# Uniqueness of Equilibrium in the Kyle (1985) Model

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**Abstract**

This note analyzes the uniqueness of the equilibrium in the one-period version of the Kyle (1985) model. It reformulates the problem as the uniqueness of the solution of an implicit functional equation.

## 1 The Kyle (1985) Model

- This is a 1-period trading model, in which trading takes place at $t = 0$, and the truth is revealed at $t = 1$.
- There is a single risky asset with normally distributed liquidation value (at $t = 1$):

$$v \sim \mathcal{N}(p_0, \sigma_v^2). \tag{1}$$

- There are 3 types of market participants:
  - Noise traders, who trade in aggregate an exogenous amount:

    $$u \sim \mathcal{N}(0, \sigma_u^2), \tag{2}$$

    such that $u$ and $v$ are independently distributed.
  - One risk-neutral trader (the "insider"), who at time $t = 0$ observes $v$, but not $u$. The insider trades at $t = 0$ a quantity $x$ which maximizes expected profit at $t = 1$, to be defined below. The optimum demand $x$ is a function of $v$:

    $$x = X(v). \tag{3}$$

  - Risk-neutral market makers, who do not observe $v$, but observe the aggregate order from the noise traders and the insider

    $$y = x + u. \tag{4}$$

    The market makers do not observe either $x$ or $u$ separately. They determine the price $p$ at which trading at $t = 0$ takes place. They set the price in a competitive way, so that the market is efficient, to be explained below. The competitive price is a function of $x + u$:

    $$p = P(x + u). \tag{5}$$

- The profit of the insider is denoted by:

$$\pi \;=\; \Pi(v,p,x) \;=\; (v-p)x. \tag{6}$$

- The equilibrium is a pair $(X,P)$, where $X$ the trading strategy of the insider, and $P$ the pricing rule of the market makers. $X$ and $P$ satisfy:

  - Market efficiency (i.e., the market makers make zero profit)

$$P(x+u) \;=\; \mathsf{E}(v \mid x+u). \tag{7}$$

  - Profit maximization:

$$X(v) \;=\; \arg\max_x \mathsf{E}\big((v-P(x+u))x\big). \tag{8}$$

- **Theorem:** There exists a unique linear equilibrium $(X,P)$. Define constants $\beta$ and $\lambda$ by:

$$\beta \;=\; \frac{\sigma_u}{\sigma_v}, \qquad \lambda \;=\; \frac{1}{2\beta} \;=\; \frac{1}{2}\frac{\sigma_v}{\sigma_u}. \tag{9}$$

  Then the equilibrium demand and price are given by

$$X(v) \;=\; \beta(v-p_0), \tag{10}$$

$$P(x+u) \;=\; p_0+\lambda(x+u). \tag{11}$$

- **Proof:** Write $P$ and $X$ as linear functions:

$$X(v) \;=\; \alpha+\beta v, \tag{12}$$

$$P(y) \;=\; \mu+\lambda y. \tag{13}$$

  Given the linear price rule $P$, the insider solves

$$\max_x \mathsf{E}\big(v-\mu-\lambda(x+u)\big)x \;=\; \max_x (v-\mu-\lambda x)x. \tag{14}$$

  The first order condition implies $v-\mu-2\lambda x = 0$, or $x = X(v) = -\frac{\mu}{2\lambda}+\frac{1}{2\lambda}v$. Therefore

$$\alpha \;=\; -\frac{\mu}{2\lambda} \qquad \text{and} \qquad \beta \;=\; \frac{1}{2\lambda}. \tag{15}$$

  The market makers set the price to satisfy

$$\mu+\lambda y \;=\; \mathsf{E}(v \mid \alpha+\beta v+u=y). \tag{16}$$

  To compute this, use the properties of the linear regression:

$$\mathsf{E}(Y|Z) \;=\; a+bZ, \tag{17}$$

  where

$$b \;=\; \frac{\mathsf{cov}(Y,Z)}{\mathsf{Var}(Z)} \qquad \text{and} \qquad a \;=\; \mu_Y - b\mu_Z. \tag{18}$$

  In our case, $Y=v$ and $Z=\alpha+\beta v+u$, so we get

$$\lambda \;=\; \frac{\beta\sigma_0^2}{\beta\sigma_0^2+\sigma_u^2} \qquad \text{and} \qquad \mu \;=\; p_0-\lambda(\alpha+\beta p_0). \tag{19}$$

  Now use the 4 equations for $\alpha$, $\beta$, $\lambda$ and $\mu$ to show that their values are indeed as stated.

