Dynamic Adverse Selection and Liquidity*

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Abstract

Does a larger fraction of informed trading generate more illiquidity, as measured by the bid–ask spread? We answer this question in the negative in the context of a dynamic dealer market where the fundamental value follows a random walk, provided we consider the long run (stationary) equilibrium. More informed traders tend to generate more adverse selection and hence larger spreads, but at the same time cause faster learning by the market makers and hence smaller spreads. This latter effect offsets the adverse selection effect when the trading frequency is equal to one, and dominates at larger frequencies.

KEYWORDS: Learning, adverse selection, dynamic model, stationary distribution.

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1 Introduction

A traditional view of market liquidity, going at least as far back as Bagehot (1971), posits that one of the causes of illiquidity is adverse selection: *"The essence of market making, viewed as a business, is that in order for the market maker to survive and prosper, his gains from liquidity-motivated transactors must exceed his losses to information motivated transactors. [...] The spread he sets between his bid and asked price affects both: the larger the spread, the less money he loses to information-motivated, transactors and the more he makes from liquidity-motivated transactors."*

This intuition has later been made precise by models such as Glosten and Milgrom (1985, henceforth GM85), in which a competitive risk-neutral dealer sequentially sets bid and ask prices in a risky asset, and makes zero expected profits in each trading round. Traders are selected at random from a population that contains a fraction ρ of informed traders, and must trade at most one unit of the asset. The asset liquidates at a value v that is constant and is either 0 or 1. In equilibrium, the bid–ask spread is wider when the informed share ρ is higher: there is more adverse selection, hence the dealer must set a larger bid–ask spread to break even.

This intuition, however, must be modified once we consider the dynamics of the bid–ask spread. A larger informed share also means that orders carry more information, which over time reduces the uncertainty about v and thus puts downward pressure on the bid–ask spread. We call this last effect "dynamic efficiency." This effect is already present in GM85, who observe that a larger informed share causes initially a larger bid–ask spread, but also causes the bid–ask spread to decrease faster to its eventual value, which is 0 (when v is fully learned).

A natural question is then: to what extent does dynamic efficiency reduce the traditional adverse selection? To answer this question, we extend the framework of GM85 to allow v to move over time according to a random walk v_t. To obtain closed-form results, we assume that the increments of v_t are normally distributed with volatility σ_v, called the fundamental volatility. We chose a moving value for two reasons. First, this is a realistic assumption in modern financial markets, where relevant information arrives essentially at a continuous rate.

Second, we want to study the long-term evolution of the bid–ask spreads, and this long-term analysis is trivial when v_t is constant, as the dealer eventually fully learns v. Note that we are interested in the long run because (as we show later) in the short run the equilibrium is similar to GM85, but in the long run it converges to the stationary equilibrium, which has novel properties.

Beside stationarity, we are also interested in the trading frequency, which we define as the number K of trading rounds between two consecutive times when the value changes. To fix ideas, in most of the paper we focus on the baseline model in which the trading frequency is $K = 1$. Then, we show that at higher frequency our main results are even stronger.

The price we must pay for this new approach is an increase in the complexity of the model. In GM85 the dealer's uncertainty is summarized by a single number: the probability that the value is equal to 1. In our model, however, the dealer's uncertainty is summarized by a probably density function, which is called the "public density." Nevertheless, we introduce the simplifying assumption that the dealer can compute correctly only the first two moments of the density, which are called the "public mean" and "public volatility," while the higher moments are ignored. With this assumption, we show that the equilibrium can be computed in closed form and converges to a unique stationary equilibrium. Moreover, we show that the exact equilibrium (when the dealer computes the whole density function) stays close to the approximate one. Thus, in the rest of the paper, we drop the term "approximate" when referring to the equilibrium.

The first property of the stationary equilibrium is that the public volatility (which is a measure of the dealer's uncertainty about v_t) is constant. The second property is that the informed share is inversely related to the public volatility. The intuition is simple: when the informed share is low, the order flow carries little information, and thus the public volatility is large.

