Equivariant Elliptic Cohomology and Rigidity

by

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ABSTRACT

We show that equivariant elliptic cohomology, as defined by I. Grojnowski, gives a natural cohomological proof of the rigidity theorem of Witten for the elliptic genus. We also state and prove a rigidity theorem for families of elliptic genera, and show the existence for spin vector bundles of a Thom class (section) in S^1-equivariant elliptic cohomology. This in turn allows us to define equivariant elliptic pushforwards with the correct properties.

Finally, we give a description of S^1-equivariant K-theory in terms of equivariant cohomology, and show that a twisted version of the Chern character becomes an isomorphism.

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Contents

1. Introduction

The classical, or level 2, elliptic genus is defined (see Landweber [18], p.56) as the Hirzebruch genus with exponential series the Jacobi sine $s(x)$. It is known to physicists as the supercharge of a supersymmetric nonlinear sigma model, and to mathematicians in connection with the mysterious field of elliptic cohomology (see Segal [23]).

A striking property of the elliptic genus is its rigidity with respect to group actions. This was conjectured by Witten in [26], where he used heuristic quantum field theory arguments to support it.

Rigorous mathematical proofs were soon given by Taubes [25], Bott & Taubes [6], and K. Liu [19]. While Bott & Taubes's proof of the rigidity theorem involved the localization formula in ordinary equivariant cohomology, Liu's proof involved the modularity properties of the elliptic genus. The question remained however whether one could find a direct connection between the rigidity theorem and elliptic cohomology.

Earlier on, Atiyah & Hirzebruch [2] had used pushforwards in equivariant K-theory to prove the rigidity of the $\hat{A}$-genus for spin manifolds. Following this idea, H. Miller [20] interpreted the equivariant elliptic genus as a pushforward in the Borel (completed) equivariant elliptic cohomology, and posed the problem of developing and using a *non-completed* S^1-equivariant elliptic cohomology, which didn't exist at that time, to prove the rigidity theorem.

In 1994 I. Grojnowski [13] proposed a noncompleted equivariant elliptic cohomology with complex coefficients. For G a compact connected Lie group he defined $\boldsymbol{E}_G^*(-)$ as a coherent holomorphic sheaf over a certain variety X_G constructed from a given elliptic curve. Grojnowski also constructed pushforwards in this theory. (At about the same time and independently, Ginzburg, Kapranov and Vasserot [11] gave an axiomatic description of equivariant elliptic cohomology.)

Given Grojnowski's construction, it seemed natural to try to use S^1-equivariant elliptic cohomology to prove the rigidity theorem. In doing so, we noticed that our proof relies on a translation and generalization of Bott & Taubes' transfer formula (see [6]). And this generalization of the transfer formula turns out to be essentially equivalent to the existence of a Thom class (or orientation) in S^1-equivariant elliptic cohomology.

One can generalize the results of this thesis in several directions. One is to extend the rigidity theorem to families of elliptic genera, which we do in Theorem 6.7. Anohter would be to generalize from $G = S^1$ to an arbitrary connected compact Lie group, or to replace complex coefficients with rational coefficients for all cohomology theories involved. Such generalizations will be treated elsewhere.

2. Statement of results

All the cohomology theories involved in this paper have complex coefficients. If X is a finite S^1-CW complex, $H^*_{S^1}(X)$ denotes its ordinary S^1-equivariant cohomology with complex coefficients (for a description of this theory see the paper [1] of Atiyah & Bott).

Let $\mathcal{E}$ be a nonsingular elliptic curve over $\mathbb{C}$. Let X be a finite S^1-CW complex (e.g. a compact S^1-manifold). Then Grojnowski defines the S^1-equivariant elliptic cohomology $\boldsymbol{E}^*_{S^1}(X)$ as a coherent analytic sheaf of superalgebras (i.e. $\mathbb{Z}/2$-graded algebras) over $\mathcal{E}$. His definition uses a choice of a global isomorphism $\mathcal{E} \cong S^1 \times S^1$. We are going to give an invariant definition of $\boldsymbol{E}^*_{S^1}(-)$, as a functor from the category of finite S^1-CW complexes to the category of sheaves of $\mathcal{O}_{\mathcal{E}}$-superalgebras.

THEOREM A. $\boldsymbol{E}^*_{S^1}(-)$ *is an* S^1*-equivariant cohomology theory with values in the category of coherent analytic sheaves of* $\mathcal{O}_{\mathcal{E}}$*-superalgebras.*

If $f : X \to Y$ is a complex oriented map between compact S^1-manifolds, Grojnowski also defines equivariant elliptic pushforwards. They are maps of sheaves of $\mathcal{O}_{\mathcal{E}}$-modules

$$f^E_! : \boldsymbol{E}^*_{S^1}(X)_{twisted} \to \boldsymbol{E}^*_{S^1}(Y)$$

satisfying properties similar to those of a pushforward. $\boldsymbol{E}^*_{S^1}(X)_{twisted}$ has the same stalks as $\boldsymbol{E}^*_{S^1}(X)$, but the gluing maps are different. (See Section 5.)

The construction of $f^E_!$ has two steps. First, one defines *local* pushforwards, at the level of stalks. Then one tries to assemble them into a sheaf map between $\boldsymbol{E}^*_{S^1}(X)$ and $\boldsymbol{E}^*_{S^1}(Y)$. This fails, because local pushforwards do not glue well. However, if one twists $\boldsymbol{E}^*_{S^1}(X)$, i.e. one changes the gluing maps, then $f^E_!$ becomes a map of sheaves from $\boldsymbol{E}^*_{S^1}(X)_{twisted}$ to $\boldsymbol{E}^*_{S^1}(Y)$.

In Grojnowski's preprint, the existence of the local pushforwards is merely stated. In spelling out the details, we realized that the proof (which is given in Corollary 5.4) implies in particular that the equivariant elliptic genus of a compact S^1-manifold is meromorphic everywhere and holomorphic at zero. More precisely, we know that the equivariant genus of an S^1-manifold can be represented as a power series in one variable $u =$ the generator of H^*BS^1. Then we have

PROPOSITION B. *The S^1-equivariant elliptic genus of a compact S^1-manifold is the Taylor expansion at zero of a function on $\mathbb{C}$ which is holomorphic at zero and meromorphic everywhere.*

This question was posed by H. Miller and answered independently by Dessai & Jung [7], who use a result in complex analysis suggested by T. Berger.

Grojnowski's construction raises a few natural questions. First, can we say more about $\boldsymbol{E}^*_{S^1}(X)_{twisted}$? The answer is given in Proposition 6.8, where we show that, up to an invertible sheaf, $\boldsymbol{E}^*_{S^1}(X)_{twisted}$ is the S^1-equivariant elliptic cohomology of the Thom space of $\nu(f)$, the stable normal bundle to f.

This suggests looking for a (Thom) section in $\boldsymbol{E}^*_{S^1}(X)_{twisted}$. More generally, given a real oriented vector bundle $V \to X$, we can twist $\boldsymbol{E}^*_{S^1}(X)$ in a similar way to obtain a sheaf, which we denote by $\boldsymbol{E}^*_{S^1}(X)^{[V]}$. When does such a Thom section in $\boldsymbol{E}^*_{S^1}(X)^{[V]}$ exist? The answer is the following key result:

THEOREM C. *If $V \to X$ is a spin S^1-vector bundle over a finite S^1-CW complex, then the element 1 in the stalk of $\boldsymbol{E}^*_{S^1}(X)^{[V]}$ at zero extends to a global section, called the* ***Thom section****.*

The sheaf $\boldsymbol{E}^*_{S^1}(X)^{[V]}$ is regarded here not on $\mathcal{E}$, but on a double cover $\tilde{\mathcal{E}}$ of $\mathcal{E}$, for reasons explained in the beginning of Section 6.

In proving Theorem C, one comes very close to the proof of the rigidity theorem given by Bott & Taubes in [6]. In fact, Theorem C is essentially a generalization of Bott & Taubes' transfer formula. Armed with this result, the rigidity theorem of Witten follows easily. But the slightly greater level of generality allows us to extend the rigidity theorem to families of elliptic genera. The question of stating and proving such a theorem was posed by H. Miller in [21].

THEOREM D. *(Rigidity for families) Let $\pi : E \to B$ be an S^1-equivariant fibration such that the fibers are spin in a compatible way, i.e. the projection map π is spin oriented. Then the elliptic genus of the family, which is $\pi_!^E(1) \in H_{S^1}^{**}(B)$, is constant as a rational function in u (i.e. if we invert the generator u of $\mathbb{C}[\![u]\!]$, over which $H_{S^1}^{**}(B)$ is a module).*

Here $H^{**}(X)$ denotes the formal Laurent series over $H^*(X)$ (as defined in Dyer [9], p. 58), and $H_{S^1}^{**}(B) = H^{**}(B \times_{S^1} ES^1)$, where $B \times_{S^1} ES^1$ is the Borel construction. $H_{S^1}^{**}(B)$ can be alternatively be thought as the completion of $H_{S^1}^*(B)$ with respect to the ideal generated by u in $H_{S^1}^*(point) = \mathbb{C}[u]$. For every finite S^1-CW complex B we have the formula

$$H_{S^1}^{**}(B) \cong H_{S^1}^*(B) \otimes_{\mathbb{C}[u]} \mathbb{C}[\![u]\!] \ .$$

Notice that, while $H_{S^1}^*(B)$ is a $\mathbb{Z}$-graded object, $H_{S^1}^{**}(B)$ is only $\mathbb{Z}/2$-graded, by its even and odd part.

A different path of research was taken in Section 4. There we show that a sheaf $\mathcal{K}_{S^1}^*(X)^{alg}$ similar to $\boldsymbol{E}_{S^1}^*(X)$, but constructed over $\mathbb{C}^\times$ instead of the elliptic curve $\mathcal{E}$, gives S^1-equivariant K-theory after taking global sections. $\mathcal{K}_{S^1}^*(X)^{alg}$ is constructed, just like $\boldsymbol{E}_{S^1}^*(X)$, by gluing together the equivariant cohomology of subspaces of X fixed by different subgroups of S^1. The map which gives the isomorphism of $K_{S^1}^*(X)$ with the global sections in $\mathcal{K}_{S^1}^*(X)^{alg}$ is some kind of "twisted" equivariant Chern character. This seems to give a satisfactory way in which the S^1-equivariant Chern character from $K_{S^1}^*(X)$ to $H_{S^1}^{**}(X)$ (which is not an isomorphism in general) can be made into an isomorphism, if the target is seen in this sheafified version over $\mathbb{C}^\times$. The fact that the result is the same when we use a general Lie group G represents work in progress.

Baum, Brylinski, and MacPherson [4] give a description of the G-equivariant K-theory of a space X as a sheaf over the space of orbits X/G, but except in very special cases they fail to show the connection with equivariant cohomology. Block and Getzler [5] construct a sheaf, but they obtain cyclic cohomology instead of equivariant K-theory.

The closest to our construction seems to be the work of Duflo and Vergne [8], but the authors cannot prove that their construction yields indeed equivariant K-theory. For one thing, they cannot prove Mayer-Vietoris, the reason being that they work directly with the sections of the sheaf, for which it is much harder to prove Mayer-Vietoris. Mayer-Vietoris is easy to show for our sheaf, though, and we do this in Proposition 3.9. Then we use the fact that for K-theory, the sheaf is algebraic coherent over an affine scheme ($\mathbb{C}^\times$), so taking global sections preserve exactness, by a classical theorem of Grothendieck. The problem with Duflo and Vergne's sheaf is that it is defined using smooth functions on the group, so we don't have the machinery of algebraic geometry to prove general results.

We should observe also that the twisting of the Chern character mentioned above is rather a translation performed over the special points. This resembles a lot the translation we have to perform while proving rigidity, which Bott & Taubes call the "transfer formula". The connection between the twisted Chern character and the transfer formula are, in our opinion, a good motivation for regarding generalized equivariant cohomology theories in this sheafified way. In fact, one could notice that the whole HKR [17] has this sheafifying touch. It is just the fact that they use a finite group, which saves them from being forced to pass to a continuous family of stalks.

3. S^1-equivariant elliptic cohomology

In this section we give, following Grojnowski [13], the construction of S^1-equivariant elliptic cohomology with complex coefficients. Given $\mathcal{E}$ an elliptic curve over $\mathbb{C}$ with structure sheaf $\mathcal{O}_{\mathcal{E}}$, S^1-equivariant elliptic cohomology is defined as a contravariant functor from the category of pairs of finite S^1-CW complexes to the abelian category of coherent analytic sheaves of $\mathcal{O}_{\mathcal{E}}$-modules: $(X, A) \to \boldsymbol{E}^*_{S^1}(X, A)$. Moreover, $\boldsymbol{E}^*_{S^1}(X)$ turns out to be a sheaf of $\mathcal{O}_{\mathcal{E}}$-superalgebras.

Let us start with an elliptic curve $\mathcal{E}$ over $\mathbb{C}$. Choose an identification of $\mathbb{C}$ with the universal cover of $\mathcal{E}$. Let θ be a local inverse around zero to the covering map $\mathbb{C} \to \mathcal{E}$. We call θ an **additive uniformizer**. Any two such uniformizers differ by a nonzero scalar multiple. Fix an additive uniformizer θ.

Definition 3.1. *Fix a neighborhood V_θ of zero in $\mathcal{E}$ such that $\theta : V_\theta \to \theta(V_\theta) \subset \mathbb{C}$ is a homeomorphism. We say that a neighborhood V of $\alpha \in \mathcal{E}$ is* ***small*** *if $t_{-\alpha}(V) \subseteq V_\theta$. $t_{-\alpha}$ is translation by $-\alpha$ in $\mathcal{E}$.*

Let X be a finite S^1-CW complex. Let $\alpha \in \mathcal{E}$. We say that α is a division point of $\mathcal{E}$ of order $n > 0$ if $n\alpha = 0$ and n is the smallest positive number with this property. If $H \subseteq S^1$ is a subgroup, denote by X^H the submanifold of X fixed by each element of H. Let $\mathbb{Z}_n \subseteq S^1$ be the cyclic subgroup of order n.

Definition 3.2.

$$X^\alpha := \begin{cases} X^{\mathbb{Z}_n}, \textit{if } \alpha \textit{ has exact order } n; \\ X^{S^1}, \textit{otherwise.} \end{cases}$$

Suppose we are given an S^1-equivariant map of pairs of S^1-CW complexes $f : (X, A) \to (Y, B)$, i.e. an S^1-equivariant map $f : X \to Y$ such that $f(A) \subseteq B$. A point $\alpha \in \mathcal{E}$ is called ***special*** *with respect to f if at least one of X^α, A^α, Y^α, B^α is not equal to X^{S^1}, A^{S^1}, Y^{S^1}, B^{S^1} respectively. When it is clear what f is, we simply call α special.*

An indexed open cover $\mathcal{U} = (U_\alpha)_{\alpha \in \mathcal{E}}$ of $\mathcal{E}$ is said to be ***adapted*** *to f if it satisfies the following conditions:*

1. *U_α is a small open neighborhood of $\alpha \in \mathcal{E}$;*
2. *If α is not special, then U_α contains no special point;*
3. *If $\alpha \neq \alpha'$ are special points, $U_\alpha \cap U_{\alpha'} = \emptyset$.*

A point $\alpha \in \mathcal{E}$ is called special with respect to the pair (X, A) if it is special with respect to the identity function $id : (X, A) \to (X, A)$. α is called special with respect to X if it is special with respect to the pair $(X, \emptyset)$.

Definition 3.3. *If (X, A) is a pair of finite S^1-CW complexes, we define the holomorphic S^1-equivariant cohomology of (X, A) to be*

$$HO^*_{S^1}(X, A) = H^*_{S^1}(X, A) \otimes_{\mathbb{C}[u]} \mathcal{O}_{\mathbb{C},0} \quad .$$

$\mathcal{O}_{\mathbb{C},0}$ *is the ring of germs of holomorphic functions at zero in the variable* u*, or alternatively it is the subring of* $\mathbb{C}[\![u]\!]$ *of convergent power series with positive radius of convergence.*

We are going to define now a sheaf $\mathcal{F} = \mathcal{F}_{\theta,\mathcal{U}}$ over $\mathcal{E}$ whose stalk at $\alpha \in \mathcal{E}$ is isomorphic to $HO^*_{S^1}(X^\alpha, A^\alpha)$. Recall that, in order to give a sheaf $\mathcal{F}$ over a topological space, it is enough to give an open cover $(U_\alpha)_\alpha$ of that space, and a sheaf $\mathcal{F}_\alpha$ on each U_α together with isomorphisms of sheaves $\phi_{\alpha\beta} : \mathcal{F}_{\alpha|_{U_\alpha \cap U_\beta}} \longrightarrow \mathcal{F}_{\beta|_{U_\alpha \cap U_\beta}}$, such that the cocycle condition $\phi_{\beta\gamma}\phi_{\alpha\beta} = \phi_{\alpha\gamma}$ is satisfied on $U_\alpha \cap U_\beta \cap U_\gamma$. The sheaf $\mathcal{F}$ is unique up to isomorphism, with the condition $\mathcal{F}_{|_{U_\alpha}} \cong \mathcal{F}_\alpha$.

Consider an adapted open cover $\mathcal{U} = (U_\alpha)_{\alpha \in \mathcal{E}}$. Such a cover exists, because X is a finite S^1-CW complex, so the set of special points is a finite subset of $\mathcal{E}$.

