_T^*(X)$ can be regarded as a coherent sheaf over $\mathcal{C}_T = \mathrm{Spec}_m K_T^*$ (because it is a K_T^*-module). The novelty is that $K_T^*(X)$ can be completely described in terms of *ordinary* equivariant cohomology (since we will show that $\mathcal{K}_T^*(X)$ is). Let us start with a few definitions.[2].

2.1. Definitions

First we want a simpler description of $\mathcal{C}_T = \mathrm{Spec}_m K_T^*$. Denote by $\hat{T}$ the Pontrjagin dual of T, i.e. $\hat{T} = \mathrm{Hom}(T, S^1)$. For example, if $T = (S^1)^p \times G$, where G is a finite abelian group, then $\hat{T} \cong \mathbb{Z}^p \times G$ (the isomorphism is not natural, however). Denote by $\mathcal{C}$ the multiplicative algebraic group $\mathbb{C} \setminus \{0\}$. Although it might generate some confusion, we will use additive notation for $\mathcal{C}$ throughout the paper. The following straighforward lemma gives two alternate descriptions of $\mathcal{C}_T$.

Proposition 2.1. *There is a natural isomorphism of algebraic varieties*

$$\mathcal{C}_T \cong \mathrm{Hom}_{\mathbb{Z}}(\hat{T}, \mathcal{C}).$$

If T is connected (i.e. a torus), and Λ_T is its integer lattice, then there is a natural isomorphism

$$\mathcal{C}_T \cong \Lambda_T \otimes_{\mathbb{Z}} \mathcal{C}.$$

Proof. It is easy to see that $K_T^* = \mathbb{C}[\hat{T}]$, the group algebra of $\hat{T}$. We define a map

$$\nu : \mathrm{Hom}_{\mathbb{Z}}(\hat{T}, \mathcal{C}) \to \mathrm{Spec}_m \mathbb{C}[\hat{T}] = \mathcal{C}_T$$

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[2] For a similar definition in the case of equivariant elliptic cohomology, see Rosu [15]. The discussion there is only for $T = S^1$, but it generalizes easily with the same formalism as here.

by noting that $\alpha \in \text{Hom}_{\mathbb{Z}}(\hat{T}, \mathcal{C})$ extends to a non-zero $\mathbb{C}$-algebra map $\alpha' : \mathbb{C}[\hat{T}] \to \mathbb{C}$. We then take $\nu(\alpha) = \ker(\alpha')$, which is a maximal ideal of $\mathbb{C}[T]$. Since the domain and codomain of ν both take products of groups to products of varieties, it suffices to check that ν is an isomorphism when $T = S^1$ or $T = \mathbb{Z}_n$, which we leave to the reader.

For the second statement, notice that when T is a torus, there is a natural isomorphism $\hat{T} \xrightarrow{\sim} \Lambda_T^*$. □

Suppose $T \cong (S^1)^p \times G$, where G is a finite abelian group. Then one can apply the previous proposition to obtain that $\mathcal{C}_T \cong \mathcal{C}^p \times G$. This formula shows that $\mathcal{C}_T$ is in fact an algebraic group (and it is the complexification of T). Just as we did for $\mathcal{C}$, in the rest of the paper we are going to use *additive* notation for $\mathcal{C}_T$.

Definition 2.2. *Let $t_{\mathbb{C}}$ be the complexification of the Lie algebra of T. Then the exponential map* $\exp : \mathbb{C} \to \mathcal{C}$ *extends to a complex algebraic group map*

$$\exp : t_{\mathbb{C}} \to \mathcal{C}_T.$$

To be more precise, this map is the composite map $t_{\mathbb{C}} = \Lambda_T \otimes_{\mathbb{Z}} \mathbb{C} \to \Lambda_T \otimes_{\mathbb{Z}} \mathcal{C} = \mathcal{C}_{T^0} \to \mathcal{C}_T$, where T^0 is the connected component of T containing the identity. (For the identification $\Lambda_T \otimes_Z \mathcal{C} = \mathcal{C}_{T^0}$, use the previous proposition.) Note that when T is connected, the exponential map is surjective.

Definition 2.3. *We call a neighborhood U of zero in $\mathcal{C}_T$ "small" if the above defined exponential map, $\exp : t_{\mathbb{C}} \to \mathcal{C}_T$, has a local inverse on U. We call a neigborhood V of zero in $t_{\mathbb{C}}$ "small" if it is of the form $\exp^{-1}(U)$, for U a small neighborhood of zero in $\mathcal{C}_T$.*

Let $\mathcal{A}$ be a collection of compact subgroups of T. Define a relation on $\mathcal{C}_T$ as follows: $\alpha \prec_{\mathcal{A}} \beta$ if, for any $H \in \mathcal{A}$, $\beta \in \mathcal{C}_H$ implies $\alpha \in \mathcal{C}_H$. The relation $\prec_{\mathcal{A}}$ is reflexive and transitive, but not antisymmetric, so it is not an order relation. When it is clear what $\mathcal{A}$ is, we will omit it and write simply $\alpha \prec \beta$. The next definition singles out a special class of open covers of $\mathcal{C}_T$, called adapted covers. These will be used below in the definition of the sheaf $\mathcal{K}_T^*(X)$.

Definition 2.4. *Let $\mathcal{A}$ be a collection of compact subgroups of T, and let $\mathcal{U} = (U_\alpha)_{\alpha \in \mathcal{C}_T}$ be an open cover indexed by the points of $\mathcal{C}_T$. Then $\mathcal{U}$ is called "adapted to $\mathcal{A}$" if it satisfies the following conditions:*

1. $\alpha \in U_\alpha$, *and $U_\alpha - \alpha$ is small.*
2. *If $U_\alpha \cap U_\beta \neq \emptyset$, then either $\alpha \prec \beta$ or $\beta \prec \alpha$.*
3. *If $\alpha \prec \beta$, and for some $H \in \mathcal{A}$ $\alpha \in \mathcal{C}_H$ but $\beta \notin \mathcal{C}_H$, then $U_\beta \cap \mathcal{C}_H = \emptyset$.*

4. If $U_\alpha \cap U_\beta \neq \emptyset$, and both α and β belong to $\mathcal{C}_H$ for some $H \in \mathcal{A}$, then α and β belong to the same connected component of $\mathcal{C}_H$.

Proposition 2.5. *If $\mathcal{A}$ is a finite collection of compact subgroups of T, then there exists a cover $\mathcal{U}$ of $\mathcal{C}_T$ adapted to $\mathcal{A}$. Any refinement of $\mathcal{U}$ is still adapted.*

Proof. Define $\mathcal{H} = \{\mathcal{C}_H \mid H \in \mathcal{A}\}$, and $\mathcal{H}^0 =$ the set of all connected components of the elements in $\mathcal{H}$. Put a metric on $\mathcal{C}_T$ which yields its usual topology. Denote this metric by "dist".

Let $\alpha \in \mathcal{C}_T$. If $\alpha \in C$ for all $C \in \mathcal{H}^0$ (this is possible only when T is connected), then choose U_α such that $\alpha \in U_\alpha$, and $U_\alpha - \alpha$ is small. If, on the contrary, there exists a connected component $C \in \mathcal{H}^0$ such that $\alpha \notin C$, then take U_α a ball of center α and radius d, with

$$d < \tfrac{1}{2} \min_{D \in \mathcal{H}^0, \alpha \notin D} \operatorname{dist}(\alpha, D),$$

and such that $U_\alpha - \alpha$ is small.

We show that $\mathcal{U} = (U_\alpha)_{\alpha \in \mathcal{C}_T}$ is adapted: Condition 1 is trivially satisfied. To prove Condition 2, let α and β be such that $U_\alpha \cap U_\beta \neq \emptyset$. Suppose we have neither $\alpha \prec \beta$, nor $\beta \prec \alpha$. Then by the definition of $\prec$ there exist two compact subgroups K and L of T such that $\alpha \in \mathcal{C}_K \setminus \mathcal{C}_L$ and $\beta \in \mathcal{C}_L \setminus \mathcal{C}_K$. But from the definition of U_α it follows that U_α is a ball of center α and radius $d < \tfrac{1}{2} \operatorname{dist}(\alpha, \mathcal{C}_L) \leq \tfrac{1}{2} \operatorname{dist}(\alpha, \beta)$. Similarly, U_β is a ball of center β and radius less than $\tfrac{1}{2} \operatorname{dist}(\alpha, \beta)$, so U_α and U_β cannot possibly intersect, contradiction.

Condition 3 is obviously satisfied, by construction.

Finally, to show Condition 4, let $\alpha, \beta \in \mathcal{C}_H$ be such that $U_\alpha \cap U_\beta \neq \emptyset$. Suppose α and β belong to different connected components of $\mathcal{C}_H$. Then by the same type of reasoning as above, it follows that the radii of U_α and U_β are smaller than $\tfrac{1}{2} \operatorname{dist}(\alpha, \beta)$, so U_α and U_β cannot possibly intersect, which again leads to a contradiction. $\qquad\square$

Let $\alpha \in \mathcal{C}_T$. The construction of $\mathcal{C}_T$ is functorial, so if H is any compact subgroup of T, we get an inclusion map $\mathcal{C}_H \to \mathcal{C}_T$. If $\alpha \in \operatorname{Im}(\mathcal{C}_H \to \mathcal{C}_T)$, we say that $\alpha \in \mathcal{C}_H$. For $\alpha \in \mathcal{C}_H$, denote by $H(\alpha)$ the smallest compact subgroup H of T such that $\alpha \in \mathcal{C}_H$. The fact that there exists a smallest H such that $\alpha \in \mathcal{C}_H$ is implied by the formula $\mathcal{C}_K \cap \mathcal{C}_L = \mathcal{C}_{K \cap L}$, which follows from an easy diagram chase. This also implies that $H(\alpha)$ is the intersection of all compact subgroups H such that $\alpha \in \mathcal{C}_H$. Another immediate consequence is the following formula, which will be useful later:

(1) $H(\alpha) \subseteq K \iff \alpha \in \mathcal{C}_K.$

Let X be a space with a T-action. If K is a compact subgroup of T, denote by $X^K \subseteq X$ the subspace of points fixed by K. Also, if $\alpha \in \mathcal{C}_T$, define
$$X^\alpha = X^{H(\alpha)}.$$
Now we want to define the notion of a cover adapted to a finite T-CW complex. So let X be a finite T-CW complex. We know that there exists a finite collection $\mathcal{A} = (H_i)_i$ of compact subgroups of T such that X has an equivariant cell decomposition of the form $X = \bigcup_i \mathbb{D}^{n_i} \times (T/H_i)$. Here we denoted by $\mathbb{D}^n$ the open disk in dimension n, and by $\mathbb{D}^0$ a point. The group T acts trivially on $\mathbb{D}^{n_i}$, and by left multiplication on T/H_i. Notice that if K is a compact subgroup of T, then the subcomplex of X fixed by K has a decomposition of the form $X^K = \bigcup_{i:K \subseteq H_i} \mathbb{D}^{n_i} \times (T/H_i)$. Taking $K = H(\alpha)$, we get

$$(2) \qquad X^\alpha = \bigcup_{i:\alpha \in \mathcal{C}_{H_i}} \mathbb{D}^{n_i} \times (T/H_i) \ .$$

We say that the cover $\mathcal{U} = (U_\alpha)_{\alpha \in \mathcal{C}_T}$ of $\mathcal{C}_T$ is "adapted to X" if $\mathcal{U}$ is adapted to the collection $\mathcal{A}$ of the isotropy groups H_i appearing in a T-equivariant cell decomposition of X. Since this collection is finite, Proposition 2.5 implies that there always exists a cover adapted to X.