A surprising property of the stationary equilibrium is that the informed share has no effect on the bid–ask spread. To understand this result, consider a small informed share, say 1%. Suppose a buy order arrives, and the dealer estimates how much to update the public mean (in equilibrium this update is half of the bid–ask spread). There are two opposite effects. First, it is very unlikely that the buy order comes from an informed trader (with only 1% chance). This is the "adverse selection effect": a low informed share makes the dealer less concerned about adverse selection, which leads to a smaller update of the public mean, and hence decreases the bid–ask spread. But, second, if the buy order does come from an informed trader, a large public volatility translates into the dealer knowing that, on average, the informed trader must have observed a value far above the public mean. This is the "dynamic efficiency effect": a low informed share leads to a larger update of the public mean, and hence increases the bid–ask spread. At the other end, a large informed share means that the dealer learns well about the asset value (the public volatility is small), and therefore the bid–ask spread tends to be small.

Our main result can be summarized as follows: the dynamic efficiency effect exactly offsets the adverse selection effects when the trading frequency is $K = 1$, and dominates when $K > 1$. To get more intuition, consider the role of trading in relation to the stationarity of the equilibrium. Note that the bid–ask spread in our model is determined by how much the dealer updates the public mean after a buy or a sell order. Consider an equilibrium which is not necessarily stationary but the trading frequency is $K = 1$. If there was no order flow at t, then the dealer's uncertainty (the public volatility) would increase from t to $t+1$ as the asset value diffuses. But the order flow at t contains information and hence reduces the uncertainty at $t+1$. In a stationary equilibrium the uncertainty increase caused by diffusion must cancel the uncertainty decrease caused by order flow. Thus, as the value diffusion is independent of the informed share, the information content of the order flow must also be independent of the informed share. This implies that, when $K = 1$, the size of public mean updates (and of the bid–ask spread) is independent of the informed share.

The exact offsetting depends crucially on the asset value changing every trading round. When $K > 1$, the increase in learning coming from the frequent order flow implies that the dynamic efficiency effect dominates the adverse selection effect. Moreover, a larger informed share induces faster learning by the dealer, and therefore the average stationary bid-ask spread is decreasing in the informed share. The strength of this dependence is increasing in the trading frequency K, as dynamic efficiency has more time to reduce the bid-ask spread.

An important result is that, for any initial public volatility, the equilibrium converges to the stationary equilibrium.[1] E.g., suppose the initial public volatility is very large. Then, initially the order flow is very informative to the dealer, and the public volatility starts decreasing toward its stationary value. The same is true for the bid–ask spread, which in a nonstationary equilibrium is always proportional to the public volatility. This phenomenon is similar to the GM85 equilibrium, except that there the stationary public volatility and bid–ask spread are both 0. This shows that the nonstationary equilibrium (the "short run") resembles GM85, while the stationary equilibrium (the "long run") is different and produces novel insights.

Studying the equilibrium behavior after various types of shocks provides a few testable implications. First, consider a positive shock to the informed share (e.g., the stock is now studied by more hedge funds). Then, the adverse selection effect suddenly becomes stronger, and as a result the bid–ask spread temporarily increases. In the long run, though, the bid–ask spread reverts to its stationary value, which does not change. At the same time, the public volatility gradually decreases to its new level, which is lower due to the increase in the informed share. Second, consider a negative shock to the current public volatility (e.g., public news about the current asset value). Then, the bid–ask spread follows the public volatility and drops immediately, after which it increases gradually to its old stationary level. Third, consider a positive shock to the fundamental volatility (e.g., all future uncertainty about the asset increases). Then, the bid–ask spread follows the public volatility and increases gradually to its new stationary level.

Based on our results, the picture on dynamic adverse selection that emerges is that liquidity is more strongly affected not by the informed share (the intensive margin), but by the fundamental volatility (the extensive margin). By contrast, price discovery (measured by the public volatility) is strongly affected by both the informed share and fundamental volatility. This suggests that the presence of privately informed traders can be more precisely identified by proxies of the current level of uncertainty, rather than by illiquidity measures such as the bid–ask spread (which is used by Collin-Dufresne and Fos, 2015).