Definition 3.4. *Define a sheaf* $\mathcal{F}_\alpha$ *on* U_α *by declaring for any open* $U \subseteq U_\alpha$

$$\mathcal{F}_\alpha(U) := H^*_{S^1}(X^\alpha, A^a a) \otimes_{\mathbb{C}[u]} \mathcal{O}_\mathcal{E}(U - \alpha) \ \ ,$$

where the map $\mathbb{C}[u] \to \mathcal{O}_\mathcal{E}(U - \alpha)$ *is given by the sending* u *to the uniformizer* θ *(we use the smallness of* U_α *here).* $U - \alpha$ *represents the translation of* U *by* $-\alpha$*, and* $\mathcal{O}_\mathcal{E}(U - \alpha)$ *is the ring of holomorphic functions on* $U - \alpha$*. The restriction maps of the sheaf are defined so that they come from those of the sheaf* $\mathcal{O}_\mathcal{E}$*.*

First we notice that we can make $\mathcal{F}_\alpha$ into a sheaf of $\mathcal{O}_{\mathcal{E}\,|U_\alpha}$-modules: if $U \subseteq U_\alpha$, we want an action of $f \in \mathcal{O}_\mathcal{E}(U)$ on $\mathcal{F}_\alpha(U)$. The translation map $t_\alpha : U - \alpha \to U$, which takes u to $u + \alpha$ gives a translation $t^*_\alpha : \mathcal{O}_\mathcal{E}(U) \to \mathcal{O}_\mathcal{E}(U - \alpha)$, which takes $f(u)$ to $f(u + \alpha)$. Then we take the result of the action of $f \in \mathcal{O}_\mathcal{E}(U)$ on $\mu \otimes g \in \mathcal{F}_\alpha(U) = H^*_{S^1}(X^\alpha, A^\alpha) \otimes_{\mathbb{C}[u]} \mathcal{O}_\mathcal{E}(U - \alpha)$ to be $\mu \otimes t^*_\alpha f \cdot g$. Moreover, $\mathcal{F}_\alpha$ is coherent because, since X^α and A^α are finite, $H^*_{S^1}(X^\alpha, A^\alpha)$ is a finitely generated $\mathbb{C}[u]$-module.

Now for the second part of the definition of $\mathcal{F}$, we have to glue the different sheaves $\mathcal{F}_\alpha$ we have just constructed. If $U_\alpha \cap U_\beta \neq \emptyset$ we need to define an isomorphism of sheaves $\phi_{\alpha\beta} : \mathcal{F}_{\alpha|_{U_\alpha \cap U_\beta}} \longrightarrow \mathcal{F}_{\beta|_{U_\alpha \cap U_\beta}}$ which satisfies the cocycle condition. Recall that we started with an adapted open cover $(U_\alpha)_{\alpha \in \mathcal{E}}$. Because of the condition 3 in Definition 3.2, α

and β cannot be both special, so we only have to define $\phi_{\alpha\beta}$ when, say, β is not special. So assume $X^\beta = X^{S^1}$. Consider an arbitrary open set $U \subseteq U_\alpha \cap U_\beta$.

Definition 3.5. *Define $\phi_{\alpha\beta}$ on U as the composite of the following isomorphisms:*

$$(*)\qquad \begin{array}{rcl} \mathcal{F}_\alpha(U) & = & H^*_{S^1}(X^\alpha, A^\alpha) \otimes_{\mathbb{C}[u]} \mathcal{O}_{\mathcal{E}}(U-\alpha) \\ & \xrightarrow{\sim} & H^*_{S^1}(X^\beta, A^\beta) \otimes_{\mathbb{C}[u]} \mathcal{O}_{\mathcal{E}}(U-\alpha) \\ & \xrightarrow{\sim} & (H^*(X^\beta, A^\beta) \otimes_{\mathbb{C}} \mathbb{C}[u]) \otimes_{\mathbb{C}[u]} \mathcal{O}_{\mathcal{E}}(U-\alpha) \\ & \xrightarrow{\sim} & H^*(X^\beta, A^\beta) \otimes_{\mathbb{C}} \mathcal{O}_{\mathcal{E}}(U-\alpha) \\ & \xrightarrow{\sim} & H^*(X^\beta, A^\beta) \otimes_{\mathbb{C}} \mathcal{O}_{\mathcal{E}}(U-\beta) \\ & \xrightarrow{\sim} & H^*_{S^1}(X^\beta, A^\beta) \otimes_{\mathbb{C}[u]} \mathcal{O}_{\mathcal{E}}(U-\beta) \\ & = & \mathcal{F}_\beta(U)\ . \end{array}$$

The map on the second row from the top is the natural one: $i^* \otimes 1$, where $i : (X^\beta, A^\beta) \to (X^\alpha, A^\alpha)$ is the inclusion. $i^* \otimes 1$ is an isomorphism because

a) If α is not special, then $X^\alpha = X^{S^1} = X^\beta$, and similarly $A^\alpha = A^\beta$, so $i^* \otimes 1$ is the identity.

b) If α is special, then either $X^\alpha \neq X^\beta$ or $A^\alpha \neq A^\beta$. However, we have $(X^\alpha)^{S^1} = X^{S^1} = X^\beta$, and similarly $(A^\alpha)^{S^1} = A^\beta$. Then we can use the Atiyah–Bott localization theorem in equivariant cohomology from [1]. This says that $i* : H^*_{S^1}(X^\alpha, A^\alpha) \to H^*_{S^1}(X^\beta, A^\beta)$ is an isomorphism after inverting u. So it is enough to show that θ is invertible in $\mathcal{O}_{\mathcal{E}}(U-\alpha)$, because this would imply that i^* becomes an isomorphism after tensoring with $\mathcal{O}_{\mathcal{E}}(U-\alpha)$ over $\mathbb{C}[u]$. Now, because α is special, the condition 2 in Definition 3.2 says that $\alpha \notin U_\beta$. But $U \subseteq U_\alpha \cap U_\beta$, so $\alpha \notin U$, hence $0 \notin U - \alpha$. This is equivalent to θ being invertible in $\mathcal{O}_{\mathcal{E}}(U-\alpha)$.

The isomorphism on the third row comes from the isomorphism $H^*_{S^1}(X^\beta, A^\beta) = H^*(X^\beta, A^\beta) \otimes_{\mathbb{C}} \mathbb{C}[u]$, since since X^β and A^β are fixed by the S^1-action. The isomorphism on the fifth row is given by the translation $t^*_{\beta-\alpha} : \mathcal{O}_{\mathcal{E}}(U-\alpha) \to \mathcal{O}_{\mathcal{E}}(U-\beta)$.

Remark 3.6. To simplify notation, we can describe $\phi_{\alpha\beta}$ as the composite of the following two maps:

$$\begin{array}{c} H^*_{S^1}(X^\alpha, A^\alpha) \otimes_{\mathbb{C}[u]} \mathcal{O}_{\mathcal{E}}(U-\alpha) \\ \downarrow i^* \\ H^*_{S^1}(X^\beta, A^\beta) \otimes_{\mathbb{C}[u]} \mathcal{O}_{\mathcal{E}}(U-\alpha) \\ \downarrow t^*_{\beta-\alpha} \\ H^*_{S^1}(X^\beta, A^\beta) \otimes_{\mathbb{C}[u]} \mathcal{O}_{\mathcal{E}}(U-\beta) \end{array}\quad .$$

By the first map we really mean $i^* \otimes 1$. The second map is not $1 \otimes t^*_{\beta-\alpha}$, because $t^*_{\beta-\alpha}$ is not a map of $\mathbb{C}[u]$-modules. However, we use $t^*_{\beta-\alpha}$ as a shorthand for the corresponding composite map specified in $(*)$.

One checks easily now that $\phi_{\alpha\beta}$ satisfies the cocycle condition: Suppose we have three open sets U_α, U_β and U_γ such that $U_\alpha \cap U_\beta \cap U_\gamma \neq \emptyset$. Because our cover was chosen to be adapted, at least two out of the three pairs (X^α, A^α), (X^β, A^β) and (X^γ, A^γ) are equal to (X^{S^1}, A^{S^1}). Thus the cocycle condition reduces essentially to $t^*_{\gamma-\beta} t^*_{\beta-\alpha} = t^*_{\gamma-\alpha}$. This completes the definition of $\mathcal{F} = \mathcal{F}_{\theta,\mathcal{U}}$. One can check easily that $\mathcal{F}$ is a coherent analytic sheaf of $\mathcal{O}_{\mathcal{E}}$-superalgebras.

We can remove the dependence of $\mathcal{F}$ on the adapted cover $\mathcal{U}$ as follows: Let $\mathcal{U}$ and $\mathcal{V}$ be two covers adapted to (X, A). Then any common refinement $\mathcal{W}$ is going to be adapted as well, and the corresponding maps of sheaves $\mathcal{F}_{\theta,\mathcal{U}} \to \mathcal{F}_{\theta,\mathcal{W}} \leftarrow \mathcal{F}_{\theta,\mathcal{V}}$ are isomoprhisms on stalks, hence isomorphisms of sheaves. Therefore we can omit the subscript $\mathcal{U}$, and write $\mathcal{F} = \mathcal{F}_\theta$.

Next we want to show that $\mathcal{F}_\theta$ is independent of the choice of the additive uniformizer θ.

Proposition 3.7. *If θ and θ' are two additive uniformizers, then there exists an isomorphism of sheaves of $\mathcal{O}_{\mathcal{E}}$-superalgebras $f_{\theta\theta'} : \mathcal{F}_\theta \to \mathcal{F}_{\theta'}$. If θ'' is a third additive uniformizer, then $f_{\theta'\theta''} f_{\theta\theta'} = \pm f_{\theta\theta''}$.*

Proof. We modify slightly the notations used in Definition 3.4 to indicate the dependence on θ:

$$\mathcal{F}^{\theta}_{\alpha}(U) := H^*_{S^1}(X^\alpha, A^a a) \otimes^{\theta}_{\mathbb{C}[u]} \mathcal{O}_{\mathcal{E}}(U-\alpha) \ .$$

(u is sent to θ via the algebra map $\mathbb{C}[u] \to \mathcal{O}_{\mathcal{E}}(U-\alpha)$). If θ' is another additive uniformizer, there exists a nonzero constant $c \in \mathbb{C}$ such that $\theta = c\theta'$. Choose a square root of c and denote it by $c^{1/2}$ (we need the square root because u is in homogeneous degree 2). For a homogeneous element $x \in H^*_{S^1}(X^\alpha, A^a a)$, define a map $f_{\theta\theta',\alpha} : \mathcal{F}^{\theta}_{\alpha}(U) \to \mathcal{F}^{\theta'}_{\alpha}(U)$ by

$$x \otimes^{\theta} g \mapsto c^{|x|/2} x \otimes^{\theta'} g \ .$$

$|x|$ is the homogeneous degree of x. One can easily check that $f_{\theta\theta',\alpha}$ is a map of sheaves of $\mathcal{O}_{\mathcal{E}}$-superalgebras. We also have $\phi^{\theta'}_{\alpha\beta} \circ f_{\theta\theta',\alpha} = f_{\theta\theta',\beta} \circ \phi^{\theta}_{\alpha\beta}$, which means that the maps $f_{\theta\theta',\alpha}$ glue to define a map of sheaves $f_{\theta\theta'} : \mathcal{F}_{\theta} \to \mathcal{F}_{\theta'}$.

The equality $f_{\theta'\theta''} f_{\theta\theta'} = \pm f_{\theta\theta''}$ comes from $(\theta'/\theta'')^{1/2}(\theta/\theta')^{1/2} = \pm(\theta/\theta'')^{1/2}$. □

Definition 3.8. *The S^1-equivariant elliptic cohomology of the pair (X, A) is defined to be the sheaf $\mathcal{F}$, which, according to the previous results does not depend on the adapted open cover $\mathcal{U}$ or on the additive uniformizer θ.*

For $\boldsymbol{E}^*_{S^1}(-)$ to be a cohomology theory, we also need naturality. Let $f : (X, A) \to (Y, B)$ be an S^1-equivariant map of pairs of finite S^1-CW complexes. We want to define a map of sheaves $f^* : \boldsymbol{E}^*_{S^1}(Y, B) \to \boldsymbol{E}^*_{S^1}(X, A)$ with the properties that $1^*_{(X,A)} = 1_{\boldsymbol{E}^*_{S^1}(X,A)}$ and $(fg)^* = g^* f^*$. Choose $\mathcal{U}$ an open cover adapted to f, and θ an additive uniformizer of $\mathcal{E}$. Since f is S^1-equivariant, for each α we get by restriction a map of pairs $f_\alpha : (X^\alpha, A^\alpha) \to (Y^\alpha, B^\alpha)$. This induces a map

$$H^*_{S^1}(Y^\alpha, B^\alpha) \otimes_{\mathbb{C}[u]} \mathcal{O}_{\mathcal{E}}(U-\alpha) \xrightarrow{f^*_\alpha \otimes 1} H^*_{S^1}(X^\alpha, A^\alpha) \otimes_{\mathbb{C}[u]} \mathcal{O}_{\mathcal{E}}(U-\alpha) \ .$$

To get our global map f^*, we only have to check that the maps f^*_α glue well, i.e. that they commute with the gluing maps $\phi_{\alpha\beta}$. This follows easily from the naturality of ordinary equivariant cohomology, and from the naturality in (X, A) of the isomorphism

$H^*_{S^1}(X^{S^1}, A^{S^1}) \cong H^*(X^{S^1}, A^{S^1}) \otimes_{\mathbb{C}} \mathbb{C}[u]$. Now we are in the position to state the theorem–definition of Grojnowski S^1-equivariant elliptic cohomology.

Theorem 3.9. $\boldsymbol{E}^*_{S^1}(-)$ *is an* S^1*-equivariant cohomology theory with values in the category of coherent analytic sheaves of* $\mathcal{O}_{\mathcal{E}}$*-superalgebras.*

Proof. First we have to define the coboundary map $\delta : \boldsymbol{E}^*_{S^1}(A) \to \boldsymbol{E}^{*+1}_{S^1}(X, A)$. This is obtained by gluing the maps

$$H^*_{S^1}(A^\alpha) \otimes_{\mathbb{C}[u]} \mathcal{O}_{\mathcal{E}}(U - \alpha) \xrightarrow{\delta_\alpha \otimes 1} H^{*+1}_{S^1}(X^\alpha, A^\alpha) \otimes_{\mathbb{C}[u]} \mathcal{O}_{\mathcal{E}}(U - \alpha) \ ,$$

where $\delta_\alpha : H^*_{S^1}(A^\alpha) \to H^{*+1}_{S^1}(X^\alpha, A^\alpha)$ is the usual coboundary map. The maps $\delta_\alpha \otimes 1$ glue well, because δ_α is natural.

To check the usual axioms of a cohomology theory (naturality, exact sequence of a pair, and excision) for $\boldsymbol{E}^*_{S^1}(-)$, recall that it was obtained by gluing the sheaves $\mathcal{F}_\alpha$ along the maps $\phi_{\alpha\beta}$. Since the sheaves $\mathcal{F}_\alpha$ were defined using $H^*_{S^1}(X^\alpha, A^\alpha)$, the properties of ordinary S^1-equivariant cohomology pass on to $\boldsymbol{E}^*_{S^1}(X, A)$. □

This proves THEOREM A stated in Section 2. Here perhaps we should mention that one can make $\boldsymbol{E}^*_{S^1}(-)$ to take values in the category of coherent *algebraic* sheaves over $\mathcal{E}$ rather than the category of coherent *analytic* sheaves. This follows from a theorem of Serre which says that the the categories of coherent holomorphic sheaves and coherent algebraic sheaves over a projective variety (in particular over $\mathcal{E}$) are equivalent. (See for example [12], Theorem A, p. 75.)

Now the description we gave above for $\boldsymbol{E}^*_{S^1}(X)$ was good to prove that $\boldsymbol{E}^*_{S^1}(-)$ is a cohomology theory, but it is hard to work with it in practice. This is because the open cover $(U_\alpha)_{\alpha \in \mathcal{E}}$ has too many elements. To remedy this, we are going to use a finite cover of $\mathcal{E}$: Start with an adapted open cover $(U_\alpha)_{\alpha \in \mathcal{E}}$. Recall that the set of special points with respect to X is finite. Denote this set by $\{\alpha_1, \dots, \alpha_n\}$. Denote by $U_{\alpha_0} := \mathcal{E} \setminus \{\alpha_1, \dots, \alpha_n\}$. To simplify notation, denote by $U_i := U_{\alpha_i}$, for all $0 \leq i \leq n$.