Next we discuss a few useful results in equivariant cohomology. We start with a well-known proposition which says that the ring of coefficients of (complex) T-equivariant cohomology is the polynomial algebra on $t_\mathbb{C}$, the complex Lie algebra of T.

Proposition 2.6. *If T is an abelian compact Lie group, there is a natural isomorphism*
$$S(t_\mathbb{C}^*) \xrightarrow{\sim} H_T^*,$$
where $S(-)$ denotes the symmetric algebra, and $t_\mathbb{C}^*$ is the dual of $t_\mathbb{C}$.

Proof. $\hat{T} = \mathrm{Hom}(T, S^1)$ is the group of irreducible characters of T, so for $\lambda \in \hat{T}$ consider the complex vector bundle
$$V_\lambda = ET \times_T \mathbb{C},$$
over the classifying space BT, where the map $T \to \mathbb{C}$ is given by λ. Then the first Chern class of V_λ gives a natural isomorphism $c_1 : \hat{T} \to H^2(BT, \mathbb{Z})$.

If T is a torus, we saw in Proposition 2.1 that $\hat{T}$ can be identified with the dual of the integer lattice Λ_T^*, so by tensoring the map c_1 with $\mathbb{C}$, we get the natural isomorphism $c_1 : t_\mathbb{C}^* = \Lambda_T^* \otimes \mathbb{C} \to H^2(BT, \mathbb{C})$. Taking symmetric products, we get the desired isomorphism.

If T is a general (non-connected) compact abelian Lie group, the isomorphism still holds, since both domain and codomain depend only on the connected component of T containing the identity, which is a torus. $\qquad \square$

We now define an algebra homomorphism

$$h : H_T^* \to \mathcal{O}_{t_\mathbb{C},0}^h$$

by taking a polynomial in $H_T^* = S(t_\mathbb{C}^*)$ and sending it to its germ at zero. The map is injective, so we can consider H_T^* as a subring of $\mathcal{O}_{t_\mathbb{C},0}^h$. Let V be a small neighborhood of zero in $t_\mathbb{C}$. Then, since the ring $H_T^* \subset \mathcal{O}_{t_\mathbb{C},0}^h$ consists of the germs of global holomorphic functions on $t_\mathbb{C}$, the map h factors through the inclusion $\mathcal{O}_{t_\mathbb{C}}^h(V) \hookrightarrow \mathcal{O}_{t_\mathbb{C},0}^h$, so we can define a map, also denoted by h,

$$h : H_T^* \to \mathcal{O}_{t_\mathbb{C}}^h(V).$$

Let U be a small neighborhood of zero in $\mathcal{C}_T$, and let $V = \exp^{-1}(U)$, where $\exp : t_\mathbb{C} \to \mathcal{C}_T$ is the exponential map. Via the exponential, we have the following identifications: $\mathcal{O}_{\mathcal{C}_T,0}^h \xrightarrow{\sim} \mathcal{O}_{t_\mathbb{C},0}^h$ and $\mathcal{O}_{\mathcal{C}_T}^h(U) \xrightarrow{\sim} \mathcal{O}_{\mathcal{C}_T}^h(V)$.

Definition 2.7. *Let U be a small neighborhood of zero in $\mathcal{C}_T$. Via the identifications above, we define the following two maps, and denote them also by h (the second one is the correstriction of the first):*

$$h : H_T^* \to \mathcal{O}_{\mathcal{C}_T,0}^h \quad \text{and } h : H_T^* \to \mathcal{O}_{\mathcal{C}_T}^h(U).$$

Now we define a few cohomology theories that we are going to use throughout the paper. Let X be a finite T-CW complex. We define the holomorphic T-equivariant cohomology of X to be

$$\mathbf{H}_T^*(X) = H_T^*(X) \otimes_{H_T^*} \mathcal{O}_{\mathcal{C}_T,0}^h,$$

where the map $h : H_T^* \to \mathcal{O}_{\mathcal{C}_T,0}^h$ is given in Definition 2.7. It is indeed a cohomology theory, because by Proposition 2.8 the map $H_T^* \to \mathcal{O}_{\mathcal{C}_T,0}^h$ is flat. The theory is not $\mathbb{Z}$-graded anymore; however, it can be thought of as $\mathbb{Z}/2$-graded, by its even and odd part.

Let $H_T^{**}(X)$ be the completion of $H_T^*(X)$ with respect to the augmentation ideal $I = \ker(H_T^* \to \mathbb{C})$. Since $H_T^*(X)$ is a finitely generated module over the Noetherian ring H_T^*, a simple result on completions (see for example Matsumura [14], Theorem 55) implies that

$$H_T^{**}(X) \cong H_T^*(X) \otimes_{H_T^*} H_T^{**}.$$

The ring H_T^* is a polynomial ring, so we have the following well-known results from algebra (they are sometimes called *GAGA* results, since they first appeared in Serre's *GAGA* [17]).

Proposition 2.8. $\mathbf{H}_T^* = \mathcal{O}_{\mathcal{C}_T,0}^h$ *and* H_T^{**} *are flat over* H_T^*. *If U is a small neighborhood of zero, then* $\mathcal{O}_{\mathcal{C}_T}^h(U)$ *is flat over* H_T^*.

Proof. All the results we mention in this proof are from Matsumura [14]. Identify the complex Lie algebra $t_{\mathbb{C}}$ with $X = \mathbb{C}^n$. We denote by $\mathcal{O}$ the algebraic structure sheaf of $\mathbb{C}^n$, and by $\mathcal{O}^h$ the analytic structure sheaf. Let $\mathcal{O}_0^{\wedge}$ be the completion of the local ring $\mathcal{O}_0$ with respect to its maximal ideal. It is sufficient to show that the natural maps $\mathcal{O}(X) \to \mathcal{O}_0^h$ and $\mathcal{O}(X) \to \mathcal{O}_0^{\wedge}$ are flat, and that for any U open in X, the map $\mathcal{O}(X) \to \mathcal{O}^h(U)$ is flat. We start by noticing that $\mathcal{O}_0^{\wedge}$ is the completion of $\mathcal{O}(X)$ with respect to its maximal ideal at zero, so by Corollary 1 of Theorem 55, we know that $\mathcal{O}_0^{\wedge}$ is flat over $\mathcal{O}(X)$. The completion of $\mathcal{O}_0^h$ with respect to its maximal ideal is also $\mathcal{O}_0^{\wedge}$, so $\mathcal{O}_0^h \to \mathcal{O}_0^{\wedge}$ is flat. It is in fact faithfully flat, because it is local: see Theorem 3 (4.D). Now one can check directly by the definition of flatness that having $\mathcal{O}(X) \to \mathcal{O}_0^{\wedge}$ flat and $\mathcal{O}_0^h \to \mathcal{O}_0^{\wedge}$ faithfully flat implies that $\mathcal{O}(X) \to \mathcal{O}_0^h$ is flat. Notice also that the same proof can be used to show that $\mathcal{O}_0 \to \mathcal{O}_0^h$ is flat.

Next let $U \subseteq X$ be an open set. By the local characterization of flatness, Theorem (3.J), in order to show that $\mathcal{O}(X) \to \mathcal{O}^h(U)$ is flat, we have to show that for any $x \in U$ the natural map $\mathcal{O}_x \to \mathcal{O}_x^h$ is flat. But we have already shown this when $x = 0$, and the proof for general x is the same.

Now in order to prove the proposition, just transfer the results we have proved via the exponential map, $\exp : \mathbb{C}^n = t_{\mathbb{C}} \to \mathcal{C}_T$. This is where we need U small. $\qquad \square$

In particular, it follows that $H_T^*(X)$ and $\mathbf{H}_T^*(X)$ can be regarded as subrings of $H_T^{**}(X)$.

2.2. Construction of $\mathcal{K}_T^*$

Let X be a finite T-CW complex. Fix $\mathcal{U}$ a cover adapted to X, which exists because of Proposition 2.5. We are going to define a sheaf $\mathcal{F} = \mathcal{F}_{\mathcal{U}}$ over $\mathcal{C}_T$ whose stalk at $\alpha \in \mathcal{C}_T$ is isomorphic to $\mathbf{H}_T^*(X^\alpha)$. Recall that in order to give a sheaf $\mathcal{F}$ over a topological space, it is enough to give an open cover $(U_\alpha)_\alpha$ of that space, and a sheaf $\mathcal{F}_\alpha$ on each U_α together with isomorphisms of sheaves $\phi_{\alpha\beta} : \mathcal{F}_\alpha|_{U_\alpha \cap U_\beta} \longrightarrow \mathcal{F}_\beta|_{U_\alpha \cap U_\beta}$, such that $\phi_{\alpha\alpha}$ is the identity function, and the cocycle condition $\phi_{\beta\gamma}\phi_{\alpha\beta} = \phi_{\alpha\gamma}$ is satisfied on $U_\alpha \cap U_\beta \cap U_\gamma$. If U is a subset of $\mathcal{C}_T$, denote by $U + \alpha = \{x + \alpha \mid x \in U\}$ the translation of U by α.

Definition 2.9. *Define a presheaf $\mathcal{F}_\alpha$ on U_α by taking, for any open $U \subseteq U_\alpha$,*

$$\mathcal{F}_\alpha(U) = H_T^*(X^\alpha) \otimes_{H_T^*} \mathcal{O}_{\mathcal{C}_T}^h(U - \alpha),$$

where the restriction maps are induced from those of $\mathcal{O}_{\mathcal{C}_T}^h$. The ring map $h : H_T^* \to \mathcal{O}_{\mathcal{C}_T}^h(U - \alpha)$ is given in Definition 2.7.