## 2 Uniqueness Condition

We next state a necessary condition that any equilibrium, including a non-linear one must satisfy. We have already defined an equilibrium as a pair $(X, P)$ that satisfies (5) and (6):

$$P(y) = \mathsf{E}(v \mid X(v) + u = y), \tag{20}$$

$$X(v) = \arg\max_x \mathsf{E}\big((v - P(x+u))x\big). \tag{21}$$

Start with the pricing formula (20). Since $v$ and $u$ are independent, we can write

$$P(y) = \mathsf{E}(v|u = y - X(v)) \tag{22}$$

$$= \frac{\int_{-\infty}^{+\infty} v\phi\big(\frac{v}{\sigma_v}\big)\phi\big(\frac{y-X(v)}{\sigma_u}\big)\,\mathrm{d}v}{\int_{-\infty}^{+\infty} \phi\big(\frac{v}{\sigma_v}\big)\phi\big(\frac{y-X(v)}{\sigma_u}\big)\,\mathrm{d}v}, \tag{23}$$

where

$$\phi(x) = \frac{1}{(2\pi)^{1/2}}\,\mathrm{e}^{-\frac{1}{2}x^2} \tag{24}$$

is the standard normal density.

The insider takes $P$ as given, thus the optimal strategy must satisfy the first order condition with respect to $x$:

$$\mathsf{E}\big(v - P(x+u) - xP'(x+u)\big) = 0. \tag{25}$$

Note that the expectation (integration) is over $u$, since $v$ is known by the insider. Then

$$\int_{-\infty}^{+\infty} \Big(v - P(x+u) - xP'(x+u)\Big)\phi\Big(\frac{u}{\sigma_u}\Big)\frac{1}{\sigma_u}\,\mathrm{d}u = 0. \tag{26}$$

In order to integrate by parts, we make the following assumption:

**Assumption 1.** $P(y)$ grows less fast than $\mathrm{e}^{\frac{1}{2}\left(\frac{y}{\sigma_u}\right)^2}$ at infinity.

In (26) we integrate by parts:

$$v + \frac{1}{\sigma_u}\int_{-\infty}^{+\infty}\left(-P(x+u)\phi\Big(\frac{u}{\sigma_u}\Big) + \frac{x}{\sigma_u}P(x+u)\phi'\Big(\frac{u}{\sigma_u}\Big)\right)\mathrm{d}u = 0. \tag{27}$$

Since $\phi'(x) = -x\phi(x)$, we get

$$v - \frac{1}{\sigma_u}\int_{-\infty}^{+\infty} P(x+u)\left(1 + \frac{xu}{\sigma_u^2}\right)\phi\Big(\frac{u}{\sigma_u}\Big)\,\mathrm{d}u = 0. \tag{28}$$

This is the first order condition that defines the optimal strategy $x = X(v)$. Therefore, if we define

$$R(x) = \frac{1}{\sigma_u}\int_{-\infty}^{+\infty} P(x+u)\left(1 + \frac{xu}{\sigma_u^2}\right)\phi\Big(\frac{u}{\sigma_u}\Big)\,\mathrm{d}u, \tag{29}$$

then the optimal strategy of the insider can be written as

$$X(v) = R^{-1}(v), \qquad \text{or} \quad w = R(X(w)). \tag{30}$$

Putting together (30) and (23), we obtain the following result:

**Proposition 1.** *If $(X, P)$ is an equilibrium, the function $X(w)$ satisfies*

$$w \;=\; \frac{1}{\sigma_u}\int_{-\infty}^{+\infty} \frac{\int_{-\infty}^{+\infty} v\phi\big(\frac{v}{\sigma_v}\big)\phi\big(\frac{X(w)+u-X(v)}{\sigma_u}\big)\,\mathrm{d}v}{\int_{-\infty}^{+\infty} \phi\big(\frac{v}{\sigma_v}\big)\phi\big(\frac{X(w)+u-X(v)}{\sigma_u}\big)\,\mathrm{d}v}\left(1+\frac{X(w)u}{\sigma_u^2}\right)\phi\left(\frac{u}{\sigma_u}\right)\mathrm{d}u. \tag{31}$$

This is a functional equation in $X$. All one needs to do now is to prove the following result.

**Conjecture 1.** Equation (31) has a unique solution given by the linear function in (10). Without loss of generality, assume $p_0 = 0$, $\sigma_u = \sigma_v = 1$. Then, the conjecture reduces to showing that the equation

$$w \;=\; \int_{-\infty}^{+\infty} \frac{\int_{-\infty}^{+\infty} v\phi(v)\phi\big(X(w)+u-X(v)\big)\,\mathrm{d}v}{\int_{-\infty}^{+\infty} \phi(v)\phi\big(X(w)+u-X(v)\big)\,\mathrm{d}v}\big(1+X(w)u\big)\phi(u)\,\mathrm{d}u. \tag{32}$$

has a unique solution, $X(w) = w$.