Definition 3.10. *(See Definitions 3.4 and 3.5.) On each* U_i*, with* $0 \leq i \leq n$*, we are going to define a sheaf* $\mathcal{G}$ *as follows:*

a) *If* $1 \le i \le n$*, then* $\forall U \subseteq U_i$*,*

$$\mathcal{G}_i(U) := H^*_{S^1}(X^{\alpha_i}) \otimes_{\mathbb{C}[u]} \mathcal{O}_{\mathcal{E}}(U - \alpha_i) \ \ .$$

The map $\mathbb{C}[u] \to \mathcal{O}_{\mathcal{E}}(U - \alpha_i)$ *was described in Definition 3.4.*

b) *If* $i = 0$*, then* $\forall U \subseteq U_0$*,*

$$\mathcal{G}_i(U) := H^*(X^{S^1}) \otimes_{\mathbb{C}} \mathcal{O}_{\mathcal{E}}(U) \ \ .$$

Now glue each $\mathcal{G}_i$ *to* $\mathcal{G}_0$ *via the map of sheaves* ϕ_{i0} *defined as the composite of the following isomomorphisms (*$U \subseteq U_i \cap U_0$*):* $H^*_{S^1}(X^{\alpha_i}) \otimes_{\mathbb{C}[u]} \mathcal{O}_{\mathcal{E}}(U - \alpha_i) \xrightarrow{i^* \otimes 1} H^*_{S^1}(X^{S^1}) \otimes_{\mathbb{C}[u]} \mathcal{O}_{\mathcal{E}}(U - \alpha_i) \xrightarrow{\cong} H^*(X^{S^1}) \otimes_{\mathbb{C}} \mathcal{O}_{\mathcal{E}}(U - \alpha_i) \xrightarrow{t^*_{-\alpha_i}} H^*(X^{S^1}) \otimes_{\mathbb{C}} \mathcal{O}_{\mathcal{E}}(U)$*.*

Since there cannot be three distinct U_i with nonempty intersection, there is no cocycle condition to verify.

Claim 3.11. *The sheaf* $\mathcal{G}$ *described in the previous definition is isomorphic to* $\mathcal{F}$*, thus allowing an alternative definition of* $\boldsymbol{E}^*_{S^1}(X)$*.*

Proof. One notices that $U_0 = \cup\{U_\beta \mid \beta \text{ nonspecial}\}$, because of the third condition in the definition of an adapted cover. If $U \subseteq \cup_\beta U_\beta$, a global section in $\mathcal{F}(U)$ is a collection of sections $s_\beta \in \mathcal{F}(U \cap U_\beta - \beta)$ which glue, i.e. $t^*_{\beta-\beta'} s_\beta = s_{\beta'}$. So $t^*_{-\beta} s_\beta = t^*_{-\beta'} s_{\beta'}$ in $\mathcal{G}(U \cap U_\beta \cap U_{\beta'})$, which means that we get an element in $\mathcal{G}(U)$, since the U_β's cover U. So $\mathcal{F}_{|U_0} \cong \mathcal{G}_{|U_0}$. But clearly $\mathcal{F}_{|U_i} \cong \mathcal{G}_{|U_i}$ for $1 \le i \le n$, and the gluing maps are compatible. Therefore $\mathcal{F} \cong \mathcal{G}$. □

As it is the case with any coherent sheaf of $\mathcal{O}_{\mathcal{E}}$-modules over an elliptic curve, $\boldsymbol{E}^*_{S^1}(X)$ splits (noncanonically) into a direct sum of a locally free sheaf, i.e. the sheaf of sections of some holomorphic vector bundle, and a sum of skyscraper sheaves.

Given a particular X, we can be more specific: We know that $H^*_{S^1}(X)$ splits noncanonically into a free and a torsion $\mathbb{C}[u]$-module. Given such a splitting, we can speak of the free part of $H^*_{S^1}(X)$. Denote it by $H^*_{S^1}(X)_{free}$. The map

$$H^*_{S^1}(X)_{free} \xrightarrow{i^*} H^*_{S^1}(X^{S^1})$$

is an injection of finitely generated free $\mathbb{C}[u]$-modules of the same rank (by the localization theorem). $\mathbb{C}[u]$ is a p.i.d., so by choosing appropriate bases in $H^*_{S^1}(X)_{free}$ and $H^*_{S^1}(X^{S^1})$, the map i^* can be written as a diagonal matrix $D(u^{n_1},\dots,u^{n_k})$, $n_i \geq 0$. Since $i^*1 = 1$, we can choose $n_1 = 0$.

So at the special points α_i, we have the map $H^*_{S^1}(X^{\alpha_i})_{free} \xrightarrow{i^*} H^*_{S^1}(X^{S^1})$, which in appropriate bases can be written as a diagonal matrix $D(1, u^{n_2},\dots,u^{n_k})$. This gives over $U_i \cap U_0$ the transition functions $u \mapsto D(1, u^{n_2},\dots,u^{n_k}) \in GL(n,\mathbb{C})$. However, we have to be careful since the basis of $H^*_{S^1}(X^{S^1})$ changes with each α_i, which means that the transition functions are diagonal only up to a (change of base) matrix. But this matrix is invertible over $\mathbb{C}[u]$, so we get that the free part is a sheaf of sections of a holomorphic vector bundle, and moreover we can describe it explicitly.

An interesting question is what holomorphic vector bundles one gets if X varies. Recall that holomorphic vector bundles over elliptic curves were classified by Atiyah in 1957.

Example 3.12. Calculate $\boldsymbol{E}^*_{S^1}(X)$ for $X = S^2(n)$ = the 2-sphere with the S^1-action which rotates S^2 n times around the north-south axis as we go once around S^1. If α is an n-division point, then $X^\alpha = X$. Otherwise, $X^\alpha = X^{S^1}$, which consists of two points: $\{P_+, P_-\}$, the North and the South poles. Now $H^*_{S^1}(S^2(n)) = H^*(BS^1 \vee BS^1) = \{(f,g) \in \mathbb{C}[u] \oplus \mathbb{C}[u] \mid f(0) = g(0)\}$. Denote this by $\mathbb{C}[u] \oplus_0 \mathbb{C}[u]$. $\mathbb{C}[u]$ acts diagonally. $H^*_{S^1}(X) \xrightarrow{i^*} H^*_{S^1}(X^{S^1})$ is the inclusion $\mathbb{C}[u] \oplus_0 \mathbb{C}[u] \hookrightarrow \mathbb{C}[u] \oplus \mathbb{C}[u]$.

Choose the bases

a) $\{(1,1),(u,0)\}$ of $\mathbb{C}[u] \oplus_0 \mathbb{C}[u]$;

b) $\{(1,1),(1,0)\}$ of $\mathbb{C}[u] \oplus \mathbb{C}[u]$.

Then $H^*_{S^1}(X) \xrightarrow{\sim} \mathbb{C}[u] \oplus \mathbb{C}[u]$ by $(P(u),Q(u)) \mapsto (P, \frac{Q-P}{u})$, and $H^*_{S^1}(X^{S^1}) \xrightarrow{\sim} \mathbb{C}[u] \oplus \mathbb{C}[u]$ by $(P(u),Q(u)) \mapsto (P, Q-P)$. Hence i^* is given by the diagonal matrix $D(1,u)$. So $\boldsymbol{E}^*_{S^1}(X)$ looks locally like $\mathcal{O}_{\mathbb{C}P^1} \oplus \mathcal{O}_{\mathbb{C}P^1}(-1)$. This happens at all the n-division points of $\mathcal{E}$, so $\boldsymbol{E}^*_{S^1}(X) \cong \mathcal{O}_{\mathcal{E}} \oplus \mathcal{O}_{\mathcal{E}}(\Delta)$, where Δ is the divisor which consists of all n-division points of $\mathcal{E}$, with multiplicity 1.

One can also check that the sum of all n-division points is zero, so by Abel's theorem the divisor Δ is linearly equivalent to $-n^2 \cdot 0$. Thus $\boldsymbol{E}^*_{S^1}(S^2(n)) \cong \mathcal{O}_{\mathcal{E}} \oplus \mathcal{O}_{\mathcal{E}}(-n^2 \cdot 0)$. We stress that the decomposition is only true as sheaves of $\mathcal{O}_{\mathcal{E}}$-modules.

Remark 3.13. Notice that $S^2(n)$ is the Thom space of the S^1-vector space $\mathbb{C}(n)$, where z acts on $\mathbb{C}$ by complex multiplication with z^n. This means that the Thom isomorphism doesn't hold in S^1-equivariant elliptic cohomology, because $\boldsymbol{E}^*_{S^1}(point) = \mathcal{O}_{\mathcal{E}}$, while the reduced S^1-equivariant elliptic cohomology of the Thom space $\tilde{\boldsymbol{E}}^*_{S^1}(S^2(n)) = \mathcal{O}_{\mathcal{E}}(-n^2 \cdot 0)$.

Coming back to a previous observation, from the work of Serre, GAGA [24], we know that the categories of coherent holomorphic sheaves and coherent algebraic sheaves over a projective variety are equivalent. Hence we could replace $\boldsymbol{E}^*_{S^1}(X)$, which is holomorphic, by its algebraic correspondent via the GAGA functor. This means that the S^1-equivariant elliptic cohomology functor $\boldsymbol{E}^*_{S^1}(-)$ can be made to take values in the category of algebraic coherent sheaves over the elliptic curve $\mathcal{E}$.

However, for the purposes of Section4, we would like to describe the algebraic version of $\boldsymbol{E}^*_{S^1}(-)$ as a subsheaf of the holomorphic one. For this, it is enough to have a criterion which tells us when a local section in the holomorphic sheaf is algebraic. So, to work in complete generality, let $\mathcal{F}^{hol}$ be a holomorphic coherent sheaf over the projective variety X. Denote by $\mathcal{O}_X^{hol}$ and $\mathcal{O}_X^{alg}$ the holomorphic and the algebraic structure sheaves of X, respectively. If $\mathcal{G}^{alg}$ is some algebraic coherent sheaf over X, then we can associate canonically to it a holomorphic one by $\mathcal{G}^{alg} \mapsto \mathcal{G}^{hol} := \mathcal{G}^{alg} \otimes_{\mathcal{O}_X^{alg}} \mathcal{O}_X^{hol}$. For such a sheaf $\mathcal{G}^{hol}$ there is a canonical notion of an algebraic section over any Zariski open $U \subseteq X$: $s \in \mathcal{G}^{hol}(U)$ is algebraic if and only if it is in the image of the inclusion map $\mathcal{G}^{alg}(U) \hookrightarrow \mathcal{G}^{hol}(U)$.

Now coming back to $\mathcal{F}^{hol}$, by GAGA we know that there is an algebraic coherent sheaf $\mathcal{G}^{alg}$ over X such that $\mathcal{F}^{hol}$ holomorphically isomorphic via Φ to $\mathcal{G}^{hol} = \mathcal{G}^{alg} \otimes_{\mathcal{O}_X^{alg}} \mathcal{O}_X^{hol}$. Let $s \in \mathcal{F}^{hol}(U)$ be some holomorphic section over the Zariski open U. We say that s is algebraic if $\Phi(s)$ is algebraic in $\mathcal{G}^{hol}(U)$. To check that this definition is independent of the choice of $\mathcal{G}$ and Φ, let $\mathcal{H}$ and Ψ be other two similar choices. Then $\Psi \circ \Phi^{-1}$ is a holomorphic section in the holomorphic coherent sheaf $\mathrm{Hom}(\mathcal{G}^{hol}, \mathcal{H}^{hol})$. For this sheaf we have a well defined notion of an algebraic section, and by applying GAGA again, it follows that a global holomorphic section has to be algebraic. This means that $\Psi \circ \Phi^{-1}$ takes algebraic sections to algebraic sections, so our definition of an algebraic section of $\mathcal{F}^{hol}$ is well defined. So we can define now the algebraic version of $\mathcal{F}^{hol}$ by

$$\mathcal{F}^{alg}(U) := \{s \in \mathcal{F}^{hol}(U) \mid s \text{ is algebraic}\} \ .$$

4. S^1-Equivariant K-Theory

The purpose of this section is to motivate the definition we gave for the S^1-equivariant elliptic cohomology. We are going to show that a sheaf construction entirely parallel to the one in the previous section, but using the multiplicative algebraic group $\mathbb{C}^\times$ instead of the elliptic curve $\mathbb{C}/\Lambda$, yields indeed S^1-equivariant K-theory, after taking the global sections. Again the division points of the algebraic group (for $\mathbb{C}^\times$ they are the roots of 1) will come to play an essential role.

For S^1-equivariant cohomology the appropriate algebraic group is $\mathbb{C}$ with addition, so here there is only one division point: zero. Therefore, $H^*_{S^1}(X)$ itself can be regarded as a sheaf, whose only special part lies at zero (everywhere else the sheaf is built using $H^*_{S^1}(X^{S^1})$). In this sense, equivariant cohomology is the simplest such equivariant theory, and one can expect other theories to be built out of it.

This sheaf construction of S^1-equivariant K-theory obtained by gluing together the S^1-equivariant cohomology of various subcomplexes X^α of X also answers in a satisfactory way the problem that the equivariant Chern character $K^*_{S^1}(X) \xrightarrow{ch_{S^1}} H^{**}_{S^1}(X)$ fails to be and isomorphism, although the nonequivariant Chern character $K^*(X) \xrightarrow{ch} H^{**}(X)$ is. (Of course, we work at least with rational coefficients.) In fact, we will see that a suitably modified equivariant Chern character gives indeed an isomorphism, when we use instead of $H^{**}_{S^1}(X)$ the above mentioned sheaf construction.

We are going to discuss elsewhere the full story on the description of G-equivariant K-theory using G-equivariant cohomology. For the purposes of this paper however, we will restrict ourselves to the case when $G = S^1$.

A good reference concerning equivariant K-theory, is G. Segal's paper [22]. If X is a finite S^1-CW complex, then its S^1-equivariant K-theory $K^*_{S^1}(X)$ is a finitely generated module over $K^*_{S^1}(pt) = R(S^1)$, the representation ring of S^1. If λ is the representation of S^1 given by the inclusion $S^1 \hookrightarrow \mathbb{C}^\times = \operatorname{End}\mathbb{C}$, then $R(S^1) = \mathbb{C}[\lambda, \lambda^{-1}]$, the ring of Laurent polynomials in λ. But $\operatorname{Spec}\mathbb{C}[\lambda^{\pm 1}] = \mathbb{C}^\times$, hence one can regard $K^*_{S^1}(X)$ as a coherent algebraic sheaf over $\mathbb{C}^\times$ (see Hartshorne [14]). The stalk of this sheaf at a point a in $\mathbb{C}^\times$ is the localization $K^*_{S^1}(X)_{(\lambda-a)}$ with respect to the maximal ideal in $\mathbb{C}[\lambda^{\pm 1}]$ generated by $\lambda - a$.

On the other hand, we will see that by simply transposing the definition of $\boldsymbol{E}^*_{S^1}(X)$ using $\Lambda = 2\pi i\mathbb{Z} \subset \mathbb{C}$ instead of a lattice in $\mathbb{C}$, one obtains a holomorphic coherent sheaf $\mathcal{K}^*_{S^1}(X)^{hol}$ over $\mathbb{C}^\times$. Notice that this is built essentially out of the equivariant cohomology of subcomplexes X^α of X fixed by different subgroups of S^1. The sheaf $\mathcal{K}^*_{S^1}(X)^{hol}$ extends naturally to $\mathbb{C}P^1$, so by GAGA we have a well defined notion of an algebraic section on any Zariski open of $\mathbb{C}P^1$ (see the discussion at the end of Section 3). So we denote by $\mathcal{K}^*_{S^1}(X)^{alg}$ the sheaf of algebraic sections of $\mathcal{K}^*_{S^1}(X)^{hol}$, and we can use the same notation for its restriction to the Zariski open $\mathbb{C}^\times$.

So we have two coherent algebraic sheaves over $\mathbb{C}^\times$, and it is natural to try to compare them. In fact, they turn out to be isomorphic. For this, we will define a natural multiplicative map

$$K^*_{S^1}(X) \xrightarrow{\boldsymbol{ch}_{S^1}} \Gamma\mathcal{K}^*_{S^1}(X)^{alg} \quad ,$$

which is built, not surprisingly , out of the equivariant Chern character of the X^α's. However, over the division points of $\mathbb{C}^\times$, i.e. over the roots of 1, the equivariant Chern character has to be twisted in a certain sense. Or, rather, one should call it a translation of the Chern character. This bears a striking resemblance to the translation we have to perform while dealing with the rigidity of the elliptic genus. There the phenomenon is called "transfer", from the transfer formula of Bott & Taubes [6]. This resemblance indicates that sheafifying equivariant cohomology theories is not as unnatural as it might first seem.

To check that a map of S^1-equivariant cohomology theories is an isomorphism over the finite S^1-CW complexes, we only have to check that it gives an isomorphism on the "equivariant points", i.e. on the orbits of the form S^1/H, with $H \subseteq S^1$ a subgroup.

In fact, to understand better the constructions used in this section, it is a good idea to start by looking at what happens for $X = S^1/\mathbb{Z}_n$. Recall that the equivariant Chern character

$$K^*_{S^1}(X) \xrightarrow{ch_{S^1}} H^{**}_{S^1}(X)$$

is defined, if $E \to X$ is a complex S^1-vector bundle over a finite finite S^1-CW complex, by $ch_{S^1}(E) = e^{x_1} + \cdots + e^{x_n}$, where $x_1, \ldots, x_n$ are the equivariant Chern roots of E (see Definition A.3 in the Appendix). ch_{S^1} is multiplicative, but unlike the nonequivariant case it fails to be a rational isomorphism.