Proposition 2.10. $\mathcal{F}_\alpha$ *is a coherent sheaf of* $\mathcal{O}^h_{\mathcal{C}_T}$*-modules.*

Proof. First we show that $\mathcal{F}_\alpha$ is a sheaf of H^*_T-modules. If $(U_i)_i$ is an open cover of a topological space Y, denote by $U_{ij} = U_i \cap U_j$, etc. Then a presheaf $\mathcal{G}$ is a sheaf if and only if for any $m > 0$ and any finite cover $(U_i)_{i=1\ldots m}$ the following sequence is exact

$$0 \longrightarrow \mathcal{G}(Y) \xrightarrow{r_0} \prod_i \mathcal{G}(U_i) \xrightarrow{r_1} \prod_{i<j} \mathcal{G}(U_{ij}) \xrightarrow{r_2} \cdots \longrightarrow \mathcal{G}(U_{1\ldots m}) \longrightarrow 0,$$

where $r_0(s)_i = s_{|U_i}$, $r_1\left(\prod_i s_i\right)_{i,j} = s_{i\,|U_{ij}} - s_{j\,|U_{ij}}$, etc. Denote this sequence by $M(\mathcal{G})^\bullet$. Since $\mathcal{O}^h_{\mathcal{C}_T}$ is a sheaf of H^*_T-modules on $Y = U_\alpha$, the sequence $M(\mathcal{O}^h_{\mathcal{C}_T})^\bullet$ is exact. Now, by Proposition 2.8, each H^*_T-module in the sequence $M(\mathcal{O}^h_{\mathcal{C}_T})^\bullet$ is flat. Therefore, $M(\mathcal{O}^h_{\mathcal{C}_T})^\bullet$ can be thought as a flat resolution of the last term to the right, $\mathcal{O}^h_{\mathcal{C}_T}(U_{1\ldots m})$. Using the Flat Resolution Lemma (see Weibel [19], Lemma 3.2.8), one can easily show that by tensoring $M(\mathcal{O}^h_{\mathcal{C}_T})^\bullet$ with any H^*_T-module, we also get an exact sequence. In particular, by tensoring with $H^*_T(X^\alpha)$ we obtain $M(\mathcal{F}_\alpha)^\bullet$, which therefore is exact. But that means that $\mathcal{F}_\alpha$ is a sheaf (of H^*_T-modules).

To show that $\mathcal{F}_\alpha$ is a sheaf of $\mathcal{O}^h_{\mathcal{C}_T}$-modules, if $U \subseteq U_\alpha$, we need an action of $f \in \mathcal{O}^h_{\mathcal{C}_T}(U)$ on $\mathcal{F}_\alpha(U)$. The translation map $t_\alpha : U - \alpha \to U$, which takes x to $x + \alpha$ gives a map $t^*_\alpha : \mathcal{O}^h_{\mathcal{C}_T}(U) \to \mathcal{O}^h_{\mathcal{C}_T}(U - \alpha)$, which sends $f(u)$ to $f(u+\alpha)$. Then we take the result of the action of $f \in \mathcal{O}^h_{\mathcal{C}_T}(U)$ on $\mu \otimes g \in \mathcal{F}_\alpha(U) = H^*_T(X^\alpha) \otimes_{H^*_T} \mathcal{O}^h_{\mathcal{C}_T}(U - \alpha)$ to be $\mu \otimes (t^*_\alpha f \cdot g)$. It is an easy exercise to show that $\mathcal{F}_\alpha$ is coherent: one uses that $H^*_T(X^\alpha)$ is a finitely generated H^*_T-module. $\square$

Next, for the second part of the definition of $\mathcal{F}$, we glue the different sheaves $\mathcal{F}_\alpha$. Let $\alpha, \beta \in \mathcal{C}_T$ such that $U_\alpha \cap U_\beta \neq \emptyset$. Since the cover $\mathcal{U}$ is adapted to $\mathcal{A}$, Condition 2 from Definition 2.4 implies that either $\alpha \prec \beta$ or $\beta \prec \alpha$. Without loss of generality we can assume that $\alpha \prec \beta$. Then it is clear that $H(\alpha) \subseteq H(\beta)$, and hence $X^\beta \subseteq X^\alpha$. Denote by $i : X^\beta \to X^\alpha$ the inclusion map. Now we want to investigate conditions in which the restriction map $i^* : H^*_T(X^\alpha) \to H^*_T(X^\beta)$ becomes an isomorphism. The next proposition is esentially the Localization Theorem of Borel and Hsiang (see Atiyah and Bott [1]).

Proposition 2.11. *Let X be a finite T-CW complex, and $\mathcal{U} = (U_\alpha)_{\alpha\in\mathcal{C}_T}$ a cover adapted to X. Let $\alpha \prec \beta$ be two points in $\mathcal{C}_T$. Let $U \subseteq U_\alpha \cap U_\beta$, and assume $U_\alpha \cap U_\beta$ nonempty. Then the restriction map*

$$i^* \otimes 1 : H^*_T(X^\alpha) \otimes_{H^*_T} \mathcal{O}^h_{\mathcal{C}_T}(U - \alpha) \to H^*_T(X^\beta) \otimes_{H^*_T} \mathcal{O}^h_{\mathcal{C}_T}(U - \alpha)$$

is an isomorphism.

Proof. The case when $X^\alpha = X^\beta$ is trivial, so we assume the contrary. Then there exists a compact subgroup L of T such that $\alpha \in \mathcal{C}_L$ and $\beta \notin \mathcal{C}_L$. Then Condition 3 from Definition 2.4 implies that $U_\beta \cap \mathcal{C}_L = \emptyset$. But $\mathcal{C}_L = \alpha + \mathcal{C}_L$, so we deduce that $(U - \alpha) \cap \mathcal{C}_L = \emptyset$. This means that for every $x \in U - \alpha$ we have $x \notin \mathcal{C}_L$.

Now we will prove that whenever $x \notin \mathcal{C}_L$, i^* becomes an isomorphism if tensored with $\mathcal{O}^h_{\mathcal{C}_T, x}$. This is enough for us, because we can regard the statement to be proved as a statement about sheaves, and we are proving the isomorphism stalkwise.

Now notice that Equation (2) implies that X^α is built from X^β by adding cells of the form $\mathbb{D}^n \times (T/L)$, with $\alpha \in \mathcal{C}_L$ and $\beta \notin \mathcal{C}_L$. So it is enough to show that, when tensored with $\mathcal{O}^h_{\mathcal{C}_T, x}$ over H^*_T, the ring $H^*_T\big(\mathbb{D}^n \times (T/L), S^{n-1} \times (T/L)\big)$ becomes zero. Or, equivalently, it is enough to have $H^*_T(T/L) = H^*_L$ become zero after tensoring with $\mathcal{O}^h_{\mathcal{C}_T, x}$. We therefore proceed to show that $H^*_L \otimes_{H^*_T} \mathcal{O}^h_{\mathcal{C}_T, x}$ is the zero algebra.

Via the correspondence $\log : t_\mathbb{C} \to \mathcal{C}_T$, it is enough to show that if $l_\mathbb{C}$ is the complexification of the Lie algebra of L, and $x \notin l_\mathbb{C}$, then $H^*_L \otimes_{H^*_T} \mathcal{O}^h_{t_\mathbb{C}, x} = 0$. Denote by I_L the kernel of the surjective map of algebras $H^*_T \to H^*_L$. It is clear that I_L kills H^*_L as an H^*_T-module. The vector space $l_\mathbb{C}$ is the vanishing set of I_L, so since $x \notin l_\mathbb{C}$, there exists a polynomial p in I_L such that $p(x) \neq 0$. This means that p is invertible in $\mathcal{O}^h_{t_\mathbb{C}, x}$, hence $1 \otimes p$ is invertible in $H^*_L \otimes_{H^*_T} \mathcal{O}^h_{t_\mathbb{C}, x}$. By the balancing property of the tensor product, we know that $1 \otimes p = 0$, because p goes to zero via the map $H^*_T \to H^*_L$. So we found an element in $H^*_L \otimes_{H^*_T} \mathcal{O}^h_{t_\mathbb{C}, x}$ which is invertible and zero at the same time. This can only happen if $H^*_L \otimes_{H^*_T} \mathcal{O}^h_{t_\mathbb{C}, x}$ is the zero algebra. $\square$

Let us now consider the translation map $t_{\beta - \alpha} : U - \beta \to U - \alpha$. This induces a map $t^*_{\beta - \alpha} : \mathcal{O}^h_{\mathcal{C}_T}(U - \alpha) \to \mathcal{O}^h_{\mathcal{C}_T}(U - \beta)$. Eventually we want to produce a translation map $H^*_T(X^\beta) \otimes_{H^*_T} \mathcal{O}^h_{\mathcal{C}_T}(U - \alpha) \to H^*_T(X^\beta) \otimes_{H^*_T} \mathcal{O}^h_{\mathcal{C}_T}(U - \beta)$. But the problem is that $t^*_{\beta - \alpha}$ is not a map of H^*_T-modules, so we cannot simply tensor it with the identity map of $H^*_T(X^\beta)$. However, we hope to find some other group K so that $t^*_{\beta - \alpha}$ is a map of H^*_K-modules.

Lemma 2.12. *Let X be a finite T-CW complex, and $\mathcal{U} = (U_\alpha)_{\alpha \in \mathcal{C}_T}$ a cover adapted to X. Let $\alpha \prec \beta$ be two points in $\mathcal{C}_T$ and $U \subseteq U_\alpha \cap U_\beta$, and assume $U_\alpha \cap U_\beta$ nonempty. Let $H = H(\beta)$ and $K = T/H$. Then the translation map*

$$t^*_{\beta - \alpha} : \mathcal{O}^h_{\mathcal{C}_T}(U - \alpha) \to \mathcal{O}^h_{\mathcal{C}_T}(U - \beta)$$

is a map of H^*_K-algebras. Here the H^*_T-algebras $\mathcal{O}^h_{\mathcal{C}_T}(U - \alpha)$ and $\mathcal{O}^h_{\mathcal{C}_T}(U - \beta)$ are regarded as H^*_K-algebras via the natural algebra map $H^*_K \to H^*_T$ induced by the quotient map $T \to K$.