[1] This convergence is not entirely obvious, as it is possible in principle for the public volatility to grow indefinitely, with no finite limit.

Related Literature

Our paper contributes to the literature of dynamic models of adverse selection.[2] To our knowledge, this paper is the first to study the effect of stationarity in dealer models of the Glosten and Milgrom (1985) type.[3] By contrast, several stationary models of the Kyle (1985) type are analyzed for instance by Chau and Vayanos (2008) and Caldentey and Stacchetti (2010). The focus of these models, however, is not liquidity but price discovery: in the continuous-time limit the market becomes strong-form efficient, as the insider trades infinitely aggressively on infinitesimal value changes.[4] Note that in these papers there is a single insider, and thus one cannot use them to study the effect of more informed trading on liquidity.

Our main result, that more competition among informed traders can lead to better liquidity, points to a few related papers. Vives (1995) provides a non-stationary model of the Kyle (1985) type with a fixed value and a continuum of risk averse insiders. As the number of trading ("tâtonnement") rounds increases, price quotations become more informative, the market is deeper, and the informed agents react by trading more intensely and revealing their private information faster. This effect is similar to the dynamic efficiency effect in GM85 or in our model. The difference is that we prove the improvement in liquidity in a stationary model, and we measure illiquidity by the bid–ask spread (instead of the Kyle lambda measure in Vives (1995)). Roşu (2020) provides a model of a limit order market in which informed traders can choose between providing liquidity (with a limit order) and demanding liquidity (with a market order). In that paper, a larger informed fraction of informed trading contributes to better liquidity, in part because informed traders can also supply liquidity (and in equilibrium they do). The effect is however, not strong. Lester, Shourideh, Venkateswaran, and Zetlin-Jones (2018) proposes a different channel and a different result from ours. They show that more frequent trading (or more competition among dealers) makes traders' behavior less dependent on asset quality, and as a result dealers learn about the asset quality more

[2]See, e.g., the surveys of Vives (2008) and Foucault, Pagano, and Röell (2013) and the references therein.

[3]Glosten and Putnins (2016) study the welfare effect of the informed share in the Glosten and Milgrom (1985) model, but they do not consider the effect of stationarity.

[4]In both these models, the continuous-time limit does not exist, as one obtains an indeterminacy (infinite multiplied by 0). However, the illiquidity, i.e., the price impact coefficient, remains finite and constant, just as it does in the Kyle (1985) model.

slowly, and set wider bid–ask spreads to compensate for the increase in uncertainty.

The paper speaks to the literature on the identification of informed trading and in particular on the identification of insider trading. Collin-Dufresne and Fos (2015, 2016) show both empirically and theoretically that times when insiders trade coincide with times when liquidity is actually stronger (and in particular bid–ask spreads decline). They attribute this finding to the action of discretionary insiders who trade when they expect a larger presence of liquidity (noise) traders. During those times the usual positive effect of noise traders on liquidity dominates, and thus bid–ask spreads decline despite an increase in informed trading. By contrast, our effect works even when the noise trader activity is constant over time, as long as there is enough time for the equilibrium to become stationary.

The paper is organized as follows. In Section 2, we describe the model. In Section 3, we analyze the baseline model (where the trading frequency is equal to 1) and show the existence of an exact equilibrium. We next analyze the (approximate) equilibrium and show that it converges to a unique stationary equilibrium. In Section 4, we analyze the equilibrium when the trading frequency is higher than 1. In Section 5, we discuss several model assumptions, as well as the robustness of our main results. Section 6 concludes. All proofs are in the Appendix. The Internet Appendix contains a discussion of general dealer models, and an application to a model in which the fundamental value switches randomly between 0 and 1.