Consider now $X = S^1/\mathbb{Z}_n$,

$$K^*_{S^1}(S^1/\mathbb{Z}_n) \xrightarrow{\sim} R(\mathbb{Z}_n) = \mathbb{C}[\mathbb{Z}_n] = \mathbb{C}[\lambda^{\pm 1}]/(\lambda^n - 1) \ \ .$$

This isomorphism sends a complex S^1-vector bundle to its fiber over a point. Since $S^1/\mathbb{Z}_n$ is fixed by $\mathbb{Z}_n$, the fiber is a representation of $\mathbb{Z}_n$. The inverse of the above isomorphism sends a $\mathbb{Z}_n$-module V to its Borel construction $S^1 \times_{\mathbb{Z}_n} V \to S^1/\mathbb{Z}_n$. $\mathbb{C}[\mathbb{Z}_n]$ is the group ring of $\mathbb{Z}_n$. The generator λ represents the bundle $S^1 \times_{\mathbb{Z}_n} \mathbb{C}(1) \to S^1/\mathbb{Z}_n$, where $\mathbb{C}(1)$ is the standard representation of S^1 restricted to $\mathbb{Z}_n$.

$$H^*_{S^1}(S^1/\mathbb{Z}_n) = H^*(S^1/\mathbb{Z}_n \ \times_{S^1} \ ES^1) = H^*(ES^1/\mathbb{Z}_n) = H^*(B\mathbb{Z}_n) = \mathbb{C} \ \ .$$

$B\mathbb{Z}_n$ is the classifying space of $\mathbb{Z}_n$. $H^*(B\mathbb{Z}_n;\mathbb{Z}) = \mathbb{Z}[u]/(nu)$ is torsion in degree higher than zero, so the complex cohomology $H^*(B\mathbb{Z}_n) = \mathbb{C}$.

One can check that $\mathbb{C}[\mathbb{Z}_n]$ splits as a $\mathbb{C}$-algebra into a direct product $\mathbb{C} \oplus \cdots \oplus \mathbb{C}$, n copies. This splitting is given by the idempotents $I_\epsilon = \frac{1}{n}(1 + \epsilon^{-1}\lambda + \cdots + \epsilon^{-(n-1)}\lambda^{n-1})$, one for each root of unity ϵ. Say $\epsilon_n = e^{\frac{2\pi i}{n}}$ is the generator of $\mathbb{Z}_n$. Then the direct product decomposition of $\mathbb{C}[\mathbb{Z}_n]$ can be rewritten as

$$\mathbb{C}[\mathbb{Z}_n] \xrightarrow{\sim} \mathbb{C}_{(1)} \oplus \mathbb{C}_{(\epsilon_n)} \oplus \mathbb{C}_{(\epsilon_n^2)} \oplus \cdots \oplus \mathbb{C}_{(\epsilon_n^{n-1})} \ \ .$$

To describe this isomorphism, take $\lambda \in \mathbb{C}[\mathbb{Z}_n]$ and multiply it by the idempotents I_ϵ. Then $\lambda I_\epsilon = \epsilon I_\epsilon$. So the map is

$$\lambda \mapsto (1, \epsilon_n, \epsilon_n^2, \dots, \epsilon_n^{n-1}) \ .$$

Now $ch_{S^1}(\lambda) = e^{c_1(\lambda)_{S^1}} = e^0 = 1$, since $c_1(\lambda)_{S^1} \in H^2_{S^1}(S^1/\mathbb{Z}_n) = 0$. This means that, if we want to have an isomorphism

$$K^*_{S^1}(S^1/\mathbb{Z}_n) \xrightarrow{\sim} H^*_{S^1}(S^1/\mathbb{Z}_n)_{(1)} \oplus H^*_{S^1}(S^1/\mathbb{Z}_n)_{(\epsilon_n)} \oplus \cdots \oplus H^*_{S^1}(S^1/\mathbb{Z}_n)_{(\epsilon_n^{n-1})} \ ,$$

we need to send λ not to $(ch_{S^1}(\lambda), \dots, ch_{S^1}(\lambda)) = (1, \dots, 1)$, but to

$$(ch_{S^1}(\lambda), t^*_{\epsilon_n} ch_{S^1}(\lambda), \dots, t^*_{\epsilon_n^{n-1}} ch_{S^1}(\lambda)) = (1, \epsilon_n, \epsilon_n^2, \dots, \epsilon_n^{n-1}) \ .$$

We will define this translation $t^*_\epsilon ch(-)_{S^1}$ of the Chern character later, but at least we know that, when $X = S^1/\mathbb{Z}_n$, $t^*_\epsilon ch_{S^1}(\lambda)$ should equal ϵ.

So, if $X = S^1/\mathbb{Z}_n$, one checks easily that (see Definition 3.2) $X^\alpha = X$ if α is an n'th root of unity, and $X^\alpha = \emptyset$ otherwise. This means that, if we define a sheaf $\mathcal{K}$ over $\mathbb{C}^\times$ such that the stalk at α is $\mathcal{K}_\alpha = HO^*_{S^1}(X)$, this will be a skyscraper sheaf with nonzero stalks only at the n'th roots of unity. At an n'th root of unity α, $\mathcal{K}_\alpha = \mathbb{C}[u]/(u) \otimes_{\mathbb{C}[u]} \mathcal{O}_{\mathbb{C},0} = \mathcal{O}_{\mathbb{C},0}/(u) = \mathbb{C}$. Thus $\Gamma\mathcal{K} = \mathbb{C} \oplus \cdots \oplus \mathbb{C}$, n copies, and "ch_{S^1}", the twisted equivariant Chern character map mentioned above is an isomorphism $K^*_{S^1}(S^1/\mathbb{Z}_n) \xrightarrow{\sim} \Gamma\mathcal{K}$.

This points to the general strategy we will take. Consider X a finite S^1-CW complex. Define a coherent sheaf $\mathcal{K}^*_{S^1}(X)^{hol}$ over $\mathbb{C}^\times$ in the same way we defined $\boldsymbol{E}^*_{S^1}(X)^{hol}$ over $\mathcal{E} = \mathbb{C}/\Lambda$ in Section 3. The results of that section apply also to $\mathcal{K}^*_{S^1}(X)^{hol}$, including the simpler description 3.10 and the existence of the algebraic version. We simply have to make Λ mean not a lattice in $\mathbb{C}$, but the subgroup $2\pi i\mathbb{Z} \subset \mathbb{C}$. There is one problem though: GAGA is not valid over $\mathbb{C}^\times$, since $\mathbb{C}^\times$ is not a projective variety. But luckily we notice that $\mathcal{K}^*_{S^1}(X)^{hol}$ has a natural extension over $\mathbb{C}P^1$. This can be seen more easily using the simpler description 3.10, which uses only a finite cover of $\mathbb{C}/\Lambda$. In this case the sheaf is trivial over the U_0, the complement of the special points, so we can use the same gluing maps $\phi_{\alpha\beta}$ to extend the sheaf at 0 and ∞. We will use the same notation $\mathcal{K}^*_{S^1}(X)^{hol}$ for the sheaf extended over $\mathbb{C}P^1$.

Now $\mathbb{C}P^1$ is projective, so GAGA applies as in the end of Section 3 to allow us to define what algebraic sections of $\mathcal{K}^*_{S^1}(X)^{hol}$ mean, on all Zariski opens of $\mathbb{C}P^1$.

Definition 4.1. *If U is a Zariski open in $\mathbb{C}P^1$,*

$$\mathcal{K}^*_{S^1}(X)^{alg}(U) := \{s \in \mathcal{K}^*_{S^1}(X)^{hol}(U) \mid s \text{ is algebraic}\} \ .$$

This is a coherent algebraic sheaf over $\mathbb{C}P^1$, but we can also use the same notation to denote its restriction to $\mathbb{C}^\times$.

Proposition 4.2. *$\mathcal{K}^*_{S^1}(X)^{alg}(-)$ is an S^1-equivariant cohomology theory with values in the category of coherent algebraic sheaves of $\mathcal{O}_{\mathbb{C}^\times}$-superalgebras. The sections over $\mathbb{C}^\times$, $\Gamma\mathcal{K}^*_{S^1}(X)^{alg}(-)$ is also an S^1-equivariant cohomology theory, with values in the category of $\mathbb{C}[\lambda^{\pm 1}]$-superalgebras.*

Proof. The first part of the proposition has the same proof as Proposition 3.9. For the second part, notice that taking global sections over $\mathbb{C}^\times$ is an exact functor, due to the vanishing of the higher sheaf cohomology groups over the affine variety $\mathbb{C}^\times$. (See Hartshorne [14], p. 215.) □

The next step is to provide a multiplicative map of S^1-equivariant cohomology theories

$$K^*_{S^1}(X) \xrightarrow{\boldsymbol{ch}_{S^1}} \Gamma\mathcal{K}^*_{S^1}(X)^{alg} \ .$$

Take X a finite S^1-CW complex. Given a complex S^1-vector bundle $E \to X$, we would like to associate a section in $\mathcal{K}^*_{S^1}(X)^{alg}$. We are going to construct a section in $\mathcal{K}^*_{S^1}(X)^{hol}$, and then show that the section is algebraic. If $\alpha_1, \ldots, \alpha_r$ are the special points corresponding to X (so $X^{\alpha_i} \neq X^{S^1}$), we choose an open cover $U_0, U_1, \ldots, U_r$ as in the discussion before Definition 3.10, with $U_0 = \mathbb{C}^\times \setminus \{\alpha_1, \ldots, \alpha_r\}$.

Over U_0 consider the section $s_0 = ch_{S^1}(E_{|X^{S^1}})$, where $E_{|X^{S^1}}$ is the restriction of the bundle E to the fixed point subspace X^{S^1}. $E_{|X^{S^1}}$ splits as a complex S^1-vector bundle into a direct sum

$$E_{|X^{S^1}} = E(m_1) \oplus \cdots \oplus E(m_p) \ .$$

$E(m_j)$ is a complex S^1-vector bundle, say of rank r_j, where $g \in S^1$ acts by complex multiplication with g^{m_j}. Since we are working over the space X^{S^1}, which has a trivial S^1-action, we can calculate $ch_{S^1}(E_{|X^{S^1}})$. So let $x_i = w_i + m_j u$ be the equivariant Chern roots of the bundle $E(m_j)$, where w_i are its ordinary (nonequivariant) Chern roots. Then

$$ch_{S^1}E(m_j) = \sum_{i=1}^{r_j} e^{w_i + m_j u} = \sum_{i=1}^{r_j} e^{w_i} e^{m_j u} = chE(m_j) \cdot e^{m_j u} \ ,$$

where $chE(m_j) \in H^*(X^{S^1})$ is the ordinary Chern character of $E(m_j)$. So, using the direct sum decomposition of $E_{|X^{S^1}}$,

$$ch_{S^1}(E_{|X^{S^1}}) = \sum_{j=1}^{p} chE(m_j) \cdot e^{m_j u} \ .$$

Over $\mathbb{C}^\times$ we use the variable $\lambda = e^u$ given by the map $\mathbb{C} \xrightarrow{\exp} \mathbb{C}^\times = \mathbb{C}/2\pi i\mathbb{Z}$, so

$$ch_{S^1}(E_{|X^{S^1}}) = \sum_{j=1}^{p} chE(m_j) \cdot \lambda^{m_j} \ .$$

If $\alpha = \alpha_j$ is a special point, thus a primitive n'th root of unity for some n, we have to see if $ch_{S^1}(E_{|X^{S^1}})$ glues via the sheaf map $\phi_{j0} = t^*_{-\alpha} \circ i^* \otimes 1$ (see Definition 3.10) to some section over U_j. Or, equivalently, we have to see if $t^*_\alpha ch_{S^1}(E_{|X^{S^1}})$, which is well-defined, lifts via i^* to an element in $H^*_{S^1}(X^\alpha) \otimes_{\mathbb{C}[u]} \mathcal{O}_{\mathbb{C},0}$. It is enough to take germs, since we can always restrict the open cover.

Now, if $\alpha = e^{\tilde{\alpha}}$,

$$t^*_\alpha ch_{S^1}(E_{|X^{S^1}}) = \sum_{j=1}^{p} \sum_{i=1}^{r_j} e^{w_i + m_j u + m_j \tilde{\alpha}} \ ,$$

where $x_i = w_i + m_j$ are the equivariant Chern roots of $E_{|X^{S^1}}$. We know that $e^{n\tilde{\alpha}} = 1$, so m_j only matters modulo n: Write $m_j = q_j \cdot n + r_j$, $0 \leq r_j < n - 1$; then $e^{m_j \tilde{\alpha}} = e^{r_j \tilde{\alpha}}$.

Hence

$$t_\alpha^* ch_{S^1}(E_{|X^{S^1}}) = \sum_{k=1}^{n-1}\sum_i e^{x_i+k\tilde{\alpha}} = \sum_{k=1}^{n-1}\sum_i e^{x_i}\cdot\alpha^k \quad ,$$

where x_i are the equivariant Chern roots of $E_{|X^{S^1}}$.

Look at the bundle $E_{|X^\alpha}$. Since X^α is fixed by the action of $\mathbb{Z}_n$, we have the (fiberwise) decomposition $E_{|X^\alpha} = V(0)\oplus V(1)\oplus\cdots\oplus V(n-1)$, corresponding to the irreducible representations of $\mathbb{Z}_n$. $V(k)$ is a complex S^1-vector bundle over X^α, and its restriction to X^{S^1} decomposes into a direct sum of bundles of the form $E(m_j)$ with the numbers m_j having the same remainder modulo n. Then we have just proved the following "transfer formula":

Proposition 4.3. *If $i : X^{S^1}\to X^\alpha$ is the inclusion, then*

$$t_\alpha^* ch_{S^1}(E_{|X^{S^1}}) = i^*\sum_{k=1}^{n-1} ch_{S^1}V(k)\cdot\alpha^k \quad .$$

Definition 4.4.

a) *The translation by α of $ch_{S^1}(E_{|X^\alpha})$ is defined as*

$$t_\alpha^* ch_{S^1}(E_{|X^\alpha}) := \sum_{k=1}^{n-1} ch_{S^1}V(k)\cdot\alpha^k \quad .$$

b) *We have just showed that the sections $t_\alpha^* ch_{S^1}(E_{|X^\alpha})$ and $ch_{S^1}(E_{|X^{S^1}})$ glue to a global section in $\mathcal{K}_{S^1}^*(X)^{hol}$ over $\mathbb{C}^\times$. Denote it by $\boldsymbol{ch}_{S^1}(E)$. It is an object of $\Gamma\mathcal{K}_{S^1}^*(X)^{hol}$, where we understand by Γ sections over $\mathbb{C}^\times$.*

Proposition 4.5. *$\boldsymbol{ch}_{S^1}(E)$ is actually an algebraic map, i.e. it lies in $\Gamma\mathcal{K}_{S^1}^*(X)^{alg}$.*

Proof. Consider what happens when try to extend $\boldsymbol{ch}_{S^1}(E)$ to a section in $\mathcal{K}_{S^1}^*(X)^{hol}$ over the whole $\mathbb{C}P^1$. On U_0 we saw that $\boldsymbol{ch}_{S^1}(E)$ is equal to

$$ch_{S^1}(E_{|X^{S^1}}) = \sum_{j=1}^{p} chE(m_j)\cdot\lambda^{m_j} \quad ,$$

with λ the complex variable on $\mathbb{C}^\times = \operatorname{Spec}\mathbb{C}[\lambda^{\pm 1}]$. Clearly this extends as a meromorphic section over $\mathbb{C}P^1$. But a global meromorphic section of $\mathcal{K}_{S^1}^*(X)^{hol}$ by GAGA has to be in fact rational. So the section $\boldsymbol{ch}_{S^1}(E)$ over $\mathbb{C}^\times$ is holomorphic and rational, therefore algebraic. $\square$

Proposition 4.6. *The map we have just defined*

$$K^*_{S^1}(X) \xrightarrow{\boldsymbol{ch}_{S^1}} \Gamma\mathcal{K}^*_{S^1}(X)^{alg}$$

is an algebra map.

Proof. It is immediate that $\boldsymbol{ch}_{S^1}$ is additive, because the usual Chern character is additive.

For the multiplicativity part, we have to show that $t^*_\alpha ch_{S^1}(E\otimes F)_{|X^\alpha} = t^*_\alpha ch_{S^1}(E_{|X^\alpha})\cdot t^*_\alpha ch_{S^1}(F_{|X^\alpha})$. From the decomposition into eigenspaces of the $\mathbb{Z}_n$-action, we can assume that $E_{|X^\alpha} = V(i)$ and $F_{|X^\alpha} = W(j)$, with $0 \leq i, j < n-1$.

On $V\otimes W$ $g \in S^1$ acts as $g\cdot(v\otimes w) = g^i v \otimes g^j w = g^{i+j} v\otimes w$, so $V(i)\otimes W(j) = (V\otimes W)(i+j)$. Then $t^*_\alpha ch_{S^1}(V(i)\otimes W(j)) = t^*_\alpha ch_{S^1}(V\otimes W)(i+j) = ch_{S^1}(V\otimes W)\alpha^{i+j} = ch_{S^1}(V) ch_{S^1}(W)\alpha^i\alpha^j = t^*_\alpha ch_{S^1} V(i)\cdot t^*_\alpha ch_{S^1} W(j)$. $\square$

Theorem 4.7. *The twisted equivariant Chern character map*

$$K^*_{S^1}(-) \xrightarrow{\boldsymbol{ch}_{S^1}} \Gamma\mathcal{K}^*_{S^1}(-)^{alg}$$

is a multiplicative isomorphism of S^1-cohomology theories.