Proof. Since $\beta \in \mathcal{C}_H$, it follows that $\alpha \in \mathcal{C}_H$. Let H^0 be the connected component of H containing the identity. Because both α and β belong to $\mathcal{C}_H$, Condition 4 from Definition 2.4 implies that α and β belong to the same connected component of $\mathcal{C}_H$, or equivalently that $\beta - \alpha \in \mathcal{C}_{H^0}$. Let $h_{\mathbb{C}}$ be the complexification of the Lie algebra of H. The exponential map $\exp : h_{\mathbb{C}} \to \mathcal{C}_{H^0}$ is a surjective group homomorphism, because H^0 is a torus. Consider an element in the coset $\exp^{-1}(\beta - \alpha) \subset h_{\mathbb{C}}$. Denote it also by $\beta - \alpha$. Let Γ be the global sections functor. Using the exponential map, it is enough to show that

$$t_{\beta-\alpha}^* : \Gamma \mathcal{O}_{t_{\mathbb{C}}}^h \to \Gamma \mathcal{O}_{t_{\mathbb{C}}}^h$$

is a map of H_K^*-modules. This is equivalent to showing that $t_{\beta-\alpha}^*(p) = p$ for all $p \in H_K^*$. But this is easy, since we can identify H_K^* with the ring of polynomial functions $f(u)$ in H_T^* such that $f(u + a) = f(u)$ for any a in the complexification of the Lie algebra of K. $\qquad \square$

Here is a well-known result in equivariant cohomology theory (see for example Atiyah and Bott [1] for ordinary cohomology and Segal [16] for K-theory).

Proposition 2.13. *Let $h_T^*(-)$ be either ordinary equivariant cohomology, or equivariant K-theory. Let Z be a finite T-CW complex which is fixed by a compact subgroup H of T. Let $K = T/H$. Then there exists a natural isomorphism*

$$h_K^*(Z) \otimes_{h_K^*} h_T^* \xrightarrow{\sim} h_T^*(Z) \ .$$

The last step in the construction of the gluing maps $\phi_{\alpha\beta}$ is the following proposition:

Proposition 2.14. *Under the hypotheses of Lemma 2.12, the translation map $t_{\beta-\alpha}^*$ extends to a natural isomorphism*

$$\tau_{\beta-\alpha}^* : H_T^*(X^\beta) \otimes_{H_T^*} \mathcal{O}_{\mathcal{C}_T}^h(U - \alpha) \to H_T^*(X^\beta) \otimes_{H_T^*} \mathcal{O}_{\mathcal{C}_T}^h(U - \beta).$$

Proof. X^β is fixed by $H = H(\beta)$, so by Proposition 2.13 we know that $H_T^*(X^\beta) \otimes_{H_T^*} \mathcal{O}_{\mathcal{C}_T}^h(U-\alpha) \cong H_K^*(X^\beta) \otimes_{H_K^*} \mathcal{O}_{\mathcal{C}_T}^h(U-\alpha)$, with $K = T/H$. Therefore it is enough to define a map

$$\phi : H_K^*(X^\beta) \otimes_{H_K^*} \mathcal{O}_{\mathcal{C}_T}^h(U - \alpha) \to H_K^*(X^\beta) \otimes_{H_K^*} \mathcal{O}_{\mathcal{C}_T}^h(U - \beta).$$

But $t_{\beta-\alpha}^*$ was shown in Lemma 2.12 to be a map of H_K^*-algebras, so we can just take $\phi = 1 \otimes t_{\beta-\alpha}^*$.

More formally, we define $\tau_{\beta-\alpha}^*$ as the composition of the following isomorphisms: $H_T^*(X^\beta) \otimes_{H_T^*} \mathcal{O}_{\mathcal{C}_T}^h(U - \alpha) \to H_K^*(X^\beta) \otimes_{H_K^*} \mathcal{O}_{\mathcal{C}_T}^h(U - \alpha) \xrightarrow{1 \otimes t_{\beta-\alpha}^*} H_K^*(X^\beta) \otimes_{H_K^*} \mathcal{O}_{\mathcal{C}_T}^h(U - \beta) \to H_T^*(X^\beta) \otimes_{H_T^*} \mathcal{O}_{\mathcal{C}_T}^h(U - \beta).$ $\square$

Definition 2.15. *Under the same hypotheses as Lemma 2.12, define $\phi_{\alpha\beta}$:* $\mathcal{F}_\alpha(U) \to \mathcal{F}_\beta(U)$ *as the composite map* $H_T^*(X^\alpha) \otimes_{H_T^*} \mathcal{O}_{\mathcal{C}_T}^h(U - \alpha) \xrightarrow{i^* \otimes 1}$ $H_T^*(X^\beta) \otimes_{H_T^*} \mathcal{O}_{\mathcal{C}_T}^h(U - \alpha) \xrightarrow{\tau_{\beta-\alpha}^*} H_T^*(X^\beta) \otimes_{H_T^*} \mathcal{O}_{\mathcal{C}_T}^h(U - \beta)$, *where the second map was defined above.*

Proposition 2.16. $\phi_{\alpha\beta}$ *is an isomorphism.*

Proof. The first map is an isomorphism by Proposition 2.11, and the second map is an isomorphism by Proposition 2.14. $\qquad\square$

Proposition 2.17. *The maps* $\phi_{\alpha\beta}$ *verify the cocycle condition, i.e.* $\phi_{\beta\gamma} \circ \phi_{\alpha\beta} = \phi_{\alpha\gamma}$, *on every triple intersection* $U_\alpha \cap U_\beta \cap U_\gamma \neq \emptyset$.

Proof. Without loss of generality, we can assume $\alpha \prec \beta \prec \gamma$. We need to show that the following diagram is commutative:

$$
\begin{array}{c}
H_T^*(X^\alpha) \otimes_{H_T^*} \mathcal{O}_{\mathcal{C}_T}^h(U - \alpha) \\
\quad\downarrow{\scriptstyle i^* \otimes 1} \\
H_T^*(X^\beta) \otimes_{H_T^*} \mathcal{O}_{\mathcal{C}_T}^h(U - \alpha) \\
\quad\downarrow{\scriptstyle \tau_{\beta-\alpha}^*} \qquad\qquad \searrow^{i^* \otimes 1} \\
H_T^*(X^\beta) \otimes_{H_T^*} \mathcal{O}_{\mathcal{C}_T}^h(U - \beta) \qquad H_T^*(X^\gamma) \otimes_{H_T^*} \mathcal{O}_{\mathcal{C}_T}^h(U - \alpha) \\
\quad\downarrow{\scriptstyle i^* \otimes 1} \qquad\qquad \swarrow_{\tau_{\beta-\alpha}^*} \\
H_T^*(X^\gamma) \otimes_{H_T^*} \mathcal{O}_{\mathcal{C}_T}^h(U - \beta) \\
\quad\downarrow{\scriptstyle \tau_{\gamma-\beta}^*} \\
H_T^*(X^\gamma) \otimes_{H_T^*} \mathcal{O}_{\mathcal{C}_T}^h(U - \gamma)
\end{array}
$$

In this diagram the only map which has not been defined is $\tau_{\beta-\alpha}^* : H_T^*(X^\gamma)$ $\otimes_{H_T^*} \mathcal{O}_{\mathcal{C}_T}^h(U - \alpha) \to H_T^*(X^\gamma) \otimes_{H_T^*} \mathcal{O}_{\mathcal{C}_T}^h(U - \beta)$. But since $\beta \prec \gamma$ it follows that $H(\beta)$ fixes X^γ, so we can replace X^β by X^γ in Corollary 2.14, and obtain a map which we also denote by $\tau_{\beta-\alpha}^*$.

Since $H = H(\beta)$ fixes X^β and X^γ, we can use Proposition 2.13 to replace (to pick an example) $H_T^*(X^\beta) \otimes_{H_T^*} \mathcal{O}_{\mathcal{C}_T}^h(U - \beta)$ by $H_K^*(X^\beta) \otimes_{H_K^*} \mathcal{O}_{\mathcal{C}_T}^h(U - \beta)$, where $K = T/H$. We have to show that the following diagram

is commutative:

$$H_K^*(X^\beta) \otimes_{H_K^*} \mathcal{O}_{\mathcal{C}_T}^h(U - \alpha)$$

$$\downarrow 1 \otimes t_{\beta-\alpha}^* \qquad \qquad \xrightarrow{\quad i^* \otimes 1 \quad}$$

$$H_K^*(X^\beta) \otimes_{H_K^*} \mathcal{O}_{\mathcal{C}_T}^h(U - \beta) \qquad \qquad H_K^*(X^\gamma) \otimes_{H_K^*} \mathcal{O}_{\mathcal{C}_T}^h(U - \alpha)$$

$$\downarrow i^* \otimes 1 \qquad \qquad \swarrow 1 \otimes t_{\beta-\alpha}^*$$

$$H_K^*(X^\gamma) \otimes_{H_K^*} \mathcal{O}_{\mathcal{C}_T}^h(U - \beta)$$

Now observe that both composites are equal to $i^* \otimes t_{\beta-\alpha}^*$. $\qquad\qquad \square$

Definition 2.18. *Let X be a finite T-CW complex, and $\mathcal{U} = (U_\alpha)_{\alpha \in \mathcal{C}_T}$ an adapted cover of $\mathcal{C}_T$. We then define a sheaf $\mathcal{F} = \mathcal{K}_T^*(X)$ on $\mathcal{C}_T$ by gluing the sheaves $\mathcal{F}_\alpha$ defined in 2.9 via the gluing maps $\phi_{\alpha\beta}$ defined in 2.15.*

Notice that we can remove the dependence of $\mathcal{K}_T^*(X)$ on the cover $\mathcal{U}$ as follows: Let $\mathcal{U}$ and $\mathcal{V}$ be two covers adapted to X. Then any common refinement $\mathcal{W}$ is going to be adapted as well, and the corresponding maps of sheaves $\mathcal{F}_{\mathcal{U}} \to \mathcal{F}_{\mathcal{W}} \leftarrow \mathcal{F}_{\mathcal{V}}$ are isomorphisms on stalks, hence isomorphisms of sheaves. Therefore we can omit the subscript $\mathcal{U}$.

