COMMENTS ON PROSPECT THEORY

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ABSTRACT. This note presents a critique of prospect theory, and develops a model for comparison of two simple lotteries, i.e. of the form $(x_1, p_1; x_2, p_2; \ldots; x_n, p_n)$. The main contention is that in most cases classical utility theory can properly compare two simple lotteries $(x_1, p_1; x_2, p_2; \ldots; x_n, p_n)$ and $(x'_1, p'_1; x'_2, p'_2; \ldots; x'_n, p'_n)$. The only exception appears when $p_1 \approx p'_1$, and either $p_i = p'_i$ or $x_i = x'_i = 0$ for $i > 1$. In that case, a utility function defined as in prospect theory, but using a simpler weighting function $\pi(p)$, can be used to compare the two lotteries.

Prospect theory, as described in Kahneman and Tversky [1] (see also [3]), has two main features: first, that choice is based on gains and losses, as compared to a reference point; second, that the utility function is nonlinear in the probabilities. This is in contrast with classical utility theory, where choice is based on the probabilities of the final outcomes, and the utility function is linear in the probabilities.

It seems that the current literature concentrates almost solely on the first feature of Prospect theory, namely on the framing effect and on the fact that losses loom larger than gains. Kahneman and Tversky attribute this idea to H. Markowitz [2]. I will only quote their conclusion ([1], p. 274):

> [...] people normally perceive outcomes as gains and losses, rather than as final states of wealth or welfare. Gains and losses, of course, are defined relative to some neutral reference point. The reference point usually corresponds to the current asset position, in which case gains and losses coincide with the actual amounts that are received and paid. However, the location of the reference point, and the consequent coding of outcomes as gains or losses, can be affected by the formulation of the offered prospects, and by the expectations of the decision maker.

The second aspect of Prospect Theory, less discussed in the literature (and which Kahneman and Tversky present as the novel feature of their theory) is the existence of a weighting function $\pi(p)$, which makes the utility function nonlinear in the probabilities. For a lottery (a.k.a. prospect) of the form $(x, p; y, q; 0, r)$, with $p+q+r = 1$, Kahneman and Tversky define their version of the utility function by

$$U(x, p; y, q) := \begin{cases} \pi(p)u(x) + \pi(q)u(y), & \text{if either } p+q < 1 \text{ or } xy \leq 0 \\ u(y) + \pi[u(x) - u(y)], & \text{if } p+q = 1, \text{ and either } x > y > 0 \text{ or } x < y < 0. \end{cases}$$

where for simplicity 0 was omitted from the expression of the lottery. By comparing examples of two such lotteries, one can derive the following properties of π:

(1) $\pi : [0,1] \to [0,1]$ is strictly increasing on $[0,1]$ and continuous on $[0,1]$. $\pi(0) = 0$, and $\pi(1) = 1$.
(2) $\pi(p) > p$ for small values of p.
(3) (Subadditivity) $\pi(rp) > r\pi(p)$ for $0 < r < 1$ and small values of p.
(4) (Subproportionality) $\pi(pq)/\pi(p) \leq \pi(pqr)/\pi(pr)$ for all $0 < p, q, r \leq 1$.

(5) $\pi(p) + \pi(1-p) < 1$ for all $0 < p < 1$.

One can prove that in fact conditions (1), (2), and (4) imply (3).

Objections to prospect theory.

Firstly, it is hard to find any weighting function $\pi(p)$ satisfying conditions (1)–(5). In their paper Kahneman and Tversky seem to imply that such a function would be discontinuous at 0 and 1 (in fact, there are functions satisfying all conditions (1)–(5), but neither of them so far seems to have a simple formula). This would be a problem, because we want $\pi(p) \approx p$ for small p.

Secondly, the theory is developed only for lotteries of the form $(x, p; y, q; 0, r)$, and it is not clear how it can be generalized to simple lotteries of the form $(x_1, p_1; x_2, p_2; \ldots ; x_n, p_n)$.

Thirdly, the utility function U cannot be used to compare directly two lotteries, as it does in classical utility theory. Instead, before comparing two lotteries, one has to go through a complicated process, called by Kahneman and Tversky *editing*. Editing involves several phases, namely: coding, combination, segregation, cancellation, simplification, and detection of dominance. Except for coding and cancellation, which deal with the existence of a reference point, as mentioned above, the other phases try to eliminate conditions which would put extra pressure on the function π. For example, on pages 283–284, it is shown that without detection of dominance and combination, the function π would have to be linear, hence $\pi(p) = p$, which wouldn't satisfy any of the conditions (2)–(5).