Proof. Given what we know so far, it only remains to check the isomorphism part. Because of Mayer–Vietoris it is enough to verify the isomorphism on the "equivariant points" S^1/H, $H \subseteq S^1$ subgroup. If $H = S^1$, $S^1/H = pt$, and so $K^*_{S^1}(pt) = \mathbb{C}[\lambda^{\pm 1}]$. $\mathcal{K}^*_{S^1}(pt) = \mathcal{O}^{alg}_{\mathbb{C}^\times}$, the algebraic structure sheaf of $\mathbb{C}^\times$. $\Gamma\mathcal{K}^*_{S^1}(pt)^{alg} = \mathbb{C}[\lambda^{\pm1}]$, and $\boldsymbol{ch}_{S^1}$ is obviously an isomorphism, because it takes 1 to 1, and it is a map of $\mathbb{C}[\lambda^{\pm1}]$-modules (coherent sheaves over $\mathbb{C}^\times$ correspond to finitely generated $\mathbb{C}[\lambda^{\pm1}]$-modules).

Now suppose $X = S^1/\mathbb{Z}_n$. We saw before that $\mathcal{K}^*_{S^1}(S^1/\mathbb{Z}_n)^{hol}$ is a skyscraper sheaf with stalks $= \mathbb{C}$ at the n'th roots of unity. $\mathcal{K}^*_{S^1}(S^1/\mathbb{Z}_n)^{hol} = \mathcal{K}^*_{S^1}(S^1/\mathbb{Z}_n)^{alg}$, and $\Gamma\mathcal{K}^*_{S^1}(S^1/\mathbb{Z}_n)^{alg} = \mathbb{C}\oplus\cdots\oplus\mathbb{C}$, n copies. Take $\lambda = S^1\times_{\mathbb{Z}_n}\mathbb{C}(1) \to S^1/\mathbb{Z}_n$. If α is an n'th root of unity, $X^\alpha = X$. As a $\mathbb{Z}_n$-bundle, λ can be written as $\lambda = \lambda(1)$. Then

$t_\alpha^* ch_{S^1}(\lambda) = ch_{S^1}(\lambda) \cdot \alpha^1 = 1 \cdot \alpha = \alpha$. (Recall that $ch_{S^1}(\lambda) = e^{c_1(\lambda)_{S^1}} = e^0 = 1$.) But we showed before that the map

$$K_{S^1}^*(S^1/\mathbb{Z}_n) = \mathbb{C}[\mathbb{Z}_n] \xrightarrow{\sim} \mathbb{C}_{(1)} \oplus \mathbb{C}_{(\epsilon_n)} \oplus \mathbb{C}_{(\epsilon_n^2)} \oplus \cdots \oplus \mathbb{C}_{(\epsilon_n^{n-1})} \ ,$$

$\lambda \mapsto (1, \epsilon_n, \ldots, \epsilon_n^{n-1})$ gives a multiplicative isomorphism. So $K_{S^1}^*(S^1/\mathbb{Z}_n) \xrightarrow{\boldsymbol{ch}_{S^1}} \Gamma\mathcal{K}_{S^1}^*(S^1/\mathbb{Z}_n)^{alg}$ is an isomorphism of algebras. □

5. S^1-equivariant elliptic pushforwards

While the construction of $\boldsymbol{E}_{S^1}^*(X)$ depends only the elliptic curve $\mathcal{E}$, the construction of the elliptic pushforward $f_!^E$ involves an extra choice, that of a *2-division point* on $\mathcal{E}$. This is because $f_!^E$ is defined using the Jacobi sine function $s(x)$, which is associated to an elliptic curve $\mathcal{E}$ with a specified 2-division point.

More precisely, if $\mathcal{E} = \mathbb{C}/\Lambda$, where $\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ is a lattice in $\mathbb{C}$ with a specified division point $\omega_1/2$, then there exists a meromorphic function $s : \mathbb{C} \to \mathbb{C}$, which is periodic to the "doubled" lattice $\tilde{\Lambda} = \mathbb{Z}\omega_1 + 2\mathbb{Z}\omega_2$, and has simple zeroes at $0 + \tilde{\Lambda}$, $\omega_2 + \tilde{\Lambda}$, and simple poles at $\omega_1/2 + \tilde{\Lambda}$, $\omega_1/2 + \omega_2 + \tilde{\Lambda}$. This meromorphic function is unique up to a scalar, which is fixed by requiring that $\lim_{x\to 0} s(x)/x = 1$.

The Jacobi sine $s(x)$ is an elliptic function with respect to the "doubled" elliptic curve $\tilde{\mathcal{E}} = \mathbb{C}/\tilde{\Lambda}$. Notice that the construction of $\tilde{\mathcal{E}}$, given $\mathcal{E}$ and the 2-division point, is canonical (does not depend on the choice of the lattice Λ). It is easy to check that $s(x)$ has the following additional properties:

Facts 5.1.

a) $s(x)$ is odd, i.e. $s(-x) = -s(x)$. Around zero, s can be expanded as a power series $s(x) = x + a_3x^3 + a_5x^5 + \cdots$.

b) $s(x + \omega_1) = s(x); s(x + \omega_2) = -s(x)$.

c) $s(x + \omega_1/2) = a/s(x)$, $a \neq 0$.

For the construction of S^1-equivariant elliptic pushforwards we are going to follow Grojnowski [13]. Let $f : X \to Y$ be an equivariant map between compact S^1-manifolds so that the restrictions $f : X^\alpha \to Y^\alpha$ are oriented maps. Then the Grojnowski pushforward of f is a map of sheaves

$$f_!^E : \boldsymbol{E}^*_{S^1}(X)^{[f]} \to \boldsymbol{E}^*_{S^1}(Y) \ \ ,$$

where $\boldsymbol{E}^*_{S^1}(X)^{[f]}$ is the sheaf $\boldsymbol{E}^*_{S^1}(X)$ twisted by a 1-cocycle to be defined below.

The main technical ingredient in the construction of the (global i.e. sheafwise) elliptic pushforward $f_!^E : \boldsymbol{E}^*_{S^1}(X)^{[f]} \to \boldsymbol{E}^*_{S^1}(Y)$ is the (local i.e. stalkwise) elliptic pushforward $f_!^E : HO^*_{S^1}(X^\alpha) \to HO^*_{S^1}(Y^\alpha)$. For this, we need to define the elliptic Thom class of an oriented S^1-vector bundle.

Let V be a $2n$-dimensional oriented real vector bundle over a finite S^1-CW complex X. Classify its Borel construction $V_{S^1} \to X_{S^1}$ by mapping into $BSO(2n)$, and get the map $f_V : X_{S^1} \to BSO(2n)$. If V_{univ} is the universal orientable vector bundle over $BSO(2n)$, we also have a map of pairs, also denoted by $f_V : (DV_{S^1}, SV_{S^1}) \to (DV_{univ}, SV_{univ})$. As usual, DV and SV represent the disc and the sphere bundle of V, respectively.

But it is known that the pair (DV_{univ}, SV_{univ}) is homotopic to $(BSO(2n), BSO(2n-1))$. Also, we know that

$$H^*BSO(2n) = \mathbb{C}[p_1, \dots, p_n, e]/(e^2 - p_n) \ \ ,$$

where p_j is the universal j'th Pontrjagin class, and e is the universal Euler class. From the long exact sequence of the pair, it follows that $H^*(BSO(2n), BSO(2n-1))$ can be regarded as the ideal generated by e in $H^*BSO(2n)$. The class $e \in H^*(DV_{univ}, SV_{univ})$ is the universal Thom class, which we will denote by ϕ_{univ}. Then the ordinary equivariant Thom class of V is defined as the pullback class $f_V^*\phi_{univ} \in H^*_{S^1}(DV, SV)$, and we denote it by $\phi(V)_{S^1}$.

Definition 5.2. *Consider $Q(x) = s(x)/x$, where $s(x)$ is the Jacobi sine. Since $Q(x)$ is even and holomorphic around zero, Proposition A.7 gives a class $\mu_Q(V)_{S^1} \in HO^*_{S^1}(X)$. Then we define the equivariant elliptic Thom class of V to be the product $\mu_Q(V)_{S^1} \cdot \phi(V)_{S^1}$ in $H^{**}_{S^1}(DV, SV)$, and denote it by $\phi^E(V)_{S^1}$. (One can also say that $\phi^E(V)_{S^1} =$*

$s(x_1)\dots s(x_n)$, while $\phi(V)_{S^1} = x_1 \dots x_n$, where $x_1, \dots, x_n$ are the equivariant Chern rootsof V.)

Proposition 5.3. *If $V \to X$ is an even dimensional real oriented S^1-vector bundle, and X is a finite S^1-CW complex, then $\phi^E(V)_{S^1}$ actually lies in $HO^*_{S^1}(DV, SV)$.*

Proof. The only difficult part, namely that $\mu_Q(V)_{S^1}$ is holomorphic, is proved in the Appendix, in Proposition A.5. So we only need to know that the cup product

$$H^*_{S^1}(X) \otimes H^*_{S^1}(DV, SV) \to H^*_{S^1}(DV, SV)$$

extends by tensoring with $\mathcal{O}_{\mathbb{C},0}$ over $\mathbb{C}[u]$ to a map

$$HO^*_{S^1}(X) \otimes HO^*_{S^1}(DV, SV) \to HO^*_{S^1}(DV, SV) \ .$$

□

Because $Q(x) = s(x)/x$ is an invertible power series around zero, it follows that multiplication by the equivariant elliptic Thom class $\phi^E(V)_{S^1}$ gives an isomorphism $HO^*_{S^1}(X) \xrightarrow{\sim} HO^*_{S^1}(DV, SV)$, which is the Thom isomorphism in HO-theory.

Corollary 5.4. *If $f : X \to Y$ is an S^1-equivariant oriented map between compact S^1-manifolds, then there is a functorial elliptic pushforward*

$$f^E_! : HO^*_{S^1}(X) \to HO^*_{S^1}(Y) \ .$$

In the case when Y is a point, $f^E_!(1)$ is the S^1-equivariant elliptic genus of X.

Proof. Recall ([9]) that the ordinary pushforward is defined as the composition of three maps, two of which are Thom isomorphisms, and the third is a natural one, so the first statement follows from the previous corollary.

The second statement is an easy consequence of the topological Riemann–Roch theorem (see again [9]), and of the definition of the equivariant elliptic Thom class. □

Notice that, if Y is point, $HO^*_{S^1}(Y) \cong \mathcal{O}_{\mathbb{C},0}$, so the S^1-equivariant elliptic genus of X is holomorphic around zero. Also, if we replace $HO^*_{S^1}(-) = H^*_{S^1}(-) \otimes_{\mathbb{C}[u]} \mathcal{O}_{\mathbb{C},0}$

by $HM^*_{S^1}(-) = H^*_{S^1}(-) \otimes_{\mathbb{C}[u]} \mathcal{M}(\mathbb{C})$, where $\mathcal{M}(\mathbb{C})$ is the ring of global meromorphic functions on $\mathbb{C}$, the same proof as above shows that the S^1-equivariant elliptic genus of X is meromorphic in $\mathbb{C}$. This proves PROPOSITION B stated in Section 2.

The local construction of elliptic pushforwards is thus completed. We want now to assemble the pushforwards in a map of sheaves. The problem is that pushforwards do not commute with pullbacks, i.e. if

$$\begin{array}{ccc} X^\alpha & \xrightarrow{f} & Y^\alpha \\ {\scriptstyle i}\uparrow & & \uparrow{\scriptstyle j} \\ X^\beta & \xrightarrow[g]{} & Y^\beta \end{array}$$

is a commutative diagram where f, g are oriented maps between S^1-manifolds, and i, j are the inclusions, then it is not true in general that $j^* f^E_! = g^E_! i^*$.

To see the extent to which this relation fails, consider $e^E_{S^1}(X^\alpha/X^\beta)$ the S^1-equivariant Euler class of the normal bundle of the embedding $X^\beta \hookrightarrow X^\alpha$, and similarly consider $e^E_{S^1}(Y^\alpha/Y^\beta)$. Denote by

$$\lambda_{\alpha\beta} = e^E_{S^1}(X^\alpha/X^\beta)^{-1} \cdot g^* e^E_{S^1}(Y^\alpha/Y^\beta) \ ,$$

and assume for the moment that $e^E_{S^1}(X^\alpha/X^\beta)$ is invertible, so that $\lambda_{\alpha\beta}$ exists. Then we have the following standard result:

Lemma 5.5.

$$j^* f^E_! \mu^\alpha = g^E_! (i^* \mu^\alpha \cdot \lambda_{\alpha\beta}) \ .$$

Proposition 5.6. *Let $f : X \to Y$ be an S^1-map such that the induced maps $f : X^\alpha \to Y^\alpha$ and $g : X^\beta = X^{S^1} \to Y^\beta = Y^{S^1}$ are oriented. Let U be a small neighborhood of α but not containing α. Then $\lambda_{\alpha\beta}$ exists in $H^*_{S^1}(X^\beta) \otimes_{\mathbb{C}[u]} \mathcal{O}_{\mathcal{E}}(U - \beta)$, and the following* *diagram is commutative:*

$$\begin{array}{ccc}
H^*_{S^1}(X^\alpha)\otimes_{\mathbb{C}[u]}\mathcal{O}_{\mathcal{E}}(U-\alpha) & \xrightarrow{f_!^E} & H^*_{S^1}(Y^\alpha)\otimes_{\mathbb{C}[u]}\mathcal{O}_{\mathcal{E}}(U-\alpha) \\
\downarrow{\scriptstyle i^*\cdot\lambda_{\alpha\beta}} & & \downarrow{\scriptstyle j^*} \\
H^*_{S^1}(X^\beta)\otimes_{\mathbb{C}[u]}\mathcal{O}_{\mathcal{E}}(U-\alpha) & \xrightarrow{g_!^E} & H^*_{S^1}(Y^\beta)\otimes_{\mathbb{C}[u]}\mathcal{O}_{\mathcal{E}}(U-\alpha) \\
\downarrow{\scriptstyle t^*_{\beta-\alpha}} & & \downarrow{\scriptstyle t^*_{\beta-\alpha}} \\
H^*_{S^1}(X^\beta)\otimes_{\mathbb{C}[u]}\mathcal{O}_{\mathcal{E}}(U-\beta) & \xrightarrow{g_!^E} & H^*_{S^1}(Y^\beta)\otimes_{\mathbb{C}[u]}\mathcal{O}_{\mathcal{E}}(U-\beta)
\end{array}$$

Here the notation "$i^*\cdot\lambda_{\alpha\beta}$" means: apply i^* first, and then multiply the result with $\lambda_{\alpha\beta}$.

Proof. Denote by $W=$ the normal bundle of the embedding $X^\beta = X^{S^1}\to X^\alpha$. Let us show that, if $\alpha\notin U$, then $e^E_{S^1}(W)$ is invertible in $H^*_{S^1}(X^\beta)\otimes_{\mathbb{C}[u]}\mathcal{O}_{\mathcal{E}}(U-\alpha)$. Denote by w_i the nonequivariant Chern roots of W, and by m_i the corresponding rotation numbers of W (i.e. the weights of the S^1-action). Since $X^\beta = X^{S^1}$, $m_i\neq 0$. Also, the S^1-equivariant Euler class of W is given by

$$e_{S^1}(W) = (w_1+m_1u)\dots(w_r+m_ru) = m_1\dots m_r(u+w_1/m_1)\dots(u+w_r/m_r)\ .$$

But w_i are nilpotent, so $e_{S^1}(W)$ is invertible as long as u is invertible. Now $\alpha\notin U$ translates to $0\notin U-\alpha$, which implies that the image of u via the map $\mathbb{C}[u]\to\mathcal{O}_{\mathcal{E}}(U-\alpha)$ is indeed invertible. To deduce now that $e^E_{S^1}(W)$, the elliptic S^1-equivariant Euler class of W, is also invertible, recall that $e^E_{S^1}(W)$ and $e_{S^1}(W)$ differ by a class defined using the power series $s(x)/x = 1+a_3x^2+a_5x^4+\cdots$, which is invertible for U small enough.

So $\lambda_{\alpha\beta}$ exists, and by the previous Lemma, the uppper part of our diagram is commutative. The lower part is trivially commutative.