Theorem 2.19. $\mathcal{K}_T^*(-)$ *is a T-equivariant cohomology theory on finite complexes, with values in the category of coherent holomorphic sheaves of $\mathbb{Z}/2$-graded $\mathcal{O}_{\mathcal{C}_T}^h$-algebras.*

Proof. We start by showing that the construction of $\mathcal{K}_T^*(X)$ is natural. Let X and Y be two finite T-CW complexes, and $f : X \to Y$ a T-equivariant map between them. We want to define a map of sheaves $f^* : \mathcal{K}_T^*(Y) \to \mathcal{K}_T^*(X)$ with the properties that $1_X^* = 1_{\mathcal{K}_T^*(X)}$ and $(fg)^* = g^* f^*$. Consider two T-cell decompositions of X and Y, and let $\mathcal{A}$ be the collection of all compact subgroups L of T such that $\mathbb{D}^n \times (T/L)$ appears in the cell decomposition of either X or Y. Let $\mathcal{U}$ be a cover adapted to $\mathcal{A}$. Since f is T-equivariant, for each α we get by restriction a map $f_\alpha : X^\alpha \to Y^\alpha$. This induces for all $U \subseteq U_\alpha$ a map $f_\alpha^* \otimes 1 : H_T^*(Y^\alpha) \otimes_{H_T^*} \mathcal{O}_{\mathcal{C}_T}^h(U - \alpha) \to H_T^*(X^\alpha) \otimes_{H_T^*} \mathcal{O}_{\mathcal{C}_T}^h(U - \alpha)$, which commutes with restrictions. Therefore, we obtain a sheaf map $f_\alpha^* : \mathcal{F}_\alpha(Y) \to \mathcal{F}_\alpha(X)$. To get our global map f^*, we only have to check that the maps f_α^* glue well, i.e. that they commute with the gluing maps $\phi_{\alpha\beta}$. This follows easily from the naturality of ordinary equivariant cohomology, and from the naturality in X of the isomorphism $H_T^*(X^\beta) \cong H_K^*(X^\beta) \otimes_{H_K^*} H_T^*$. (See Proposition 2.13 with $Z = X^\beta$, $H = H(\beta)$ and $K = T/H$.)

Let (X, A) be a pair of finite T-CW complexes, i.e. A is a closed subspace of X and the inclusion map $A \to X$ is T-equivariant. We then define

$\mathcal{K}_T^*(X, A)$ as the kernel of the map $j^* : \mathcal{K}_T^*(X/A) \to \mathcal{K}_T^*(point)$, where $j : point = A/A \to X/A$ is the inclusion map. If $f : (X, A) \to (Y, B)$ is a map of pairs of finite T-CW complexes, then $f^* : \mathcal{K}_T^*(Y, B) \to \mathcal{K}_T^*(X, A)$ is defined as the unique map induced on the corresponding kernels from $f^* : \mathcal{K}_T^*(Y) \to \mathcal{K}_T^*(X)$.

The last definition we need is of the coboundary map. If (X, A) is a pair of finite T-CW complexes, we want to define $\delta : \mathcal{K}_T^*(A) \to \mathcal{K}_T^{*+1}(X, A)$. This is obtained by gluing the maps

$$\delta_\alpha \otimes 1 : H_T^*(A^\alpha) \otimes_{H_T^*} \mathcal{O}_{\mathcal{C}_T}^h(U - \alpha) \to H_T^{*+1}(X^\alpha, A^\alpha) \otimes_{H_T^*} \mathcal{O}_{\mathcal{C}_T}^h(U - \alpha),$$

where $\delta_\alpha : H_T^*(A^\alpha) \to H_T^{*+1}(X^\alpha, A^\alpha)$ is the usual coboundary map. The maps $\delta_\alpha \otimes 1$ glue well, because δ_α is natural.

To check the usual axioms of a cohomology theory (naturality, exact sequence of a pair, and excision) for $\mathcal{K}_T^*(-)$, recall that it was obtained by gluing the sheaves $\mathcal{F}_\alpha$ along the maps $\phi_{\alpha\beta}$. Since the sheaves $\mathcal{F}_\alpha$ were defined using $H_T^*(X^\alpha)$, the properties of ordinary T-equivariant cohomology pass on to $\mathcal{K}_T^*(-)$, as long as tensoring with $\mathcal{O}_{\mathcal{C}_T}^h(U - \alpha)$ over H_T^* preserves exactness. But this is implied by Proposition 2.8. $\square$

Now, $\mathcal{C}_T = \mathrm{Spec}_m K_T^*$ is a nonsingular affine complex variety, so it is Stein. By a generalization of Theorem B of Cartan, the sheaf cohomology vanishes on $\mathcal{C}_T$ in positive dimensions for any coherent sheaf. (See Gunning and Rossi [13].) This implies that Γ, the global sections functor, is exact, so we get the following result:

Corollary 2.20. *$\Gamma \mathcal{K}_T^*(-)$ is an T-equivariant cohomology theory on finite complexes with values in the category of $\mathbb{Z}/2$-graded algebras.*

We want to investigate a few more useful properties of $\mathcal{K}_T^*(-)$.

Proposition 2.21. *The value of the theory on a point is given by $\mathcal{K}_T^* \cong \mathcal{O}_{\mathcal{C}_T}^h$.*

Proof. Notice that if X is a point, translation by α induces an isomorphism of the corresponding sheaf $\mathcal{F}_\alpha$ on U_α to $\mathcal{O}_{\mathcal{C}_T |U_\alpha}^h$. Via this isomorphism, the gluing map $\phi_{\alpha\beta}$ is the usual restriction of holomorphic functions. Therefore $\mathcal{K}_T^* \cong \mathcal{O}_{\mathcal{C}_T}^h$. $\square$

Proposition 2.22. *Let Γ be the global sections functor. Then the ring $\Gamma \mathcal{K}_T^*$ is faithfully flat over K_T^*.*

Proof. Recall that we defined $\mathcal{C}_T$ as the complex algebraic variety $\mathrm{Spec}_m K_T^*$. As in the proof of Proposition 2.8, denote by $\mathcal{O}$ the algebraic structure sheaf of $\mathcal{C}_T$, and by $\mathcal{O}^h$ the analytic structure sheaf. Let $X = \mathcal{C}_T$. We need to show that the natural map $\mathcal{O}(X) \to \mathcal{O}^h(X)$ is faithfully flat. We first

show it is flat: By the local characterization of flatness (e.g. Theorem (3.J) in Matsumura [14]), it is enough to show that for any $x \in U$, the natural map $\mathcal{O}_x \to \mathcal{O}_x^h$ is flat. But this is true, and the proof goes exactly as in Proposition 2.8.

To prove that $\mathcal{O}(X) \to \mathcal{O}^h(X)$ is in fact faithfully flat, we also need to show that the induced map of spectra of maximal ideals, $\mathrm{Spec}_m \, \mathcal{O}^h(X) \to \mathrm{Spec}_m \, \mathcal{O}(X)$ is surjective (see Theorem 3 (4.D) in [14]). This is clearly true, since maximal ideals of $\mathcal{O}(X)$ and $\mathcal{O}^h(X)$ correspond to the points $x \in X$.
$\square$

Since $\mathcal{K}_T^*(X)$ was obtained from gluing the sheaves $\mathcal{F}_\alpha$, its stalks are the same as those of $\mathcal{F}_\alpha$.

Proposition 2.23. *If X is a finite T-CW complex and $\alpha \in \mathcal{C}_T$, then the stalk of the $\mathcal{K}_T^*(X)$ at α is*

$$\mathcal{K}_T^*(X)_\alpha = H_T^*(X^\alpha) \otimes_{H_T^*} \mathcal{O}_{\mathcal{C}_T,0}^h \ .$$

Proposition 2.24. *Let A and B are compact abelian Lie groups, X is a finite A-CW complex, and Y is a finite B-CW complex. Then*

$$\Gamma \mathcal{K}_{A \times B}^*(X \times Y) \cong \Gamma \mathcal{K}_A^*(X) \otimes_{\mathbb{C}} \Gamma \mathcal{K}_B^*(Y).$$

Proof. This follows from $\mathcal{C}_{A \times B} \cong \mathcal{C}_A \times \mathcal{C}_B$ and from $H_{A \times B}^*(X \times Y) \cong H_A^*(X) \otimes_{\mathbb{C}} H_B^*(Y)$, where the last isomorphism comes from the Künneth formula.
$\square$

We now return to the usual equivariant K-theory, and define its holomorphic extension

$$\mathbf{K}_T^*(X) = K_T^*(X) \otimes_{K_T^*} \Gamma \mathcal{K}_T^*.$$

Since $K_T^*(-)$ is a T-equivariant cohomology theory, it follows from Lemma 2.22 that $\mathbf{K}_T^*(-)$ is a T-equivariant cohomology theory, with values in $\mathbb{Z}/2$-graded algebras.

3. The equivariant Chern character

Let T be a compact abelian Lie group and X a finite T-CW complex. In this section we construct a natural isomorphism

$$CH_T : \mathbf{K}_T^*(X) \to \Gamma \mathcal{K}_T^*(X).$$

The main ingredient in the definition of CH_T is the construction of a natural ring map $CH_T : K_T^0(X, \mathbb{Z}) \to \Gamma \mathcal{K}_T^*(X)$. Let $E \to X$ be a complex T-vector bundle. We want $CH_T(E)$ to be a section in $\mathcal{K}_T^*(X)$, so we need

to know its germ at a point $\alpha \in \mathcal{C}_T$. The germ $CH_T(E)_\alpha$ should be an element of $\mathbf{H}_T^*(X^\alpha)$, so we would be quite tempted to define it as the equivariant Chern character ch_T of the restriction of E to X^α (we will define the equivariant Chern character in the next subsection). However, there are a few problems. The first one is that in order to get ch_T to take values in $\mathbf{H}_T^*(-)$, we need some sort of holomorphicity result on ch_T. This is done in Lemma 3.1. The second problem is that when we vary $\alpha \in \mathcal{C}_T$, the equivariant Chern character does not glue well. To solve this problem, we modify the definition of CH_T by twisting ch_T accordingly: see Definition 3.4.

3.1. Preliminaries

Let $E \to X$ be a complex T-vector bundle. Then the equivariant Chern character $ch_T(E)$ can be defined as $ch(E_T)$, where ch is the usual Chern character, and $E_T \to X_T$ is the Borel construction. This a priori is an element of $H_T^{**}(X)$, because the Chern character is defined using the power series e^x (see the Appendix of Rosu [15] for a discussion of equivariant characteristic classes). So we get a map

$$ch_T : K_T^*(X) \to H_T^{**}(X).$$

Also, denote the n-th Chern class of E_T by $c_n(E)_T$. This belongs to the noncompleted ring $H_T^*(X)$.

Lemma 3.1. *Let $E \to X$ be a complex T-vector bundle over a finite T-CW complex. Then its T-equivariant Chern character, $ch_T(E)$, which a priori is an element of $H_T^{**}(X)$, actually lies in $\mathbf{H}_T^*(X)$.*

Proof. First assume that E is a line bundle. Then from the definition of the Chern character we get $ch_T(E) = e^{c_1(E)_T}$. We have already seen that $c_1(E)_T \in H_T^*(X)$. Since X is finite, $H_T^*(X)$ is a finite module over H_T^*. Let $a_1, \ldots, a_m$ be a set of generators for $H_T^*(X)$. Choose a set of elements $f_{ij}^k \in H_T^*$ so that

$$a_i \cdot a_j = \sum_{k=1}^{m} f_{ij}^k a_k \quad \forall\, i, j, k \in 1, \ldots, m.$$

Denote by $c = c_1(E)_T$. The element c is in $H_T^*(X)$, so there exist elements $g_i \in H_T^*$ such that

$$c = g_1 a_1 + \cdots + g_m a_m.$$

We can also calculate $c^2 = \sum_k \left(\sum_{i,j} g_i g_j f_{ij}^k \right) a_k$, $c^3 = \sum_p \left(\sum_{i,j,k,l} g_i g_j g_l f_{ij}^k f_{kl}^p \right) a_p$, etc.