Already from the editying process it is clear that U cannot be considered a true utility function, since when applied to a lottery it is not defined independently of the other lotteries. Rather, prospect theory looks more like it is trying to give an algorithm to compare between two given simple lotteries.

Alternative theory.

With the proviso that the utility function u is S-shaped, i.e. concave in the positive domain and convex in the negative domain,[1] I claim that classical utility theory holds most of the time. The exceptions appear when we try to compare two simple lotteries, $(x_1, p_1; x_2, p_2; \ldots ; x_n, p_n)$ and $(x'_1, p'_1; x'_2, p'_2; \ldots ; x'_n, p'_n)$, where $p_1 \approx p'_1$, and either $p_i = p'_i$ or $x_i = x'_i = 0$ for all $i > 1$. Then the utility function designed to compare the two lotteries should be changed to

$$U(x_1, p_1; x_2, p_2; \ldots ; x_n, p_n) := \pi(p_1)u(x_1) + p_2 u(x_2) + \ldots + p_n u(x_n).$$

The weighting function π is required to satisfy only conditions (1)–(3), with (2) and (3) satisfied for all $0 < p < 1$. Examples of such functions include $\pi(p) = p^{1-\alpha}$, $0 < \alpha < 1$, which are all continuous on $[0, 1]$.

The reason why we introduce $\pi(p_1)$ instead of p_1 is given by the same principle that underlies risk aversion: overweighting of certainty (or rather underweighting of uncertainty). When p_1 is very close to p'_1, the difference in probability is not considered significant, so the individual would rather concentrate on the comparison of x_1 with x'_1.

To see how this works, consider the two lotteries (x, p) and (x', p'), with $x < x'$ and $p > p'$. Assume classical utility theory doesn't distinguish between them, i.e.

[1] Kahneman and Tversky show that u has to satisfy some more properties, namely $u(-x) > -u(x)$ and $u'(-x) > u'(x)$, corresponding to the fact that "losses loom larger than gains".

$pu(x) = p'u(x')$. The alternative version of prospect theory, however, predicts that (x', p'), which has the larger gain x', will be preferred to (x, p): condition (3) implies $\pi(p)/p < \pi(p')/p'$, hence $\pi(p)u(x) < \pi(p')u(x')$. This is confirmed by experimental data: see for example [1], p. 267, with $(x, p) = (3000, 0.002)$ and $(x', p') = (6000, 0.001)$.

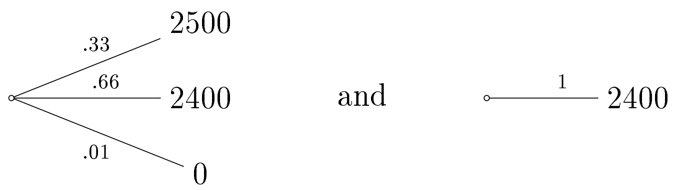
Verification.

Let us consider now the experiments which Kahneman and Tversky invoke to show the inadequacy of classical utility theory, and on which they build prospect theory.

The experiments typically give a pair (A) and (B) of choice problems. The number of respondents who answered each problem is denoted by N, and the percentage who chose each option is given in parentheses.

In problem (A) one has to choose between

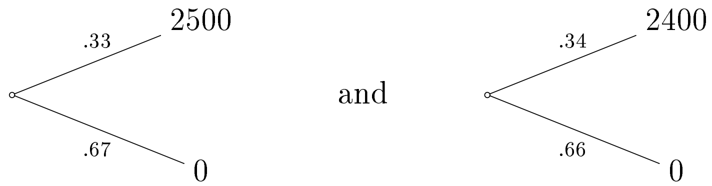
(A)



$N = 72$ I (18%) II (82%)

In problem (B) one has to choose between

(B)



$N = 72$ I (83%) II (17%)

Classical utility theory predicts for (A) that $.33u(2500) + .66u(2400) < u(2400)$, hence $.33u(2500) < .34u(2400)$. For (B) one obtains $.33u(2500) > .34u(2400)$. It is clear that the two relations cannot hold simultaneously, so utility theory fails to explain this experiment.