Now, while i^* were essentially the gluing maps in the sheaf $\mathcal{F} = \boldsymbol{E}^*_{S^1}(X)$, we think of the maps $i^*\cdot\lambda_{\alpha\beta}$ as giving a twisted sheaf, denoted by $\mathcal{F}^{[f]}$. $\mathcal{F}$ was obtained by gluing the sheaves $\mathcal{F}_\alpha$ over an adapted open cover $(U_\alpha)_{\alpha\in\mathcal{E}}$. The gluing maps $\phi_{\alpha\beta}$ were defined in Section 3 as the composite of a few maps. □

Definition 5.7. *The twisted gluing functions* $\phi_{\alpha\beta}^{[f]}$ *are defined as the composition of the following three maps*

$$H_{S^1}^*(X^\alpha)\otimes_{\mathbb{C}[u]}\mathcal{O}_{\mathcal{E}}(U-\alpha)$$

$$i^*\otimes 1$$

$$H_{S^1}^*(X^\beta)\otimes_{\mathbb{C}[u]}\mathcal{O}_{\mathcal{E}}(U-\alpha)$$

$$\cdot\lambda_{\alpha\beta}$$

$$H_{S^1}^*(X^\beta)\otimes_{\mathbb{C}[u]}\mathcal{O}_{\mathcal{E}}(U-\beta)$$

$$t^*_{\beta-\alpha}$$

$$H_{S^1}^*(X^\beta)\otimes_{\mathbb{C}[u]}\mathcal{O}_{\mathcal{E}}(U-\beta)\ \ ,$$

where $\lambda_{\alpha\beta}=e^E_{S^1}(X^\alpha/X^\beta)^{-1}\cdot f^*e^E_{S^1}(Y^\alpha/Y^\beta)$*. For explanations regarding the third map, read Remark 3.6.* $\phi_{\alpha\beta}^{[f]}$ *satisfy the cocycle condition, because if* β *and* γ *are not special,* $\phi^{[f]}_{\beta\gamma}=t^*_{\gamma-\beta}$*, and as in the case of* $\phi_{\alpha\beta}$ *the cocycle condition reduces to* $t^*_{\gamma-\beta}t^*_{\beta-\alpha}=t^*_{\gamma-\alpha}$*.*

The sheaf $\boldsymbol{E}^*_{S^1}(X)^{[f]}$ *is now defined by gluing the same sheaves* $\mathcal{F}_\alpha$*, but using the new functions* $\phi^{[f]}_{\alpha\beta}$*.*

Proposition 5.8. *If* $f:X\to Y$ *is a map of compact* S^1*-manifolds such that the restrictions* $f:X^\alpha\to Y^\alpha$ *are oriented* $\forall\alpha\in\mathcal{E}$*, then the commutativity of the diagram in Proposition 5.6 gives a map of coherent sheaves over* $\mathcal{E}$

$$f_!^E:\boldsymbol{E}^*_{S^1}(X)^{[f]}\to\boldsymbol{E}^*_{S^1}(Y)\ \ .$$

This is the **Grojnowski pushforward** of f in $\boldsymbol{E}^*_{S^1}(-)$. It is functorial in a certain sense (see [13]), and is a map of $\boldsymbol{E}^*_{S^1}(Y)$-modules, i.e.

$$f_!^E(\mu\cdot f^*\nu)=f_!^E\mu\cdot\nu\ \ .$$

6. Rigidity of the elliptic genus

As in the beginning of the previous section, let $\mathcal{E}=\mathbb{C}/\Lambda$ be a nonsingular elliptic curve over $\mathbb{C}$ together with a 2-division point. We saw that we can associate to this data a double cover $\tilde{\mathcal{E}}$ of $\mathcal{E}$, such that the Jacobi sine function $s(x)$, which appears in the definition of the elliptic pushforward, is an elliptic function with respect to $\tilde{\mathcal{E}}$.

In this section we discuss the rigidity phenomenon in the context of equivariant elliptic cohomology. If X is a compact spin S^1-manifold, a theorem of Edmonds (see [10]) says that the fixed point submanifolds $X^{\mathbb{Z}_n}$ are oriented for all $n \in \mathbb{N}$. X^{S^1} is also oriented, because the normal bundle of the embedding $X^{S^1} \hookrightarrow X$ has a complex structure. Since X^α is oriented $\forall \alpha \in \mathcal{E}$, the map $\pi : X \to$ *point* satisfies the hypothesis of Proposition 5.8, so we get a Grojnowski pushforward

$$\pi_!^E : \boldsymbol{E}^*_{S^1}(X)^{[\pi]} \to \boldsymbol{E}^*_{S^1}(point) = \mathcal{O}_{\mathcal{E}} \ \ .$$

We will see that the rigidity phenomenon amounts to finding a global (Thom) section in the sheaf $\boldsymbol{E}^*_{S^1}(X)^{[\pi]}$. Since $s(x)$ is not a well-defined function on $\mathcal{E}$, we cannot expect to find such a global section. However, if we take the pullback of the sheaf $\boldsymbol{E}^*_{S^1}(X)^{[\pi]}$ along the covering map $\tilde{\mathcal{E}} \to \mathcal{E}$, we'll show that the new sheaf has global section.

Convention. From this point on, all the sheaves $\mathcal{F}$ will be considered over $\tilde{\mathcal{E}}$, i.e. we will replace them by the pullback of $\mathcal{F}$ via the map $\tilde{\mathcal{E}} \to \mathcal{E}$.

For our purposes, however, we need a more general version of $\boldsymbol{E}^*_{S^1}(X)^{[\pi]}$.

Proposition 6.1. *Let V be a spin S^1-vector bundle over the finite S^1-CW complex X. Let $n \in \mathbb{N}$. Then $V^{\mathbb{Z}_n}$ and V^{S^1} are oriented, and there exist oriented vector bundles V/V^{S^1} and $V^{\mathbb{Z}_n}/V^{S^1}$ over X^{S^1} and $V/V^{\mathbb{Z}_n}$ over $X^{\mathbb{Z}_n}$ such that*

$$V_{|X^{\mathbb{Z}_n}} = V^{\mathbb{Z}_n} \oplus V/V^{\mathbb{Z}_n}; V_{|X^{S^1}} = V^{S^1} \oplus V/V^{S^1}; V^{\mathbb{Z}_n}_{|X^{S^1}} = V^{S^1} \oplus V^{\mathbb{Z}_n}/V^{S^1}$$

as oriented bundles.

Proof. The decompositions of these three restriction bundles come from the fact that the groups $\mathbb{Z}_n$ in the first case, and S^1 in the other two cases act on the fibers and decompose them as representations into a trivial and nontrivial part.

Now we define orientations for the different bundles involved. $\mathbb{Z}_n$ preserves the spin structure of V, so we can apply Lemma 10.3 from [6], and deduce that $V^{\mathbb{Z}_n}$ is oriented. (It is interesting to notice that Bott & Taubes prove this result at the level of generality that we need, although they only use it in the special case when X is an S^1-manifold, and $V = TX$.) $V_{|X^{\mathbb{Z}_n}}$ is oriented, so $V/V^{\mathbb{Z}_n}$ gets an induced orientation. V/V^{S^1} has a complex structure, because its rotation numbers are all nontrivial. However, for computational reasons, we do not choose the complex structure on V/V^{S^1} where all rotation numbers are positive, but we choose a complex orientation depending on n: namely one for which rotation numbers m_j satisfy $m_j = nq_j + r_j$ with $0 \leq r_j \leq \frac{n}{2}$; if $r_j = 0$ or $r_j = \frac{n}{2}$, we can always arrange $m_j > 0$ (thus fixing the complex structure), but for $0 < r_j < \frac{n}{2}$ the choice of m_j is forced on us, and we may have $m_j < 0$. Since V/V^{S^1} is oriented, V^{S^1} gets an induced orientation. Now $V^{\mathbb{Z}_n}_{|X^{S^1}}$ is oriented, because $V^{\mathbb{Z}_n}$ is; so $V^{\mathbb{Z}_n}/V^{S^1}$ gets an induced orientation. □

Definition 6.2. *As in Definition 5.7, we define* $\phi^{[V]}_{\alpha\beta}$ *as the composition of three maps, where the second one is multiplication by* $\lambda_{\alpha\beta} = e^E(V^\alpha/V^\beta)^{-1}$. *(*$V^\alpha/V^\beta$ *is oriented as in the previous Proposition.)* $\phi^{[V]}_{\alpha\beta}$ *satisfies the cocycle condition, so by gluing the sheaves* $\mathcal{F}_\alpha$ *using* $\phi^{[V]}_{\alpha\beta}$, *we obtain a new sheaf which we denote by* $\boldsymbol{E}^*_{S^1}(X)^{[V]}$.

Notice that, if we take the map $\pi : X \to$ *point* as above, for $V = TX$ we have $\boldsymbol{E}^*_{S^1}(X)^{[V]} = \boldsymbol{E}^*_{S^1}(X)^{[f]}$. We now proceed to prove THEOREM C.

Theorem 6.3. *If* $V \to X$ *is a spin* S^1*-vector bundle over a finite* S^1*-CW complex, then the element 1 in the stalk of* $\boldsymbol{E}^*_{S^1}(X)^{[V]}$ *at zero extends to a global section, called the* ***Thom section****.*

Proof. To simplify notation, we are going to identify $\mathcal{E}$ with $\mathbb{C}/\Lambda$, where $\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ is a lattice in $\mathbb{C}$. It is a good idea to think of points in $\mathcal{E}$ rather as points in $\mathbb{C}$, and of $\boldsymbol{E}^*_{S^1}(X)$ as the pullback on $\mathbb{C}$ via $\mathbb{C} \to \mathbb{C}/\Lambda$. Then we call $\alpha \in \mathbb{C}$ a division point if there is an integer $n > 0$ such that $n\alpha \in \Lambda$. The smallest such n is called the order of α.

Now $\boldsymbol{E}^*_{S^1}(X)^{[V]}$ was obtained by gluing the sheaves $\mathcal{F}_\alpha$ along the adapted open cover $(U_\alpha)_{\alpha\in\mathcal{E}}$. So to give a global section μ of $\boldsymbol{E}^*_{S^1}(X)^{[V]}$ is the same as to give global sections μ_α of $\mathcal{F}_\alpha$ such that they glue, i.e. $\phi^{[V]}_{\alpha\beta}\mu_\alpha = \mu_\beta$ for any α and β with $U_\alpha \cap U_\beta \neq \emptyset$. From Definition 6.2, to give μ is the same as to give $\mu_\alpha \in HO^*_{S^1}(X^\alpha)$ so that

$t^*_{\beta-\alpha}(i^*\mu_\alpha \cdot e^E_{S^1}(V^\alpha/V^\beta)^{-1}) = \mu_\beta$, or $i^*\mu_\alpha \cdot e^E_{S^1}(V^\alpha/V^\beta)^{-1} = t^*_{\beta-\alpha}\mu_\beta$ (i the inclusion $X^\beta \hookrightarrow X^\alpha$). Because μ is supposed to globalize 1, we know that $\mu_0 = 1$. This implies that $\mu_\beta = t^*_\beta e^E_{S^1}(V/V^\beta)^{-1}$ for β in a small neighborhood of $0 \in \mathbb{C}$.

In fact, it turns out that this formula for μ_β is valid for all $\beta \in \mathbb{C}$, as long as β is not special. This means we have to check that $\mu_\beta = t^*_\beta e^E_{S^1}(V/V^\beta)^{-1}$ exists in $HO^*_{S^1}(X^\beta)$ as long as β is not special. β not special means $X^\beta = X^{S^1}$. Then consider the bundle V/V^{S^1}. Because V^{S^1} is fixed by the S^1-action, V/V^{S^1} decomposes into a direct sum $V(m_1) \oplus V(m_2) \oplus \cdots \oplus V(m_r)$. m_j are the weights of the S^1-action, and they are called rotation numbers. We can choose all m_j positive, which gives a complex structure on V/V^{S^1}, where $g \in S^1$ acts on $V(m)$ by complex multiplication with g^m. Then

$$\mu_\beta = t^*_\beta e^E_{S^1}(V/V^\beta)^{-1} = \prod_i s(x_i + m_j\beta)^{-1} \quad ,$$

where x_i are the equivariant Chern roots of V/V^{S^1} (see Definition A.3 in the Appendix). This expression for μ_β exists in $HO^*_{S^1}(X^\beta)$ long as $s(m_j\beta) \neq 0$. Suppose $s(m_j\beta) = 0$. Then $m_j\beta \in \Lambda$, so β is a division point, say of order n. It follows that n divides m_j, which implies $X^{\mathbb{Z}_n} \neq X^{S^1}$. But $X^\beta = X^{\mathbb{Z}_n}$, since β has order n, so $X^\beta \neq X^{S^1}$ i.e. β is special, contradiction.

The only problem is what happens at a special point $\alpha \in \mathbb{C}$, say of order n. We have to find a class $\mu_\alpha \in HO^*_{S^1}(X^\alpha)$ such that $\phi^{[V]}_{\alpha\beta}\mu_\alpha = \mu_\beta$, i.e. $t^*_{\beta-\alpha}(i^*\mu_\alpha \cdot e^E_{S^1}(V^\alpha/V^\beta)^{-1}) = t^*_\beta e^E_{S^1}(V/V^\beta)^{-1}$. Equivalently, we want a class μ_α such that $i^*\mu_\alpha = t^*_\alpha e^E_{S^1}(V/V^\beta)^{-1} \cdot e^E_{S^1}(V^\alpha/V^\beta)$, i.e. we want to lift the class $t^*_\alpha e^E_{S^1}(V/V^\beta)^{-1} \cdot e^E_{S^1}(V^\alpha/V^\beta)$ from $HO^*_{S^1}(X^\beta)$ to $HO^*_{S^1}(X^\alpha)$. If we can do that, we are done, because the class $(\mu_\alpha)_{\alpha\in\mathbb{C}}$ is a global section in $\boldsymbol{E}^*_{S^1}(X)^{[V]}$, and it extends $\mu_0 = 1$ from the stalk at zero. So it only remains to prove the following lemma, which is a generalization of the transfer formula of Bott & Taubes. $\square$

Lemma 6.4. *Let α be a special point of order n, and $V \to X$ a spin S^1-vector bundle. Let $i : X^{S^1} \to X^{\mathbb{Z}_n}$ be the inclusion map. Then there exists a class $\mu_\alpha \in HO^*_{S^1}(X^{\mathbb{Z}_n})$ such that*

$$i^*\mu_\alpha = t^*_\alpha e^E_{S^1}(V/V^{S^1})^{-1} \cdot e^E_{S^1}(V^{\mathbb{Z}_n}/V^{S^1}) \quad .$$

Proof. The main difficulty arises from the fact that X^{S^1} may have different connected components, and so may $X^{\mathbb{Z}_n}$. Otherwise, it is easy to lift the class $t_\alpha^* e_{S^1}^E(V/V^{S^1})^{-1} \cdot e_{S^1}^E(V^{\mathbb{Z}_n}/V^{S^1})$ from X^{S^1} to $X^{\mathbb{Z}_n}$ if we do not worry about signs. Indeed, we might get different signs if we have several connected components of X^{S^1} inside a connected component of $X^{\mathbb{Z}_n}$; then the class μ_α on $X^{\mathbb{Z}_n}$ would not be well defined. This is where the spin structure of V will make sure that the different signs are equal.

Fix N a connected component of X^{S^1}. Also, let P be the connected component of $X^{\mathbb{Z}_n}$ which contains N. Denote by $i : N \hookrightarrow P$. We will use the same notation for $V^{\mathbb{Z}_n}$ and V^{S^1} when we restrict them to P and N respectively. The action of $\mathbb{Z}_n$ on P is trivial, so we get a fiberwise decomposition of $V_{|P}$ by the different representations of $\mathbb{Z}_n$:

$$V_{|P} = V^{\mathbb{Z}_n} \oplus V/V^{\mathbb{Z}_n} = V^{\mathbb{Z}_n} \oplus \bigoplus_{0<k<\frac{n}{2}} V(k) \oplus V(\frac{n}{2}) \ . \tag{1}$$

Here $V^{\mathbb{Z}_n}$ and $V(\frac{n}{2})$ are real vector spaces, and $V(k)$ has a complex structure for which a generator $g = e^{2\pi i/n} \in \mathbb{Z}_n$ acts by complex multiplication with g^k. $V(\frac{n}{2}) = 0$ if n is odd. Denote by $V(K) = \bigoplus_{0<k<\frac{n}{2}} V(k)$. Then we have the following decomposition

$$V/V^{S^1} = V^{\mathbb{Z}_n}/V^{S^1} \oplus V(K)_{|N} V/V^{\mathbb{Z}_n} \oplus V(\frac{n}{2})_{|N} \ . \tag{2}$$

The orientations are chosen as follows: V is oriented by its spin structure. $V^{\mathbb{Z}_n}$, V^{S^1}, $V^{\mathbb{Z}_n}/V^{S^1}$, V/V^{S^1} have orientations as described in Proposition 6.1.

- If $V(\frac{n}{2}) \neq 0$, choose the complex orientation of $V(K)$ described above.
- If $V(\frac{n}{2}) = 0$, then $V(K) = V/V^{\mathbb{Z}_n}$, which is already oriented, so choose this orientation for $V(K)$.

All bundles appearing in 2 also have orientations coming from their complex structure (they have nonzero rotation numbers). As a notational rule, we are going to use the subscript "*or*" to indicate the "correct" orientation on the given vector space, i.e. the orientation which is induced from the spin structure on V as in Proposition 6.1. When we omit the subscript "*or*", we assume the bundle has the correct orientation. The subscript "*cx*" will indicate that we chose a complex structure on the given vector space.