Choose some coordinates on $t_{\mathbb{C}}$ such that $H_T^* = \mathbb{C}[u_1, \ldots, u_p]$. We say that a polynomial $\phi \in \mathbb{C}[u_1, \ldots, u_p]$ is dominated by another polynomial ψ if all coefficients of ψ are positive and all the coefficients of ϕ are less in absolute value than the coefficients of ψ. Also, if $\mu, \nu \in H_T^*(X)$ then we can write $\mu = \mu_1 a_1 + \cdots + \mu_m a_m$ and $\nu = \nu_1 a_1 + \cdots + \nu_m a_m$, with $\mu_i, \nu_i \in H_T^*$. We then say that μ is dominated by ν if μ_i is dominated by ν_i for all i. Similar definitions of domination can be made for the power series ring H_T^{**} and for $H_T^{**}(X)$.

Let $\lambda \in H_T^* = \mathbb{C}[u_1, \ldots, u_p]$ be a polynomial such that f_{ij}^k and g_i are dominated by λ for all i, j, k. Then one can show by induction that c^n is dominated by $(2n-2)\lambda^{2n-1}(a_1 + \cdots + a_m)$, which in turn is dominated by $2n\lambda^{2n-1}(a_1 + \cdots + a_m)$. So e^c is dominated by $1 + \sum_{n \geq 1} \frac{2}{(n-1)!} \lambda^{2n-1}(a_1 + \cdots + a_m) = 1 + \sum_{n \geq 0} \frac{2}{n!} \lambda^{2n+1}(a_1 + \cdots + a_m) = 1 + 2\lambda e^{2\lambda}(a_1 + \cdots + a_m)$. But this last element belongs to $H_T^*(X) \otimes_{H_T^*} \mathcal{O}_{t_{\mathbb{C}},0}^h = \mathbf{H}_T^*(X)$, so e^c also belongs to $\mathbf{H}_T^*(X)$.

Second, assume E is a rank n complex T-vector bundle. Let S_E be the splitting space of E (or the flag variety of E-see Bott and Tu [6]). By construction S_E comes with an equivariant map $\pi : S_E \to X$. Then the splitting principle says that the pull-back bundle $\pi^*(E)$ decomposes as a sum of T-equivariant line bundles over S_E. Say $\pi^*(E) \cong L_1 \oplus \cdots \oplus L_n$. Calculation using the Leray–Serre spectral sequence shows that

$$H_T^*(S_E) = H_T^*(X)[x_1, \ldots, x_n] / \langle \sigma_i(x_1, \ldots, x_n) - c_i(E)_T \rangle,$$

where $\sigma_i(x_1, \ldots, x_n)$ is the i'th symmetric polynomial in the x_j's. The classes $x_j = c_1(L_j)_T$ are called the Chern roots of E. Moreover, we can identify $H_T^*(X)$ as the subring of $H_T^*(S_E)$ generated by the polynomials in $H_T^*(X)[x_1, \ldots, x_n]$ which are symmetric in the x_j's. By tensoring with $\mathcal{O}_{t_{\mathbb{C}},0}^h$ or H_T^{**} the same statement is true about $\mathbf{H}_T^*(X)$ and $H_T^{**}(X)$.

Now consider $ch_T(E) = e^{x_1} \cdots e^{x_n}$. Since L_j is a line bundle and $x_j = c_1(L_j)_T$, the first part of the proof implies that $e^{x_j} \in \mathbf{H}_T^*(S_E)$ for all j. Therefore $ch_T(E) \in \mathbf{H}_T^*(S_E)$, and since it is symmetric in the x_j's it follows that $ch_T(E) \in \mathbf{H}_T^*(X)$, which is what we wanted. $\qquad \square$

We have just proved that $ch_T(E)$ is the germ of a holomorphic class, i.e. an element of $\mathbf{H}_T^*(X)$. By looking more carefully at the preceding proof, one can see in fact that we proved a stronger result:

Corollary 3.2. *With the same notations as in Lemma 3.1, $ch_T(E)$ is a global holomorphic class, i.e. an element of $H_T^*(X) \otimes_{H_T^*} \Gamma \mathcal{O}_{\mathcal{C}_T}^h$.*

Now, if we extend $ch_T(E)$ on a small neighborhood U of $0 \in \mathcal{C}_T$, we can regard it as an element of $H_T^*(X) \otimes_{H_T^*} \mathcal{O}_{\mathcal{C}_T}^h(U)$. It is important to see what happens to $ch_T(E)$ when it is translated by the map $\tau_{\beta-\alpha}^*$ from Proposition 2.14.

The basic case is when X is a point and E is given by a representation V_λ of T, with $\lambda \in \hat{T}$. Recall that $\mathcal{C}_T = \mathrm{Hom}(\hat{T}, \mathcal{C})$, and consider $\alpha \in \mathcal{C}_T$. Then we translate $ch_T(V_\lambda)$ via the map $\tau_\alpha^* = t_\alpha^* : \mathcal{O}_{\mathcal{C}_T}^h(U) \to \mathcal{O}_{\mathcal{C}_T}^h(U + \alpha)$.

Lemma 3.3. *Let T be a compact Abelian group, and T^0 the connected component containing the identity. Let $\alpha \in \mathcal{C}_{T^0}$ and $\lambda \in \hat{T}$. Then, with the notations above,*

$$t_\alpha^* ch_T(V_\lambda) = \alpha(\lambda) ch_T(V_\lambda).$$

Proof. The proof of this lemma is mostly formal, and just makes intensive use of the identifications and definitions we have made so far. Start by using the exponential map

$$\exp : t_\mathbb{C} = \Lambda_T \otimes_\mathbb{Z} \mathbb{C} \to \Lambda_T \otimes_\mathbb{Z} \mathcal{C} = \mathcal{C}_{T^0} \to \mathcal{C}_T.$$

The element α is in the image, so pick $a \in t_\mathbb{C}$ such that $\exp(a) = \alpha$. Denote by $l \in \Lambda_T^*$ the element corresponding to λ via the map $\hat{T} \to \Lambda_T^*$. Then one can apply l to $a \in \Lambda_T \otimes_\mathbb{Z} \mathbb{C}$ and get a complex number that we denote by $l(a)$. Now it is easy to check the formula $\alpha(\lambda) = \exp(l(a))$. We also know that $ch_T(V_\lambda) = \exp\big(c_1(V_\lambda)\big)$. So, via the exponential map, what we have to prove becomes

$$t_a^* c_1(V_\lambda) = l(a) + c_1(V_\lambda),$$

with the equality being now regarded in H_T^*. Let us look more closely at $c_1(V_\lambda)$. We saw in the proof of Proposition 2.6 that there is an identification $H_T^* = S(t_\mathbb{C}^*)$, and that the class $c_1(V_\lambda) \in H_T^*$ can be identified to l if this is regarded in $S(t_\mathbb{C}^*)$ via $\Lambda_T^* \subset t_\mathbb{C}^* \subset S(t_\mathbb{C}^*)$. Denote by $l(-)$ the polynomial function in $S(t_\mathbb{C}^*)$ corresponding to l. Then we have to prove that

$$t_a^* l(-) = l(a) + l(-).$$

But this is obvious, it is just saying that $l(-)$ is a linear function. $\square$

3.2. Construction of CH_T

We define a multiplicative natural map

$$CH_T : K_T^0(X, \mathbb{Z}) \to \Gamma \mathcal{K}_T^*(X).$$

Let $E \to X$ be a complex T-vector bundle. Let $\alpha \in \mathcal{C}_T$, and denote by $H = H(\alpha)$. Then $\alpha \in \mathcal{C}_H$. By Proposition 2.1, $\mathcal{C}_H \cong \mathrm{Hom}_\mathbb{Z}(\hat{H}, \mathcal{C})$, so we can think of α as a group map $\alpha : \hat{H} \to \mathcal{C}$. The space X^α has a trivial action of H, so the restriction $E_{|X^\alpha}$ of E to X^α has a fiberwise decomposition by irreducible characters of H:

$$E_{|X^\alpha} \cong \oplus_{\lambda \in \hat{H}} E(\lambda),$$

where $E(\lambda)$ is the T-vector bundle where $h \in H$ acts by complex multiplication with $\lambda(h)$.

It would be tempting to define the germ of $CH_T(E)$ at α to be ch_T $(E_{|X^\alpha})$, but these germs would not glue well to give a global section of $\mathcal{K}_T^*(X)$. Instead, we do the following:

Definition 3.4. *Let* $\alpha \in \mathcal{C}_T$ *and* $H = H(\alpha)$. *Then the germ of* $CH_T(E)$ *at* α *is defined to be*

$$CH_T(E)_\alpha = \sum_{\lambda \in \hat{H}} \alpha(\lambda) ch_T E(\lambda).$$

Proposition 3.5. *The germs* $CH_T(E)_\alpha$ *glue to a global section* $CH_T(E) \in \Gamma\mathcal{K}_T^*(X)$.

Proof. We notice that, by Lemma 3.1, $CH_T(E)_\alpha$ does indeed belong to $\mathbf{H}_T^*$ (X^α), which by Proposition 2.23 is the stalk of $\mathcal{K}_T^*(X)$ at α. Fix $(U_\alpha)_{\alpha \in \mathcal{C}_T}$ a cover of $\mathcal{C}_T$ adapted to X.