2 Environment

The model is similar to the dealer market model in GM85, except that the fundamental value moves according to a random walk:

$$v_{t+1} \;=\; v_t + \varepsilon_{t+1}, \quad \text{with} \quad \varepsilon_t \overset{IID}{\sim} \mathcal{N}\left(\cdot, 0, \sigma_v\right). \tag{1}$$

There is a single risky asset, and time is discrete and infinite. Trading in the risky asset takes place at fractional times $t \in \frac{\mathbb{N}}{K} = \{0, \frac{1}{K}, \frac{2}{K}, \ldots\}$, where $K \in \mathbb{N}_+ = \{1, 2, \ldots\}$ is called the trading frequency. The case $K = 1$ is called the baseline model.[5]

At each trading time $t \in \frac{\mathbb{N}}{K}$, a dealer posts two quotes: the ask price A_t, and the bid price B_t. Thus, a buy order at t executes at A_t, while a sell order at t executes at B_t. The dealer (referred to in the paper as "she") is risk neutral and competitive, and therefore makes 0 expected profits from each trade.

The buy or sell orders are submitted by a trading population with a fraction $\rho \in (0, 1)$ of informed traders and a fraction $1 - \rho$ of uninformed traders. At each $t \in \frac{\mathbb{N}}{K}$ a trader is selected at random from the population willing to trade, and can trade at most one unit of the asset. An uninformed trader at t is always willing to trade, and is equally likely to buy or to sell. An informed trader at t who observes the value v_t either (i) submits a buy order if $v_t > A_t$, (ii) submits a sell order if $v_t < B_t$, (iii) is not willing to trade if $v_t \in [B_t, A_t]$. If case (iii) occurs, an uninformed trader is selected, as no informed trader is willing to trade.[6]

The dealer's uncertainty about the fundamental value is summarized by the "public density," which is the density of v_t just before trading at t, conditional on all the order flow available at t, that is, the sequence of orders submitted before t. Denote by ϕ_t the public density, by μ_t its mean (called the "public mean") and by σ_t its standard deviation (called the "public volatility").

For tractability, we assume that at each time t the dealer approximates the public density with a normal density such that the first two moments are correctly computed. Thus, if the correct posterior density ϕ_t has mean μ_t and volatility σ_t, the dealer replaces it with the normal density $\phi_t^a(\cdot) = \mathcal{N}(\cdot, \mu_t, \sigma_t)$. One interpretation for this assumption is that the dealer faces a very large cost of computing moments of order higher than 2.

To examine the robustness of our results, we also define the "exact equilibrium," in which the public density is correctly computed. In that case, we assume that the initial density ϕ_0

[5] When the trading frequency is less than 1, e.g., $K = \frac{1}{m}$ with $m \in \mathbb{N}_+$, trading takes place at every positive integer multiple of m, and the model is equivalent to the baseline case in which equation (1) becomes $v_{t+m} = v_t + \eta_{t+m}$, where $\eta_{t+m} = \varepsilon_{t+1} + \cdots + \varepsilon_{t+m} \sim \mathcal{N}(\cdot, 0, \sigma_v \sqrt{m})$.

[6] Note that in GM85 both types of traders are always willing to trade: the informed because v is always outside the bid–ask spread, and the uninformed for exogenous reasons. In our model, an informed trader is not willing to trade when the value v_t is within the spread, and thus it is replaced with an uninformed trader, who for exogenous reasons is always willing to trade. In Section 5.1, we endogenize the reasons to trade, and provide more discussion on these assumptions.

is rapidly decaying at infinity.[7]

The timeline of the model at $t \in \frac{\mathbb{N}}{K}$ is as follows: (i) if t is an integer (i.e., $t \in \mathbb{N}$), the fundamental value changes to v_t; (ii) the dealer sets the ask quote A_t and the bid quote B_t; and (iii) trading takes place at the quotes set by the dealer.

3 Equilibrium in the Baseline Model

We consider the baseline model, in which the trading frequency is $K = 1$, i.e., the fundamental value changes in each trading round $t \in \mathbb{N} = \{0, 1, 2, \ldots\}$. We first show the existence of the exact equilibrium, after which we focus on the main (approximate) equilibrium in which the dealer always approximates the public density with a normal density. These two equilibria are compared in Section 5.3.