This is only intended to make calculations easier. Here is a table with the bundles of interest:

bundle with the correct orientation	bundle with the complex orientation	sign difference between the two orientations
$(V/V^{S^1})_{or}$	$(V/V^{S^1})_{cx}$	$(-1)^{\sigma}$
$(V^{\mathbb{Z}_n}/V^{S^1})_{or}$	$(V^{\mathbb{Z}_n}/V^{S^1})_{cx}$	$(-1)^{\sigma(0)}$
$V(K)_{or}$	$V(K)_{cx}$	$(-1)^{\sigma(K)}$
$i^*(V(\frac{n}{2})_{or})$	$(i^*V(\frac{n}{2}))_{cx}$	$(-1)^{\sigma(\frac{n}{2})}$

From the decomposition in (2) under the correct and the complex orientations, we deduce that

$$(-1)^{\sigma} = (-1)^{\sigma(0)}(-1)^{\sigma(K)}(-1)^{\sigma(\frac{n}{2})} \quad . \tag{3}$$

Now we want to show that there exists a class $\mu_P \in HO^*_{S^1}(P)$ such that

$$i^*\mu_P = t^*_{\alpha} e^E_{S^1}(V/V^{S^1})^{-1}_{|N} \cdot e^E_{S^1}(V^{\mathbb{Z}_n}/V^{S^1})_{|N} \quad . \tag{4}$$

From the table we deduce the following formula

$$t^*_{\alpha} e^E_{S^1}(V/V^{S^1})^{-1}_{|N} \cdot (-1)^{\sigma} t^*_{\alpha} e^E_{S^1}((V/V^{S^1})_{cx})^{-1}_{|N} \tag{5}$$

Now we calculate $t^*_{\alpha} e^E_{S^1}((V/V^{S^1})_{cx})^{-1}_{|N}$. Since the bundle V/V^{S^1} is complex over N, which is a connected space with a trivial S^1-action, we can associate the weights of the S^1-action: $m_1, \ldots, m_r$, which are also called complex rotation numbers of the S^1-action. They need not be distinct. We write $m_j = q_j \cdot n + r_j$, with $0 \le r_j < n$. Now the rotation numbers fall into classes with respect to their remainder modulo n. Define for all $0 \le k \le \frac{n}{2}$

$$I_k = \{j \in 1, \ldots, r \mid r_j = k \text{ or } n-k\} \quad .$$

I_k contains exactly the rotation numbers corresponding to the k'th term in the decomposition of $(V/V^{S^1})_{cx}$ with respect to the $\mathbb{Z}_n$-action. We get the following formula:

$$t_\alpha^* e_{S^1}^E((V/V^{S^1})_{cx})^{-1} =$$

$$\prod_{j\in I_0} s(x_j+m_j\alpha)^{-1}\cdot \prod_{\substack{0<k<n/2\\ j\in I_k}} s(x_j+m_j\alpha)^{-1}\cdot \prod_{j\in I_{n/2}} s(x_j+m_j\alpha)^{-1} \tag{6}$$

Before we analyze each term in the above formula, recall that $m_j = q_j\cdot n + r_j$, with $0\le r_j < n$.

a) $j\in I_0$: Here we chose the complex structure $(V^{\mathbb{Z}_n}/V^{S^1})_{cx}$ such that all $m_j>0$. Then, since $s(x_j+m_j\alpha) = s(x_j+q_jn\alpha) = \epsilon^{q_j}s(x_j)$, we have: $\prod_{j\in I_0} s(x_j+m_j\alpha)^{-1} = \epsilon^{\sum_{I_0} q_j}\cdot\prod_{I_0} s(x_j)^{-1} = \epsilon^{\sum_{I_0} q_j}\cdot e_{S^1}^E(V^{\mathbb{Z}_n}/V^{S^1})_{cx}^{-1} = \epsilon^{\sum_{I_0} q_j}\cdot(-1)^{\sigma(0)}\cdot e_{S^1}^E(V^{\mathbb{Z}_n}/V^{S^1})_{or}^{-1}$. So we get eventually

$$\prod_{j\in I_0} s(x_j+m_j\alpha)^{-1} = \epsilon^{\sum_{I_0} q_j}\cdot(-1)^{\sigma(0)}\cdot e_{S^1}^E(V^{\mathbb{Z}_n}/V^{S^1})_{or}^{-1}\quad . \tag{7}$$

b) $j\in I_k$, $0\le k\le \frac{n}{2}$. The complex structure on $V(k)$ is such that $g=e^{2\pi i/n}\in\mathbb{Z}_n$ acts by complex multiplication with g^k. Notice that in Porposition 6.1 we defined the complex structure on V/V^{S^1} so that for $j\in I_k$, $m_j = nq_j+k$. g acts as $g^{m_j}=g^k$, so the complex structures on $V(K)$ and $i^*V(K)$ are compatible. We have $s(x_j+m+j\alpha) = s(x_j+q_jn\alpha+k\alpha) = \epsilon^{q_j}s(x_j+k\alpha)$.

Consider μ_k the equivariant class on P corresponding to the complex vector bundle $V(k)$ with its chosen complex orientation, and the convergent power series $s(x+k\alpha)^{-1}$. Then $i^*\mu_k = \prod_{I_k} s(x_j+k\alpha)^{-1}$. Define $\mu_K = \prod_{0<k<\frac{n}{2}} = \mu_k$. We obtain $s(x_j+m_j\alpha) = s(x_j+q_jn\alpha+k\alpha) = \epsilon^{q_j}s(x_j+k\alpha)$, which implies

$$\prod_{k,j\in I_k} s(x_j+m_j\alpha)^{-1} = \epsilon^{\sum_{k,I_k} q_j}\cdot i^*\mu_K\quad . \tag{8}$$

c) $j\in I_{n/2}$. The complex structure on $i^*V(\frac{n}{2})$ is the one for which all $m_j>0$. The rotation numbers satisfy $m_j = q_jn+\frac{n}{2}$, hence $s(x_j+m_j\alpha) = \epsilon^{q_j}s(x_j+\frac{n}{2}\alpha)$. Now consider $\mu_{\frac{n}{2}}$ the equivariant characteristic class on P corresponding to the real vector bundle $V(\frac{n}{2})$

with its correct orientation, and the convergent power series $Q(x) = s(x + \frac{n}{2}\alpha)^{-1}$. $Q(x)$ satisfies $Q(-x) = s(-x + \frac{n}{2}\alpha)^{-1} = -s(x - \frac{n}{2}\alpha)^{-1} = -\epsilon s(x + \frac{n}{2}\alpha)^{-1} = (-\epsilon)Q(x)$, hence, according to Lemma A.8, $i^*\mu_{\frac{n}{2}} = (-\epsilon)^{\sigma(\frac{n}{2})} \prod_{j\in I_k} s(x_j + \frac{n}{2}\alpha)^{-1}$, where this latter class is calculated using the complex structure on $i^*V(\frac{n}{2})$. Finally we obtain

(9)
$$\prod_{j\in I_{n/2}} s(x_j + m_j\alpha)^{-1} = \epsilon^{\sum_{I_{n/2}} q_j} \cdot (-\epsilon)^{\sigma(\frac{n}{2})} \cdot i^*\mu_{\frac{n}{2}} \ .$$

Now, putting together equations (3)–(9), and defining $\mu_P := \mu_K \cdot \mu_{fracn2}$, we have proved that $t_\alpha^* e_{S^1}^E(V//V^{S^1})^{-1} = \epsilon^{\sigma(N)} \cdot e_{S^1}^E(V^{\mathbb{Z}_n}/V^{S^1})^{-1} \cdot i^*\mu_P$, or

(10)
$$t_\alpha^* e_{S^1}^E(V/V^{S^1})^{-1} \cdot e_{S^1}^E(V^{\mathbb{Z}_n}/V^{S^1}) = \epsilon^{\sigma(N)} \cdot i^*\mu_P \ ,$$

where

$$\sigma(N) = \sum_{I_0} q_j + \sum_{k,I_k} q_j + \sum_{I_{n/2}} q_j + \sigma(K) + \sigma(\frac{n}{2}) \ .$$

Notice that $\sigma(N)$ is described in terms of rotation numbers m_j of the S^1-vector bundle $(V/V^{S^1})_{cx}$. What if we consider instead m_j^*, the rotation numbers of $(V/V^{S^1})_{or}$? First, m_j^* are the same as m_j up to a sign (and a permutation). Write $m_j^* = q_j^* n + r_j^*$, $0 \le r_j^* \le \frac{n}{2}$. We have the following cases:

a) $j \in I_0$. If $m_j^* = -m_j$, then $q_j^* = -q_j$, and the parity of $\sigma(N)$ doesn't change.

b) $j \in I_k$, $0 \le k \le \frac{n}{2}$. Then we cannot have $m_j^* = -m_j$, because we chose $0 \le r_j, r_j^* \le \frac{n}{2}$

c) $j \in I_{n/2}$. If $m_j^* = -m_j$, then $m_j^* = -m_j = -q_j n - \frac{n}{2} = -(q_j+1)n + \frac{n}{2}$, so $q_j^* = -(q_j+1)$, and the parity of $\sigma(N)$ changes.

Since the orientations of $(V/V^{S^1})_{cx}$ and $(V/V^{S^1})_{or}$ differ by a parity of $\sigma(0) + \sigma(K) + \sigma(\frac{n}{2})$, we get

$$\sigma(N) = \sum_{I_0} q_j^* + \sum_{k,I_k} q_j^* + \sum_{I_{n/2}} q_j^* \ .$$

In the next lemma we will show that, for N and $\tilde{N}$ two different connected components of X^{S^1} inside P, $\sigma(N)$ and $\sigma(\tilde{N}$ are congruent modulo 2, so the class $\epsilon^{\sigma(N)} \cdot \mu_P$ is well-defined (independent of N). Now define

$$\mu_\alpha := \sum_P \epsilon^{\sigma(N)} \cdot \mu_P \in HO^*_{S^1}(X^{\mathbb{Z}_n}) = \oplus_P HO^*_{S^1}(P) \ .$$

This is a well-defined class, and since we have equation (10), Lemma 6.4 is proved. □

Lemma 6.5. *In the conditions of the previous lemma, $\sigma(N)$ and $\sigma(\tilde{N})$ are congruent modulo 2.*

Proof. The proof follows Bott & Taubes [6]. (Again they use the level of generality that we need.) Denote by $S^2(n)$ the 2-sphere with the S^1-action which rotates S^2 n times around the north-south axis as we go once around S^1. Denote by N^+ and N^- its North and South poles, respectively. Consider a path in P which connects N with $\tilde{N}$, and touches N or $\tilde{N}$ only at its endpoints. By rotating this path with the S^1-action, we obtain a subspace of P which is close to being an embedded $S^2(n)$. Even if it is not, we can still map equivariantly $S^2(n)$ onto this rotated path. Now we can pull back the bundles from P to $S^2(n)$ (with their correct orientations). The rotation numbers are the same, since the North and the South poles are fixed by the S^1-action, as are the endpoints of the path.

Therefore we have translated the problem to the case when we have the 2-sphere $S^2(n)$ and corresponding bundles over it, and we are trying to prove that $\sigma(N^+) \equiv \sigma(N^-)$ modulo 2. The only problem would be that we are not using the whole of V, but only V/V^{S^1}. However, the difference between these two bundles is V^{S^1}, whose rotation numbers are all zero, so they do not influence the result.

Now Lemma 9.2 of [6] says that any even-dimensional oriented real vector bundle W over $S^2(n)$ has a complex structure. In particular, the pullbacks of V^{S^1}, $V(K)$, and $V(\frac{n}{2})$ have complex structure, and the rotation numbers can be chosen to be the m_j^* described above. Say the rotation numbers at the South pole are $\tilde{m}_j^*$ with the obvious notation conventions. Then Lemma 9.1 of [6] says that, up to a permutation,

$m_j^* - \tilde{m}_j^* = n(q_j^* - \tilde{q}_j^*)$, and $\sum q_j^* \equiv \sum \tilde{q}_j^*$ modulo 2. But this means that $\sigma(N^+) \equiv \sigma(N^-)$ modulo 2, i.e. $\sigma(N) \equiv \sigma(\tilde{N})$ modulo 2. □

Corollary 6.6. *(The Rigidity theorem of Witten) If X is a spin manifold with an S^1-action, then the equivariant elliptic genus of X is rigid i.e. it is a constant power series.*

Proof. By the usual trick of lifting the S^1-action to a double cover of S^1, we can make the S^1-action preserve the spin structure. Then we will say that X is a spin S^1-manifold.

At the beginning of this Section, we say that if X is a compact spin S^1-manifold, i.e. the map $\pi : X \to point$ is spin, then we have the Grojnowski pushforward, which is a map of sheaves

$$\pi_!^E : \boldsymbol{E}_{S^1}^*(X)^{[\pi]} \to \boldsymbol{E}_{S^1}^*(point) = \mathcal{O}_{\mathcal{E}} \ .$$

The elliptic pushforward $\pi_!^E$, if we consider it at the level of stalks at $0 \in \mathcal{E}$, is nothing but the elliptic pushforward in $HO_{S^1}^*$-theory, as described in Corollary 5.4. So consider the element 1 in the stalk at 0 of the sheaf $\boldsymbol{E}_{S^1}^*(X)^{[\pi]} = \boldsymbol{E}_{S^1}^*(X)^{[TX]}$.

From Theorem 6.3, since TX is spin, 1 extends to a global section of $\boldsymbol{E}_{S^1}^*(X)^{[TX]}$. Denote this global section by boldface $\mathbf{1}$. Because $\pi_!^E$ is a map of sheaves, it follows that $\pi_!^E(\mathbf{1})$ is a global section of $\boldsymbol{E}_{S^1}^*(point) = \mathcal{O}_{\mathcal{E}}$, i.e. a global holomorphic function on the elliptic curve $\mathcal{E}$. But any such function has to be constant. This means that $\pi_!^E(1)$, which is the equivariant elliptic genus of X, extends to $\pi_!^E(\mathbf{1})$, which is constant. This is precisely equivalent to the elliptic genus being rigid. □

The extra generality we had in Theorem 6.3 allows us now to extend the Rigidity theorem to families of elliptic genera. This was stated as THEOREM D in Section 2.

Theorem 6.7. *(Rigidity for families) Let $F \to E \xrightarrow{\pi} B$ be an S^1-equivariant fibration such that the fibers are spin in a compatible way, i.e. the projection map π is spin oriented. Then the elliptic genus of the family, which is $\pi_!^E(1) \in H_{S^1}^{**}(B)$, is constant as a rational function in u (i.e. if we invert the generator u of $\mathbb{C}[\![u]\!]$, over which $H_{S^1}^{**}(B)$ is a module).*

Proof. We know that the map

$$\pi_!^E : \boldsymbol{E}^*_{S^1}(E)^{[\pi]} \to \boldsymbol{E}^*_{S^1}(B)$$

when regarded at the level of stalks at zero is the usual equivariant elliptic pushforward in $HO^*_{S^1}(-)$. Now $\pi_!^E(1) \in HO^*_{S^1}(B)$ is the elliptic genus of the family. We have $\boldsymbol{E}^*_{S^1}(E)^{[\pi]} \cong \boldsymbol{E}^*_{S^1}(E)^{[\tau(F)]}$, where $\tau(F) \to E$ is the bundle of tangents along the fiber.

Since $\tau(F)$ is spin, Theorem 6.3 allows us to extend 1 to the Thom section $\mathbf{1}$. Since $\pi_!^E$ is a map of sheaves, it follows that $\pi_!^E(1)$, which is the elliptic genus of the family, extends to a global section in $\boldsymbol{E}^*_{S^1}(B)$. So, if $i : B^{S^1} \hookrightarrow B$ is the inclusion of the fixed point submanifold in B, $i^*\pi_!^E(\mathbf{1})$ gives a global section in $\boldsymbol{E}^*_{S^1}(B^{S^1})$. Now this latter sheaf is trivial as a sheaf of $\mathcal{O}_{\mathcal{E}}$-modules, so any global section is constant. But i^* is an isomorphism in $HO^*_{S^1}(-)$ if we invert u. □

We saw in the previous section that, if $f : X \to Y$ is an S^1-map of compact S^1-manifolds such that the restrictions $f : X^\alpha \to Y^\alpha$ are oriented maps, we have the Grojnowski pushforward

$$f_!^E : \boldsymbol{E}^*_{S^1}(X)^{[f]} \to \boldsymbol{E}^*_{S^1}(Y) \ .$$

Also, in some cases, for example when f is a spin S^1-fibration, we saw that $\boldsymbol{E}^*_{S^1}(X)^{[f]}$ admits a Thom section. This raises the question if we can describe $\boldsymbol{E}^*_{S^1}(X)^{[f]}$ as a $\boldsymbol{E}^*_{S^1}$ of a Thom space. It turns out that, up to a line bundle over $\mathcal{E}$ (which is itself $\boldsymbol{E}^*_{S^1}$ of a Thom space), this indeed happens:

Let $f : X \to Y$ be an S^1-map as above. Embed X into an S^1-representation W, $i : X \hookrightarrow W$. (W can be also thought as an S^1-vector bundle over a point.) Look at the embedding $f \times i : X \hookrightarrow Y \times W$. Denote by $V = \nu(f)$, the normal bundle of X in this embedding (if we were not in the equivariant setup, $\nu(f)$ would be the stable normal bundle to the map f).

Proposition 6.8. *With the previous notations,*

$$\boldsymbol{E}^*_{S^1}(X)^{[f]} \cong \boldsymbol{E}^*_{S^1}(DV, SV) \otimes \boldsymbol{E}^*_{S^1}(DW, SW)^{-1} \ .$$

DV, SV are the disk and the sphere bundles of V, respectively.