Let $\alpha, \beta \in \mathcal{C}_T$ with $U_\alpha \cap U_\beta \neq \emptyset$ and $\alpha \prec \beta$. This implies $\alpha, \beta \in \mathcal{C}_{H(\beta)}$ and also $H(\alpha) \subseteq H(\beta)$. Denote by $L = H(\alpha)$ and $H = H(\beta)$. Condition 4 of Definition 2.4 implies that $\beta - \alpha \in \mathcal{C}_{H^0}$. Now we have to prove that $\phi_{\alpha\beta} CH_T(E)_\alpha = CH_T(E)_\beta$, i.e. that $\tau_{\beta-\alpha}^* \sum_{\lambda \in \hat{L}} \alpha(\lambda) ch_T E(\lambda)_{|X^\beta} = \sum_{\mu \in \hat{H}} \beta(\mu) ch_T E(\mu)$. Consider the surjective map $j : \hat{H} \to \hat{L}$ induced by the inclusion $L \hookrightarrow H$. If $\lambda \in \hat{L}$, we have $E(\lambda)_{|X^\beta} = \sum_{\mu \in j^{-1}(\lambda)} E(\lambda)$. Therefore it is enough to show that for all $\mu \in \hat{H}$ we have $\tau_{\beta-\alpha}^* \left(\alpha(\lambda) ch_T E(\mu) \right) = \beta(\mu) ch_T E(\mu)$, where $\lambda = j(\mu)$. But this is equivalent to $\tau_{\beta-\alpha}^* ch_T E(\mu) = (\beta - \alpha)(\mu) ch_T E(\mu)$. Denote by $\gamma = \beta - \alpha \in \mathcal{C}_{H^0}$. So it is enough to show that, for all $\gamma \in \mathcal{C}_{H^0}$ and $\mu \in \hat{H}$,

$$\tau_\gamma^* ch_T E(\mu) = \gamma(\mu) ch_T E(\mu).$$

Let $K = T/H$ and $Y = X^\beta$. Proposition 2.13 applied to equivariant K-theory gives a natural isomorphism

$$K_K^*(Y) \otimes_{K_K^*} K_T^* \xrightarrow{\sim} K_T^*(Y).$$

Via the identification above, we can think of $E(\mu)$ as a tensor product $F \otimes V(\mu')$, with F a K-bundle and $\mu' \in \hat{T}$ some element in the preimage of μ via the map $\hat{T} \to \hat{H}$. (At least, we know that $E(\mu)$ is generated by such elements.) Via the same identification, translation by $\gamma \in \mathcal{C}_{H^0}$ becomes

$$\tau_\gamma^* \mapsto \tau_{\pi(\gamma)}^* \otimes \tau_\gamma^*,$$

where $\pi(\tau)$ is the image of γ via the natural map $\pi : \mathcal{C}_T \to \mathcal{C}_K$, and the second γ is regarded in $\mathcal{C}_T$ via the usual inclusion $\mathcal{C}_{H^0} \to \mathcal{C}_T$. But notice

that $\pi(\tau) = 0$, because of the exact sequence $0 \to \mathcal{C}_H \to \mathcal{C}_T \to \mathcal{C}_K \to 0$. Also, $ch_T E(\mu)$ becomes $ch_K F \otimes ch_T V(\mu') \in \mathbf{H}^*_K(Y) \otimes_{\mathbf{H}^*_K} \mathbf{H}^*_T$. So via the above correspondence we have

$$\tau^*_\gamma ch_T E(\mu) \mapsto ch_K F \otimes \tau^*_\gamma ch_T V(\mu').$$

Since $\gamma \in \mathcal{C}_{H^0}$, it follows that $\gamma \in \mathcal{C}_{T^0}$, and it is sufficient for us to show that, for all $\gamma \in \mathcal{C}_{T^0}$ and $\mu' \in \hat{T}$,

$$\tau^*_\gamma ch_T V(\mu') = \gamma(\mu) ch_T V(\mu').$$

But this is directly implied by Lemma 3.3, so we are done. $\qquad\square$

We have just finished constructing a natural map $CH_T : K^0_T(X, \mathbb{Z}) \to \Gamma\mathcal{K}^*_T(X)$. By taking the suspension of X instead of X, this induces a map $CH_T : K^*_T(X, \mathbb{Z}) \to \Gamma\mathcal{K}^*_T(X)$. One can check easily that CH_T is a ring map, since ch_T is. Because $\Gamma\mathcal{K}^*_T(X)$ is a $\mathbb{C}$-algebra and CH_T is a ring map, we can now extend CH_T to a natural map of $\mathbb{C}$-algebras $CH_T : K^*_T(X) \to \Gamma\mathcal{K}^*_T(X)$. Finally, making X a point, we get a ring map $K^*_T \hookrightarrow \Gamma\mathcal{K}^*_T$, so if we extend CH_T by this, we obtain the desired natural map

$$CH_T : \mathbf{K}^*_T(X) \to \Gamma\mathcal{K}^*_T(X).$$

Theorem 3.6. *CH_T is an isomorphism of T-equivariant cohomology theories.*

Proof. Because of the Mayer-Vietoris sequence, it is enough to verify the isomorphism for "equivariant points" of the form T/L, with L a compact subgroup of T. Choose an identification

$$T = (S^1)^p \times (S^1)^q \times \prod_{i=1}^r \mathbb{Z}_{m_i}$$

such that via this identification

$$L = (S^1)^p \times \prod_{j=1}^q \mathbb{Z}_{n_i} \times \prod_{i=1}^r \mathbb{Z}_{l_i}.$$

Then $T/L = (1)^p \times \prod_{j=1}^q (S^1/\mathbb{Z}_{n_i}) \times \prod_{i=1}^r (\mathbb{Z}_{m_i}/\mathbb{Z}_{l_i})$.

We use now Proposition 2.24, which is also true if we replace $\Gamma\mathcal{K}$ by $\mathbf{K}$-theory (because it is true for K-theory). Since the map CH commutes with the isomorphisms of Proposition 2.24, it is enough to check that the following maps are isomorphisms:

(a) $CH_{S^1} : \mathbf{K}^*_{S^1} \to \Gamma\mathcal{K}^*_{S^1}$;
(b) $CH_{S^1} : \mathbf{K}^*_{S^1}(S^1/\mathbb{Z}_n) \to \Gamma\mathcal{K}^*_{S^1}(S^1/\mathbb{Z}_n)$;

(c) $CH_{\mathbb{Z}_m} : \mathbf{K}^*_{\mathbb{Z}_m}(\mathbb{Z}_m/\mathbb{Z}_l) \to \Gamma\mathcal{K}^*_{\mathbb{Z}_m}(\mathbb{Z}_m/\mathbb{Z}_l)$.

To prove (a), notice that $\mathcal{C}_{S^1} = \mathcal{C}$. Then we have $\mathbf{K}^*_{S^1} = K^*_{S^1} \otimes_{K^*_{S^1}} \Gamma\mathcal{O}^h_{\mathcal{C}} = \Gamma\mathcal{O}^h_{\mathcal{C}}$. By Proposition 2.21, $\Gamma\mathcal{K}^*_{S^1} = \Gamma\mathcal{O}^h_{\mathcal{C}}$. Now notice that, by definition, the map CH_{S^1} is the identity.

For (b), denote $X = S^1/\mathbb{Z}_n$. Then we have $\mathbf{K}^*_{S^1}(X) = K^*_{S^1}(X) \otimes_{K^*_{S^1}} \Gamma\mathcal{K}^*_{S^1} = K^*_{\mathbb{Z}_n} \otimes_{K^*_{S^1}} \Gamma\mathcal{O}^h_{\mathcal{C}}$. But we know that $K^*_{S^1} = \mathbb{C}[z^{\pm 1}]$ and $K^*_{\mathbb{Z}_n} = \mathbb{C}[z^{\pm 1}]/\langle z^n - 1\rangle$. So we deduce $\mathbf{K}^*_{S^1}(X) = \Gamma\mathcal{O}^h_{\mathcal{C}}/\langle z^n - 1\rangle$. This last ring can be identified with $\mathbb{C}[z^{\pm 1}]/\langle z^n - 1\rangle$, since the condition $z^n = 1$ makes all power series finite. In conclusion, $\mathbf{K}^*_{S^1}(X) = K^*_{\mathbb{Z}_n} = \mathbb{C}[z^{\pm 1}]/\langle z^n - 1\rangle$.

Let us now describe the sheaf $\mathcal{F} = \mathcal{K}^*_{S^1}(X)$. Let $\alpha \in \mathcal{C}$. If $\alpha \notin \mathbb{Z}_n$, $X^\alpha = \emptyset$, so the stalk of $\mathcal{F}$ at α is zero. If $\alpha \in \mathbb{Z}_n$, $X^\alpha = X$, and the stalk of $\mathcal{F}$ at α is $H^*_{\mathbb{Z}_n} \otimes_{H^*_{S^1}} \mathcal{O}^h_{\mathbb{C},0}$. But $H^*_{\mathbb{Z}_n} = \mathbb{C}$, concentrated in degree zero ($H^*_{\mathbb{Z}_n}$ is $\mathbb{Z}$-torsion in higher degrees, so the components in higher degree disappear when we tensor with $\mathbb{C}$). It follows that $\mathcal{F}$ is a sheaf concentrated at the elements of $\mathbb{Z}_n$, where it has the stalk equal to $\mathbb{C}$. Then the global sections of $\mathcal{F}$ are $\Gamma\mathcal{K}^*_{S^1}(X) = \mathbb{C} \oplus \cdots \oplus \mathbb{C}$, n copies, one for each element of $\mathbb{Z}_n$.

The map $CH_{S^1} : \mathbf{K}^*_{S^1}(X) \to \Gamma\mathcal{K}^*_{S^1}(X)$ comes from the ring map $CH_T : \mathbb{C}[z^{\pm 1}]/\langle z^n - 1\rangle \to \mathbb{C} \oplus \cdots \oplus \mathbb{C}$. Since z generates the domain of CH_T, it is enough to see where z is sent. Let $\mathbb{Z}_n = \{1, \epsilon, \epsilon^2, \ldots, \epsilon^{n-1}\}$. Then z represents the standard irreducible representation $V = V(\epsilon) = \mathbb{C}$ of $\mathbb{Z}_n$, where ϵ acts on $\mathbb{C}$ by complex multiplication with ϵ, which is regarded as an element of $\mathcal{C}$. Notice that V corresponds to the element $\lambda = \epsilon \in \widehat{\mathbb{Z}_n} = \mathbb{Z}_n$. $c_1(V)_{S^1} = 0$, because $c_1(V)_{S^1}$ lies in $H^2_{\mathbb{Z}_n} = 0$. Then $ch_{S^1}(V) = e^{c_1(V)_{S^1}} = e^0 = 1$, and the stalk of $CH_{S^1}(V)$ at $\alpha \in \mathbb{Z}_n$ is $CH_{S^1}(V)_\alpha = \alpha(\epsilon) = \alpha$. Therefore CH_{S^1} sends z to $(1, \epsilon, \epsilon^2, \ldots, \epsilon^{n-1}) \in \mathbb{C} \oplus \cdots \oplus \mathbb{C}$. One can easily check that this map is an isomorphism.