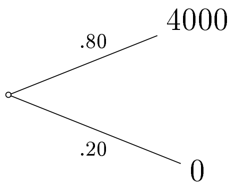
Prospect theory on the other hand attaches the weighting function π to all the probabilities involved. This yields $\pi(.33)u(2500) < [1 - \pi(.66)]u(2400)$, and $\pi(.33)u(2500) > \pi(.34)u(2400)$. This implies $\pi(.34) + \pi(.66) < 1$, which is condition (5).

The alternative theory leaves problem (A) to utility theory, implying as above $.33/.34 < u(2400)/u(2500)$. However, problem (B) is explained by comparing $\pi(.33)u(2500) > \pi(.34)u(2400)$, which is equivalent to $\pi(.33)/\pi(.34) > u(2400)/u(2500)$. For the two conditions to be compatible we need $\pi(.33)/\pi(.34) > .33/.34$, which is implied by the subadditivity condition (3) (thus avoiding condition (5)).

The explanation for the different way in which the alternative theory treats the two problems is that there are different forces acting in each case. In problem (A) it is a clear situation of risk aversion, which utility theory explains very well. In problem (B) the reversal of choice comes from the fact that the two probabilities .33 and .34 are perceived as almost equal, hence the individual tends to choose the larger gain, in this case 2500.

The next experiment is also a pair of choices:

(A)



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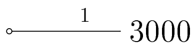
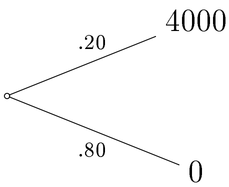


$N = 95$ I (20%) II (80%)

(B)



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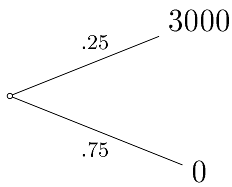


$N = 95$ I (65%) II (35%)

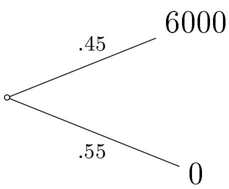
Classical utility theory implies: in problem (A) $.80u(4000) < u(3000)$; in problem (B) $.20u(4000) > .25u(3000)$. Multiplying the second inequality by 4, we obtain a contradiction with the first inequality.

Prospect theory implies: in problem (A) $\pi(.80)/\pi(1) < u(3000)/u(4000)$; in problem (B) $\pi(.20)/\pi(.25) > u(3000)/u(4000)$. Together they imply $\pi(.80)/\pi(1) < \pi(.20)/\pi(.25) = \pi(.80 \cdot \frac{1}{4})/\pi(1 \cdot \frac{1}{4})$. This is true if $\pi(pq)/\pi(p) < \pi(pqr)/\pi(pr)$ when $0 < r < 1$, which is the subproportionality condition (4).

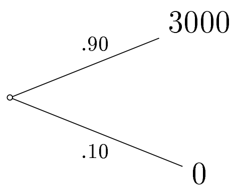
The alternative theory explains problem (A) by utility theory, which implies $.80/1 < u(3000)/u(4000)$. Since .20 and .25 are very close, problem (B) implies as in prospect theory that $\pi(.20)/\pi(.25) > u(3000)/u(4000)$. The two choices are consistent when $\pi(.20)/\pi(.25) > .80/1 = .20/.25$, which is implied again by the subadditivity condition (3).

The third experiment is the pair of choices

(A)

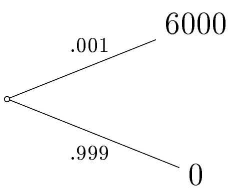


and



$N = 66$ I (14%) II (86%)

(B)



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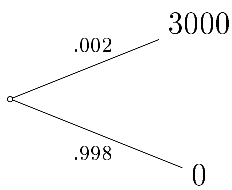
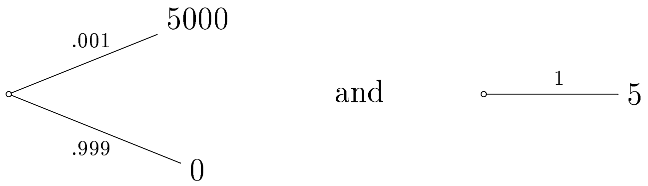


$N = 66$ I (73%) II (27%)

The explanation is the same as in the previous experiment, with the only difference that prospect theory requires for π to be subadditive as well: problem (B) implies that $\pi(.001) > \pi(.002) > u(3000)/u(6000) > 1/2$, the last inequality coming from the concavity of u. But $\pi(.001)/\pi(.002) > .001/.002$ requires condition (3).