3.1 Exact Equilibrium

We prove the existence of the exact equilibrium of the model in two steps. First, for each $t \in \mathbb{N}$ we start with an public density ϕ_t, an ask A_t, a bid $B_t < A_t$, and compute the public density ϕ_{t+1} after a buy or sell order. Second, for any public density ϕ_t we show that there exists an "ask–bid pair" (A_t, B_t), i.e., an ask A_t and a bid B_t that satisfy the dealer's pricing conditions which require that her expected profit from trading at t is 0. We show numerically that the ask–bid pair (A_t, B_t) is unique, provided that the initial density is a particular normal density called the stationary density.

Let ϕ_t be the public density of v_t before trading at t, and let $A_t > B_t$ be, respectively, the ask and bid at t (not necessarily satisfying the dealer's pricing conditions). Suppose a buy or sell order $\mathcal{O}_t \in \{\mathrm{B}, \mathrm{S}\}$ arrives at t. Let $\mathbf{1}_P$ be the indicator function, which is 1 if P is true and 0 if P is false. Conditional on $v_t = v$, the probability of observing a buy order at t is:

$$g_t(\mathrm{B}, v) \;=\; \rho \mathbf{1}_{v > A_t} + \tfrac{\rho}{2} \mathbf{1}_{v \in [B_t, A_t]} + \tfrac{1-\rho}{2}. \tag{2}$$

[7] A function f is rapidly decaying (at infinity) if it is smooth and satisfies $\lim_{v \to \pm\infty} |v|^M f_0^{(N)}(v) = 0$, where $f^{(N)}$ is the N-th derivative of f. The space $\mathcal{S}$ of rapidly decaying functions is called the Schwartz space. Any normal density belongs to $\mathcal{S}$, and the convolution of two densities in $\mathcal{S}$ also belongs to $\mathcal{S}$.

To see this, consider the following cases:

- If $v \in [B_t, A_t]$, the informed traders are not willing to trade, and an uninformed trader submits a buy order with probability $\frac{1}{2}$. Then, $g_t(\mathrm{B}, v) = \rho \times 0 + \frac{\rho}{2} \times 1 + \frac{1-\rho}{2} = \frac{1}{2}$.

- If $v \notin [B_t, A_t]$, an informed trader (chosen with probability ρ) submits a buy order with probability $\mathbf{1}_{v>A_t}$, while an uninformed trader (chosen with probability $1-\rho$) submits a buy order with probability $\frac{1}{2}$. Then, $g_t(\mathrm{B}, v) = \rho \mathbf{1}_{v>A_t} + \frac{\rho}{2} \times 0 + \frac{1-\rho}{2}$

Similarly, the probability of observing a sell order at t is:

$$g_t(\mathrm{S}, v) \ = \ \rho \mathbf{1}_{v<B_t} + \tfrac{\rho}{2} \mathbf{1}_{v\in[B_t,A_t]} + \tfrac{1-\rho}{2}. \tag{3}$$

Proposition 1 describes the evolution of the public density.

Proposition 1. *Consider a rapidly decaying public density ϕ_t, and an ask–bid pair with $A_t > B_t$. After observing an order $\mathcal{O}_t \in \{\mathrm{B}, \mathrm{S}\}$, the density of v_t is $\psi_t(v|\mathcal{O}_t)$, where:*

$$\begin{aligned} \psi_t(v|\mathrm{B}) \ &= \ \frac{\left(\rho \mathbf{1}_{v>A_t} + \frac{\rho}{2}\mathbf{1}_{v\in[B_t,A_t]} + \frac{1-\rho}{2}\right) \cdot \phi_t(v)}{\frac{\rho}{2}(1-\Phi_t(A_t)) + \frac{\rho}{2}(1-\Phi_t(B_t)) + \frac{1-\rho}{2}}, \\ \psi_t(v|\mathrm{S}) \ &= \ \frac{\left(\rho \mathbf{1}_{v<B_t} + \frac{\rho}{2}\mathbf{1}_{v\in[B_t,A_t]} + \frac{1-\rho}{2}\right) \cdot \phi_t(v)}{\frac{\rho}{2}\Phi_t(A_t) + \frac{\rho}{2}\Phi_t(B_t) + \frac{1-\rho}{2}}, \end{aligned} \tag{4}$$