For (c), denote $X = \mathbb{Z}_m/\mathbb{Z}_l$. As in (b), $\mathbf{K}^*_{\mathbb{Z}_m}(X) = K^*_{\mathbb{Z}_l} = \mathbb{C}[z^{\pm 1}]/\langle z^l - 1\rangle$, and $\Gamma\mathcal{K}^*_{\mathbb{Z}_m}(X) = \mathbb{C} \oplus \cdots \oplus \mathbb{C}$, l copies. The proof that $CH_{\mathbb{Z}_m}$ is an isomorphism is the same as above. $\qquad\square$

Appendix A. Applications

We now give applications of the construction in this paper. First we use the Chang-Skjelbred theorem in equivariant cohomology to infer the corresponding result for equivariant K-theory. Then as a corollary we show how a result about the equivariant cohomology of GKM manifolds can be extended to equivariant K-theory. Along the way we need a natural splitting that does not seem to have been noticed before in this area.

Definition A.1. *Let X be a compact T-manifold, for T a compact abelian Lie group. We say that X is equivariantly formal if the equvariant cohomology spectral sequence collapses at the E_2 term.*

Many interesting T-spaces are equivariantly formal; for example any subvariety of complex projective space preserved by a linear action, or symplectic manifold with a Hamiltonian action. Our reference for equivariantly formal spaces is [11]. We need three results about them: The first is that the map $H_T^*(X) \to H_T^*(X^T)$ is an injection. The second is that for any H compact subgroup of T, X^H is also equivariantly formal. The third is due to Chang and Skjelbred [9] (see also [8] for a proof):

Theorem A.2. *Let X be an equivariantly formal T-manifold, and let X_1 be its equivariant 1-skeleton, i.e. X_1 is the set of points in X with stabilizer of codimension at most one. We have inclusion maps $i : X^T \to X$ and $j : X^T \to X_1$. Then the map $i^* : H_T^*(X) \to H_T^*(X^T)$ is injective, and the maps i^* and $j^* : H_T^*(X_1) \to H_T^*(X^T)$ have the same image.*

The ring $H_T^*(X_1)$, in the notation of the above theorem, has not received much study. It is typically much bigger than $H_T^*(X)$, and though $H_T^*(X)$ injects into it, it does not inject into $H_T^*(X^T)$. These phenomena can be seen in the case of T^2 acting on $X = \mathbb{CP}^2$, where X^1 is a cycle of three $\mathbb{CP}^1$'s and therefore has H^1, not seen in either $H_T^*(X)$ or $H_T^*(X^T)$.

Lemma A.3. *In the notation of Theorem A.2, there is a natural identification $H_T^*(X_1) = H_T^*(X) \oplus \ker j^*$.*

Proof. By Theorem A.2 the images of the two maps $i^* : H_T^*(X) \to H_T^*(X^T)$ and $j^* : H_T^*(X_1) \to H_T^*(X^T)$ are the same. But i^* is injective, so we can identify $H_T^*(X)$ with the image of i^*. This implies that j^* factors through a map $H_T^*(X_1) \to H_T^*(X)$, and this yields a splitting $H_T^*(X_1) = H_T^*(X) \oplus \ker j^*$. $\square$

This natural splitting sheafifies, allowing us to extend both results to K-theory.

Theorem A.4. *We use the same notations as in Theorem A.2. Then $i^* : K_T^*(X) \to K_T^*(X^T)$ is injective, and the maps i^* and $j^* : K_T^*(X_1) \to K_T^*(X^T)$ have the same image.*

Proof. Let $\alpha \in \mathcal{C}_T$. Any compact T-manifold admits a decomposition as a finite T-CW complex (see for example Allday and Puppe [2]). Let $X = \bigcup_i \mathbb{D}^{n_i} \times (T/H_i)$ be such a cell decomposition. We saw that if K is a compact subgroup of T, $X^K = \sum_{i:K \subseteq H_i} \mathbb{D}^{n_i} \times (T/H_i)$. In particular, this implies that $(X_1)^\alpha = (X^\alpha)_1$.

Let $Y = X^\alpha$, which is again equivariantly formal. By Lemma A.3 there is a natural identification $H_T^*(Y_1) = H_T^*(Y) \oplus \ker j^*$. By Proposition 2.8 the map $H_T^* \to \mathbf{H}_T^*$ is flat, so tensoring with $\mathbf{H}_T^*$ over H_T^* yields a splitting $\mathbf{H}_T^*(Y_1) = \mathbf{H}_T^*(Y) \oplus \ker j^*$. Now, we observed above that $(Y_1)^\alpha = (Y^\alpha)_1$. So we finally get a splitting $\mathbf{H}_T^*\big((X_1)^\alpha\big) = \mathbf{H}_T^*(X^\alpha) \oplus \ker j^*$. This is compatible with the gluing maps of the sheaf $\mathcal{K}_T^*(X)$, so we get $\mathcal{K}_T^*(X_1) = \mathcal{K}_T^*(X) \oplus \ker j^*$.

The upshot of the above discussion is that $i^* : \mathcal{K}_T^*(X) \to \mathcal{K}_T^*(X^T)$ is injective (since it is injective on stalks), and $\mathcal{K}_T^*(X_1) = \mathcal{K}_T^*(X) \oplus \ker j^*$. The global section functor Γ is left exact, so $i^* : \Gamma\mathcal{K}_T^*(X) \to \Gamma\mathcal{K}_T^*(X^T)$ is injective and $\Gamma\mathcal{K}_T^*(X_1) = \Gamma\mathcal{K}_T^*(X) \oplus \ker j^*$. This implies that i^* and j^* have the same image in $\Gamma\mathcal{K}_T^*(X^T)$, namely $\Gamma\mathcal{K}_T^*(X)$. (Notice we couldn't have done this without using the splitting, because Γ is not right exact, so it doesn't commute with the image functor.)

Now recall that we have a natural isomorphism $CH_T : \mathbf{K}_T^*(X) \to \Gamma\mathcal{K}_T^*(X)$. Translating the above results via CH_T, we obtain that the maps $i^* : \mathbf{K}_T^*(X) \to \mathbf{K}_T^*(X^T)$ and $j^* : \mathbf{K}_T^*(X_1) \to \mathbf{K}_T^*(X^T)$ have the same image. But $\mathbf{K}_T^*(X) = K_T^*(X) \otimes_{K_T^*} \mathbf{K}_T^*$ and by Lemma 2.22 the map $K_T^* \to \mathbf{K}_T^*$ is faithfully flat. So we obtain that i^* and j^* have the same images in $K_T^*(X^T)$, which is what we wanted. $\square$

An alternative proof of Theorem A.4 can be given by noticing that the sheaf maps $i^* : \mathcal{K}_T^*(X) \to \mathcal{K}_T^*(X^T)$ and $j^* : \mathcal{K}_T^*(X_1) \to \mathcal{K}_T^*(X^T)$ have the same image (because they have the same image at the level of stalks). But the global section functor is exact, since we work with coherent sheaves over Stein manifolds (see the comment before Corollary 2.20). It follows that the maps $i^* : \mathbf{K}_T^*(X) \to \mathbf{K}_T^*(X^T)$ and $j^* : \mathbf{K}_T^*(X_1) \to \mathbf{K}_T^*(X^T)$ have the same image, and the proof proceeds as before.

In [11] a special case of this is studied, in which X^T is discrete and X_1 is a union of S^2's; these are called *balloon manifolds* or *GKM manifolds*. (An interesting example of GKM manifolds are toric varieties.) In this case it is easy to calculate the image of restriction from $H_T^*(X_1)$, by reducing it to the case of $H_T^*(S^2)$. The theorem above lets us extend this result to K-theory.

Corollary A.5. *Let X be a GKM manifold, and $i^* : K_T^*(X) \to K_T^*(X^T)$ the restriction map. Then i^* is an injection; and a class $\alpha \in K_T^*(X^T)$ is in the image if for each 2-sphere $B \subseteq X_1$ with fixed points N and S, the difference $\alpha|_N - \alpha|_S \in K_T$ is a multiple of the K-theoretic Euler class of the tangent space $T_N B$. (Technically, we can only take the Euler class once we orient $T_N B$, but either orientation leads to the same condition on α.)*

If $T = (S^1)^n$, we can identify K_T^* with Laurent polynomials, and this condition says that the difference $\alpha|_S - \alpha|_N$ of Laurent polynomials must

be a multiple of $1 - w$, where w is the weight of the action of T on $T_N B$. (Again, we can only speak of the weight w once we orient this $\mathbb{R}^2$-bundle over the point N, but it doesn't matter because being a multiple of $1 - w$ is the same as being a multiple of $1 - w^{-1}$. In most examples one has around a T-invariant almost complex structure with which to orient all these tangent spaces simultaneously.)

After finishing this paper, similar results in the algebraic case have appeared in [18], particularly with regard to the extension of Chang-Skjelbred's results to K-theory. More specifically, their Corollary 5.10 is the algebraic analogue of our Theorem A.4. Notice also that their results, remarkably, hold over $\mathbb{Z}$, while ours only hold over $\mathbb{C}$. We thank Angelo Vistoli for explaining his results to us.

We also found out that Atiyah [3] proved a Chang-Skjelbred lemma in the context of equivariant K-theory. He not only did it more or less at the same time as Chang and Skjelbred, but his results are stronger: Let G be a torus and X a compact G-manifold. Denote by X_i the equivariant i-skeleton of X. Then Atiyah shows that the long exact sequence of the pair $(X \setminus X_i, X \setminus X_{i+1})$ splits into short exact sequences $0 \longrightarrow K_G^{*-1}(X \setminus X_{i+1}) \xrightarrow{\delta} K_G^*(X_{i+1} \setminus X_i) \longrightarrow K_G^*(X \setminus X_i) \longrightarrow 0$. This in turn is equivalent to the having a long exact sequence

$$0 \longrightarrow K_G^*(X) \longrightarrow K_G^*(X^0) \longrightarrow K_G^{*+1}(X_1 \setminus X_0)$$
$$\longrightarrow K_G^{*+2}(X_2 \setminus X_1) \longrightarrow \cdots$$

As noted by Bredon [7], Atiyah's argument carries over to equivariant cohomology with compact support and rational coefficients. In this context, the exactness up to the term $K_G^{*+1}(X_1 \setminus X_0)$ is just the assertion of the Chang–Skjelbred lemma. In cohomological setting, the above sequence may be thought of as the E_1 term of the spectral sequence coming from the filtration of the Borel construction X_G by the subspaces $(X_i)_G$. Moreover, its exactness is in fact equivalent to the equivariant formality of X, i.e. to the freeness of $H_G^*(X)$. (The direction Atiyah proved is the harder one.) We thank Matthias Franz for pointing out these results to us.

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