The last experiment involves one choice, between

(A)



$N = 72$ I (72%) II (28%)

Utility theory fails to explain this experiment: suppose $.001u(5000) > u(5)$, i.e. $.001 > u(5)/u(5000)$. Concavity of u implies $u(5)/u(5000) > 5/5000 = .001$, so we obtain $.001 > .001$, contradiction.

Prospect theory requires $\pi(.001) > u(5)/u(5000)$. This inequality, together with the concavity of u imply that $\pi(.001) > .001$, which is condition (2).

The alternative theory interprets the second lottery as $(5, .001; 5, .999)$. It follows that $\pi(.001)u(5000) > \pi(.001)u(5) + .999u(5)$, which implies $\pi(.001)/[\pi(.001) + .999] > u(5)/u(5000) > 5/5000 = .001$, hence $\pi(.001) > .001$. This is condition (2) again.

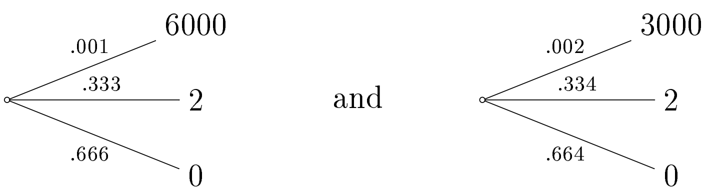
Conclusion

One sees from the four experiments that, while prospect theory requires that the weighting function π satisfy conditions (1)–(5), the alternative theory only requires (1)–(3) (with $0 < p < 1$). This allows us to choose for example $\pi(p) = p^{1-\alpha}$, $0 < \alpha < 1$, which is continuous on $[0, 1]$, and has a simple expression. (α could be called the "lottery coefficient": the larger α is, the more willing the individual is to buy a ticket to an ordinary lottery.)

Another advantage of this theory is that it avoids completely the editing phase from prospect theory. All one needs to do is to compare the probabilities of the two lotteries, and then decide if one uses utility theory or the alternative version.

Finally, one sees that the alternative theory works for all simple lotteries, i.e. of the form $(x_1, p_1; x_2, p_2; \dots ; x_n, p_n)$. For composed lotteries, it is much more complicated, since the reduction axiom does not hold, as shown by Kahneman and Tversky in [1], pp. 271–272.

A possible objection to the alternative theory is the following type of choice problems:



Presumably one who chose the 6000-lottery in the third experiment will do the same here. However, since the other probabilities p_2 and p_3 are not the same with p'_2

and p'_3 respectively, we are forced to use utility theory, which would clearly favor the second lottery. One way to solve this inconsistency would be to apply the following algorithm: Start with two lotteries, $L = (x_1, p_1; x_2, p_2; \dots ; x_n, p_n)$ and $L' = (x'_1, p'_1; x'_2, p'_2; \dots ; x'_n, p'_n)$. For each lottery compare the terms $p_i u(x_i)$. If any of them is much larger than the sum of the rest, call it a *dominant* term. If both lotteries have a dominant term, say $p_1 u(x_1)$ and $p'_1 u(x'_1)$ respectively, and if $p_1 \approx p'_1$, then we can compare only the dominant terms, but changed to $\pi(p_1)u(x_1)$ and $\pi(p'_1)u(x'_1)$. Otherwise, apply the usual theory and compare $U(L)$ with $U(L')$, where $U(L) = \pi(p_1)u(x_1) + p_2 u(x_2) + \dots + p_n u(x_n)$. If only one lottery has a dominant term, say $p_1 u(x_1)$, change only this term, as above, to $\pi(p_1)u(x_1)$, and compare the mixed utilities, as in the usual alternate theory. (This last idea can be used to explain why people buy lottery tickets, and even compare their inclinations by comparing their α's, where $\pi(p) = p^{1-\alpha}$.)

References

[1] D. Kahneman, A. Tversky, *Prospect theory: an analysis decision under risk*, Econometrica, vol. 47 (1979), pp. 263–291.

[2] H. Markowitz, *The utility of wealth*, Journal of Political Economy, vol. 60 (1952), pp. 151–158.

[3] A. Tversky, D. Kahneman, *The framing of decisions and the psychology of choice*, Science, vol. 211 (1981), pp. 453–458.