where Φ_t is the cumulative density function corresponding to ϕ_t. The public density at $t+1$ is rapidly decaying, and satisfies:

$$\phi_{t+1}(w|\mathcal{O}_t) \ = \ \int_{-\infty}^{+\infty} \psi_t(v|\mathcal{O}_t)\mathcal{N}(w-v, 0, \sigma_v)\mathrm{d}v \ = \ \Big(\psi_t(\cdot|\mathcal{O}_t) \ast \mathcal{N}(\cdot, 0, \sigma_v)\Big)(w), \tag{5}$$

where "$\ast$" denotes the convolution of two densities.

Proposition 1 shows how the public density evolves once a particular order (buy or sell) is submitted at t. Note, however, that this result does not assume anything about the ask and bid other than $A_t > B_t$, so in principle these can be chosen arbitrarily. In equilibrium, however, these prices must satisfy the dealer's pricing conditions, namely that the dealer's expected profits at t must be 0.

Next, we impose these conditions and we show how to determine the equilibrium ask and bid. Then, Proposition 1 allows us to describe the whole evolution of the public density, conditional on the initial density ϕ_0 and the sequence of orders $\mathcal{O}_0, \mathcal{O}_1, \ldots$ that have been submitted.

Let ϕ_t be the public density of v_t before trading at t. We define an "ask–bid pair" (A_t, B_t) as a pair of ask and bid satisfying the pricing conditions of the dealer. As the dealer is risk neutral and competitive, the pricing conditions are: (i) the ask A_t is the expected value of v_t conditional on a buy order at t, and (ii) the bid B_t is the expected value of v_t conditional on a sell order at t. Using the previous notation, the dealer's pricing conditions are that A_t is the mean of $\psi_t(v|\mathrm{B})$, the posterior density of v_t after observing a buy order at t; and B_t is the mean of $\psi_t(v|\mathrm{S})$, the posterior density after observing a sell order at t. Thus, the dealer's pricing conditions are equivalent to:

$$A_t \;=\; \int_{-\infty}^{+\infty} v\psi_t(v|\mathrm{B})\mathrm{d}v, \qquad B_t \;=\; \int_{-\infty}^{+\infty} v\psi_t(v|\mathrm{S})\mathrm{d}v. \tag{6}$$

For future use, we record the following straightforward result.

Corollary 1. *The pair (A_t, B_t) is an ask–bid pair if and only if the following equations are satisfied:*

$$A_t \;=\; \mu_{t+1,\mathrm{B}}, \quad B_t \;=\; \mu_{t+1,\mathrm{S}}, \quad \textit{with} \quad \mu_{t+1,\mathcal{O}_t} \;=\; \int_{-\infty}^{+\infty} w\phi_{t+1}(w|\mathcal{O}_t)\mathrm{d}w, \quad \mathcal{O}_t = \{\mathrm{B},\mathrm{S}\}. \tag{7}$$

To analyze the existence of an ask–bid pair, first we introduce some notation. Suppose μ_t is the mean of ϕ_t. For $(A, B) \in (\mu_t, \infty) \times (-\infty, \mu_t)$, define the functions:

$$\begin{aligned} F(A,B) \;&=\; \frac{\Theta_t(A)+\Theta_t(B)}{A-\mu_t} - \frac{1+\rho}{\rho} + \Phi_t(A) + \Phi_t(B), \\ G(A,B) \;&=\; \frac{\Theta_t(A)+\Theta_t(B)}{\mu_t - B} - \frac{1-\rho}{\rho} - \Phi_t(A) - \Phi_t(B), \end{aligned} \tag{8}$$

where Φ_t is the cumulative density associated to ϕ_t, and Θ_t is defined by:

$$\Theta_t(v) \;=\; \int_{-\infty}^{v} (\mu_t - w)\phi_t(w)\mathrm{d}w. \tag{9}$$

The function Θ_t is strictly positive everywhere and approaches 0 at infinity on both sides.[8]

Proposition 2 shows that the existence of an ask–bid pair is equivalent to solving a 2×2 system of nonlinear equations.