Proof. From the embedding $X \hookrightarrow Y \times W$, we have the following isomorphism of vector bundles:

$$TX \oplus V \cong f^*TY \oplus W \ .$$

So, in terms of S^1-equivariant elliptic Thom classes we have

$$e^E_{S^1}(V^\alpha/V^\beta) = e^E_{S^1}(X^\alpha/X^\beta)^{-1} \cdot f^* e^E_{S^1}(Y^\alpha/Y^\beta) \cdot e^E_{S^1}(W^\alpha/W^\beta) \ .$$

Multiplication by the equivariant elliptic Thom class $\phi^E_{S^1}(V^\alpha)$ on each stalk gives the following commutative diagram

$$\begin{array}{ccc}
H^*_{S^1}(X^\alpha) \otimes_{\mathbb{C}[u]} \mathcal{O}_{\mathcal{E}}(U-\alpha) & \xrightarrow{\cdot t^*_\alpha e^E_{S^1}(V/V^\alpha)} & H^*_{S^1}(X^\alpha) \otimes_{\mathbb{C}[u]} \mathcal{O}_{\mathcal{E}}(U-\alpha) \\
\downarrow{\scriptstyle i^*} & & \downarrow{\scriptstyle i^* \cdot e^E_{S^1}(V^\alpha/V^\beta)} \\
H^*_{S^1}(X^\beta) \otimes_{\mathbb{C}[u]} \mathcal{O}_{\mathcal{E}}(U-\alpha) & \xrightarrow{\cdot t^*_\alpha e^E_{S^1}(V/V^\beta)} & H^*_{S^1}(X^\beta) \otimes_{\mathbb{C}[u]} \mathcal{O}_{\mathcal{E}}(U-\alpha) \\
\downarrow{\scriptstyle t^*_{\beta-\alpha}} & & \downarrow{\scriptstyle t^*_{\beta-\alpha}} \\
H^*_{S^1}(X^\beta) \otimes_{\mathbb{C}[u]} \mathcal{O}_{\mathcal{E}}(U-\beta) & \xrightarrow{\cdot t^*_\beta e^E_{S^1}(V/V^\beta)} & H^*_{S^1}(X^\beta) \otimes_{\mathbb{C}[u]} \mathcal{O}_{\mathcal{E}}(U-\beta) \ .
\end{array}$$

This gives an isomorphism of sheaves

$$\boldsymbol{E}^*_{S^1}(DV, SV) \cong \boldsymbol{E}^*_{S^1}(X)^{[f]} \otimes \boldsymbol{E}^*_{S^1}(DW, SW) \ .$$

The latter sheaf $\boldsymbol{E}^*_{S^1}(DW, SW)$ has stalks $HO^*_{S^1}(DW^\alpha, SW^\alpha) \cong HO^*_{S^1}(point) = \mathcal{O}_{\mathbb{C},0}$, so it is invertible. In fact, we can identify it by the same method we used in Proposition 3.12. □

This suggests that we can define Gysin maps if we compose the Grojnowski pushforward with multiplication by a Thom section. They are well-defined and functorial again up to a line bundle.

Appendix A. Equivariant characteristic classes

The results of this section are well-known, with the exception of the holomorphicity result Proposition A.5.

Let V be a complex n-dimensional S^1-equivariant vector bundle over an S^1-CW complex X. Then to any power series $Q(x) \in \mathbb{C}[\![x]\!]$ starting with 1 we are going to associate by Hirzebruch's formalism (see [15]) a multiplicative characteristic class $\mu_Q(V)_{S^1} \in H^{**}_{S^1}(X)$. (Recall that $H^{**}_{S^1}(X)$ is the completion of $H^*_{S^1}(X)$.)

Consider the Borel construction for both V and X: $V_{S^1} = V \times_{S^1} ES^1 \to X \times_{S^1} ES^1 = X_{S^1}$. $V_{S^1} \to X_{S^1}$ is a complex vector bundle over a paracompact space, hence we have a classifying map $f_V : X_{S^1} \to BU(n)$. We know that the image via f_V^* of the universal j'th Chern class $c_j \in H^*BU(n) = \mathbb{C}[c_1, \ldots, c_n]$ is the equivariant j'th Chern class of V, $c_j(V)_{S^1}$. Now look at the product $Q(x_1)Q(x_2)\cdots Q(x_n)$. It is a power series in $x_1, \ldots, x_n$ which is symmetric under permutations of the x_j's, hence it can be expressed as another power series in the elementary symmetric functions $\sigma_j = \sigma_j(x_1, \ldots, x_n)$:

$$Q(x_1)\cdots Q(x_n) = P_Q(\sigma_1, \ldots, \sigma_n) \ .$$

Notice that $P_Q(c_1, \ldots, c_n)$ lies not in $H^*BU(n)$, but in its completion $H^{**}BU(n)$. The map f_V^* extends to a map $H^{**}BU(n) \to H^{**}(X_{S^1})$.

Definition A.1. *Given the power series $Q(x) \in \mathbb{C}[\![x]\!]$ and the complex S^1-vector bundle V over X, there is a canonical complex equivariant characteristic class $\mu_Q(V)_{S^1} \in H^{**}(X_{S^1})$, given by*

$$\mu_Q(V)_{S^1} := P_Q(c_1(V)_{S^1}, \ldots, c_n(V)_{S^1}) = f_V^* P_Q(c_1, \ldots, c_n) \ .$$

Remark A.2. If $T^n \hookrightarrow BU(n)$ is a maximal torus, then then $H^*BT^n = \mathbb{C}[x_1, \ldots, x_n]$, and the x_j's are called the universal Chern roots. The map $H^*BU(n) \to H^*BT^n$ is injective, and its image can be identified as the Weyl group invariants of H^*BT^n. The Weyl group of $U(n)$ is the symmetric group on n letters, so $H^*BU(n)$ can be identified as the subring of symmetric polynomials in $\mathbb{C}[x_1, \ldots, x_n]$. Similarly, $H^{**}BU(n)$

is the subring of symmetric power series in $\mathbb{C}[\![x_1,\dots,x_n]\!]$. Under this interpretation, $c_j = \sigma_j(x_1,\dots,x_n)$. This allows us to identify $Q(x_1)\cdots Q(x_n)$ with the element $P_Q(c_1,\dots,c_n) \in H^{**}BU(n)$.

Definition A.3. *We can write formally* $\mu_Q(V)_{S^1} = Q(x_1)\cdots Q(x_n)$*.* $x_1,\dots,x_n$ *are called the equivariant Chern roots of* V*.*

We want now to show that the class we have just constructed, $\mu_Q(V)_{S^1}$, is holomorphic in a certain sense, provided $Q(x)$ is the expansion of a holomorphic function around zero. But first, let us state a classical lemma in the theory of symmetric functions.

Lemma A.4. *Suppose* $Q(y_1,\dots,y_n)$ *is a holomorphic (i.e. convergent) power series, which is symmetric under permutations of the* y_j*'s. Then the power series* P *such that*

$$Q(y_1,\dots,y_n) = P(\sigma_1(y_1,\dots,y_n),\dots,\sigma_n(y_1,\dots,y_n)) \ ,$$

is holomorphic.

We have mentioned above that $\mu_Q(V)_{S^1}$ belongs to $H^{**}_{S^1}(X)$. This ring is equivariant cohomology tensored with power series. It contains $HO^*_{S^1}(X)$ as a subring, corresponding to the holomorphic power series.

Proposition A.5. *If* $Q(x)$ *is a convergent power series, then* $\mu_Q(V)_{S^1}$ *is a holomorphic class, i.e. it belongs to the subring* $HO^*_{S^1}(X)$ *of* $H^{**}_{S^1}(X)$*.*

Proof. We have $\mu_Q(V)_{S^1} = P(c_1(V)_{S^1},\dots,c_n(V)_{S^1})$, where we write P for P_Q.

Assume X has a trivial S^1-action. It is easy to see that $H^*_{S^1}(X) = (H^0(X)\otimes_{\mathbb{C}}\mathbb{C}[u])\oplus$ nilpotents. Hence we can write $c_j(E)_{S^1} = f_j + \alpha_j$, with $f_j \in H^0(X)\otimes_{\mathbb{C}}\mathbb{C}[u]$, and α_j nilpotent in $H^*_{S^1}(X)$. We expand $\mu_Q(V)_{S^1}$ in Taylor expansion in multiindex notation. We make the following notations: $\lambda = (\lambda_1,\cdots,\lambda_n) \in \mathbb{N}^n$, $|\lambda| = \lambda_1 + \cdots + \lambda_n$, and $\alpha^\lambda = \alpha_1^{\lambda_1}\cdots\alpha_n^{\lambda_n}$. Now we consider the Taylor expansion of $\mu_Q(V)_{S^1}$ in multiindex notation:

$$\mu_Q(V)_{S^1} = P(\dots,c_j(V)_{S^1},\dots) = \sum_\lambda \frac{\partial^{|\lambda|}P}{\partial c^\lambda}(\dots,f_j,\dots)\cdot\alpha^\lambda \ .$$

This is a finite sum, since α_j's are nilpotent. We want to show that $\mu_Q(V)_{S^1} \in HO^*_{S^1}(X)$. α^λ lies in $HO^*_{S^1}(X)$, since it lies even in $H^*_{S^1}(X)$. So we only have to show that $\frac{\partial^{|\lambda|}P}{\partial c^\lambda}(\ldots, f_j, \ldots)$ lies in $HO^*_{S^1}(X)$.

But $f_j \in H^0(X) \otimes_{\mathbb{C}} \mathbb{C}[u] = \mathbb{C}[u] \oplus \cdots \oplus \mathbb{C}[u]$, with one $\mathbb{C}[u]$ for each connected component of X. If we fix one such component N, then the corresponding component $f_j^{(N)}$ lies in $\mathbb{C}[u]$. According to Lemma A.4, P is holomorphic around $(0, \ldots, 0)$, hence so is $\frac{\partial^{|\lambda|}P}{\partial c^\lambda}$. Therefore $\frac{\partial^{|\lambda|}P}{\partial c^\lambda}(\ldots, f_j^{(N)}(u), \ldots)$ is holomorphic in u around 0, i.e. it lies in $\mathcal{O}_{\mathbb{C},0}$. Collecting the terms for the different connected components of X, we finally get

$$\frac{\partial^{|\lambda|}P}{\partial c^\lambda}(\ldots, f_j, \ldots) \in \mathcal{O}_{\mathbb{C},0} \oplus \cdots \oplus \mathcal{O}_{\mathbb{C},0} = H^0(X) \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{C},0} \ .$$

But $H^0(X) \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{C},0} \subseteq H^*(X) \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{C},0} = H^*_{S^1}(X) \otimes_{\mathbb{C}[u]} \mathcal{O}_{\mathbb{C},0} = HO^*_{S^1}(X)$, so we are done.

If the S^1-action on X is not trivial, look at the following exact sequence associated to the pair (X, X^{S^1}):

$$0 \to T \hookrightarrow H^*_{S^1}(X) \xrightarrow{i^*} H^*_{S^1}(X^{S^1}) \xrightarrow{\delta} H^{*+1}_{S^1}(X, X^{S^1}) \ ,$$

where T is the torsion submodule of $H^*_{S^1}(X)$. (The fact that $T = \ker i^*$ follows from the following arguments: on the one hand, $\ker i^*$ is torsion, because of the localization theorem; on the other hand, $H^*_{S^1}(X^{S^1})$ is free, hence all torsion in $H^*_{S^1}(X)$ maps to zero via i^*.) Also, since T is a direct sum of modules of the form $\mathbb{C}[u]/(u^n)$, it is easy to see that

$$T \otimes_{\mathbb{C}[u]} \mathcal{O}_{\mathbb{C},0} \cong T \cong T \otimes_{\mathbb{C}[u]} \mathbb{C}[\![u]\!] \ .$$

Now tensor the above exact sequence with $\mathcal{O}_{\mathbb{C},0}$ and $\mathbb{C}[\![u]\!]$ over $\mathbb{C}[u]$:

$$\begin{array}{ccccccccc}
0 & & T & & HO^*_{S^1}(X) & \overset{i^*}{} & HO^*_{S^1}(X^{S^1}) & \overset{\delta}{} & HO^{*+1}_{S^1}(X, X^{S^1}) \\
 & & & & s & & t & & \\
0 & & T & & HP^*_{S^1}(X) & \overset{i^*}{} & HP^*_{S^1}(X^{S^1}) & \overset{\delta}{} & HP^{*+1}_{S^1}(X, X^{S^1}) \ .
\end{array}$$

We know $\alpha := \mu_Q(V)_{S^1} \in HP^*_{S^1}(X)$. Then $\beta := i^*\mu_Q(V)_{S^1} = i^*\alpha$ was showed previously to be in the image of t, i.e. $\beta = t\tilde{\beta}$. $\delta\beta = \delta i^*\alpha = 0$, so $\delta t\tilde{\beta} = 0$, hence $\delta\tilde{\beta} = 0$. Thus $\tilde{\beta} \in \text{Im } i^*$, so there is an $\tilde{\alpha} \in HO^*_{S^1}(X)$ such that $\tilde{\beta} = i^*\tilde{\alpha}$. $s\tilde{\alpha}$ might not equal α, but $i^*(\alpha - \tilde{\alpha}) = 0$, so $\alpha - \tilde{\alpha} \in T$. Consider $\tilde{\alpha} + (\alpha - \tilde{\alpha}) \in HO^*_{S^1}(X)$. Now $s(\tilde{\alpha} + (\alpha - \tilde{\alpha}) = \alpha$, which shows that indeed $\alpha \in \text{Im } s = HO^*_{S^1}(X)$. $\square$

There is a similar story when V is an oriented $2n$-dimensional real S^1-vector bundle over a finite S^1-CW complex X. We classify $V_{S^1} \to X_{S^1}$ by a map $f_V : X_{S^1} \to BSO(2n)$. $H^*BSO(2n) = \mathbb{C}[p_1, \ldots, p_n]/(e^2 - p_n)$, where p_j and e are the universal Pontrjagin and Euler classes, respectively. The only problem now is that in order to define characteristic classes over $BSO(2n)$ we need the initial power series $Q(x) \in \mathbb{C}[\![x]\!]$ to be either even or odd:

Remark A.6. As in Remark A.2, if $T^n \hookrightarrow BSO(2n)$ is a maximal torus, then the map $H^*BSO(2n) \to H^*BT^n$ is injective, and its image can be identified as the Weyl group invariants of H^*BT^n. The Weyl group of $SO(2n)$ is the semidirect product of the symmetric group on n letters with $\mathbb{Z}_2$, so $H^*BSO(2n)$ can be identified as the subring of symmetric polynomials in $\mathbb{C}[x_1, \ldots, x_n]$ which are invariant under an even number of sign changes of the x_j's. A similar statement holds for $H^{**}BSO(2n)$. Under this interpretation, $p_j = \sigma_j(x_1^2, \ldots, x_n^2)$ and $e = x_1 \cdots x_n$.

So, if we want $Q(x_1) \cdots Q(x_n)$ to be interpreted as an element of $H^{**}BSO(2n)$, we need to make it invariant under an even number of sign changes. But this is clearly true if $Q(x)$ is either an even or an odd power series.

Let us be more precise:

a) $Q(x)$ is even, i.e. $Q(-x) = Q(x)$. Then there is another power series $S(x)$ such that $Q(x) = S(x^2)$, so $Q(x_1) \cdots Q(x_n) = S(x_1^2) \cdots S(x_n^2) = P_S(\ldots, \sigma_j(x_1^2, \ldots, x_n^2), \ldots) = P_S(\ldots, p_j, \ldots)$.

b) $Q(x)$ is odd, i.e. $Q(-x) = -Q(x)$. Then there is another power series $R(x)$ such that $Q(x) = xT(x^2)$, so $Q(x_1) \cdots Q(x_n) = x_1 \cdots x_n T(x_1^2) \cdots T(x_n^2) = x_1 \cdots x_n P_T(\ldots, \sigma_j(x_1^2, \ldots, x_n^2), \ldots) = e \cdot P_T(\ldots, p_j, \ldots)$.

Definition A.7. *Given the power series $Q(x) \in \mathbb{C}[\![x]\!]$ which is either even or odd, and the real oriented S^1-vector bundle V over X, there is a canonical real equivariant characteristic class $\mu_Q(V)_{S^1} \in H^{**}_{S^1}(X)$, defined by pulling back the element $Q(x_1)\cdots Q(x_n) \in H^{**}BSO(2n)$ via the classifying map $f_V : X_{S^1} \to BSO(2n)$.*

Proposition A.5 can be adapted to show that, if $Q(x)$ is a convergent power series, $\mu_Q(V)_{S^1}$ actually lies in $HO^*_{S^1}(X)$.

The next result is used in the proof of Lemma 6.4.

Lemma A.8. *Let V be an orientable S^1-equivariant even dimensional real vector bundle over X. Suppose we are given two orientations of V, which we denote by V_{or_1} and V_{or_2}. Define $\sigma = 0$ if $V_{or_1} = V_{or_2}$, and $\sigma = 1$ otherwise. Also, suppose $Q(x)$ is a power series such that $Q(-x) = \alpha Q(x)$, where $\alpha = \pm 1$. Then*

$$\mu_Q(V_{or_1}) = \alpha^\sigma \mu_Q(V_{or_2}) \ .$$

Proof.

a) If $Q(-x) = Q(x)$, $\mu_Q(V)$ is a power series in the equivariant Pontrjagin classes $p_j(V)_{S^1}$. But Pontrjagin classes are independent of the orientation, so $\mu_Q(V_{or_1}) = \mu_Q(V_{or_2})$.

b) If $Q(-x) = -Q(x)$, then $Q(x) = x\tilde{Q}(x)$, with $\tilde{Q}(-x) = \tilde{Q}(x)$. Hence $\mu_Q(V) = e_{S^1}(V)\cdot\mu_{\tilde{Q}}(V)$. $e(V)_{S^1}$ changes sign when orientation changes sign, while $\mu_{\tilde{Q}}(V)$ is invariant, because of a).

□

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