Proposition 2. *Consider a rapidly decaying public density ϕ_t, with mean μ_t. Then, the existence of an ask–bid pair is equivalent to finding a solution $(A, B) \in (\mu_t, \infty) \times (-\infty, \mu_t)$ of the system of equations:*

$$F(A, B) \;=\; 0, \qquad G(A, B) \;=\; 0. \tag{10}$$

A solution of (10) always exists.

Thus, Proposition 2 shows that an ask–bid pair exists for any public density ϕ_t. Uniqueness, however, is not guaranteed, as one could in principle manufacture a public density for which there is more that one corresponding ask–bid pair. Nevertheless, Result 1 shows numerically that the ask–bid pair is unique if the initial public density is the stationary density $\mathcal{N}(\cdot, \mu_t, \sigma_*)$, where σ_* is defined as in (18).

Result 1. Suppose the initial public density is $\phi_0(\cdot) = \mathcal{N}(\cdot, \mu_0, \sigma_0)$, with $\sigma_0 = \sigma_*$, the stationary volatility defined as in (18). Then the ask–bid pair is unique at all t, regardless of the realized order flow.

We discuss the numerical verification of this result in the Appendix. The idea is to use the fact that the existence of an ask–bid pair is equivalent to solving the equation $A - f(A) = 0$ for some smooth function of A, and show numerically that the derivative of f is less than 1 everywhere.

We finish the analysis of the exact equilibrium with a brief discussion about stationarity. Numerically, it appears that, regardless of the starting density, the exact equilibrium converges to a stationary one. However, proving any formal result about stationarity would be extremely difficult. To understand why, consider the space $\mathcal{S}$ of pairs (v, ϕ), where v is a real number (the fundamental value) and ϕ a rapidly decaying density (the public density). Then, our

[8] As ϕ_t is rapidly decaying, $\Theta_t(-\infty)$ is equal to 0. The definition of μ_t implies that $\Theta_t(+\infty) = \int_{-\infty}^{+\infty} (\mu_t - w)\phi_t(w) = \mu_t - \int_{-\infty}^{+\infty} w\phi_t(w) = 0$. Also, $\Theta_t'(v) = (\mu_t - v)\phi_t(v)$, hence $\Theta_t(v)$ is increasing below μ_t and decreasing above μ_t. As $\Theta_t(\pm\infty) = 0$, the function Θ_t is strictly positive everywhere.

exact equilibrium generates a Markov chain on the infinite-dimensional space $\mathcal{S}$: the transition from v_t to v_{t+1} is governed by the random walk equation (1), and the transition from ϕ_t to ϕ_{t+1} is governed by equation (5), where the correct ask and bid prices are determined as in Proposition 2 and the numerical Result 1. We can then define a stationary equilibrium as a distribution ψ on $\mathcal{S}$ which is invariant under the Markov transition. Note that ψ is a density on a product space that includes other densities, hence is infinite-dimensional. In Section 2 in the Internet Appendix, we consider a much simpler model in which the fundamental value is either 0 or 1, and it switches every period between these two values with probability $\nu < 1/2$. Then, the public density reduces to a single number, but even in that case, finding a stationary density on $\mathcal{S}$ is not straightforward and can be solved only numerically. Given all these difficulties, in Section 3.2 we introduce an approximate equilibrium as our main equilibrium concept, and we show that everything can be computed in closed form.

3.2 Main Equilibrium

By assumption, at each $t \in \mathbb{N}$ the dealer approximates the public density with a normal density such that the first two moments are correctly computed. Specifically, suppose that the dealer regards v_t to be distributed as:

$$\phi_t^a(v) \;=\; \mathcal{N}(v, \mu_t, \sigma_t). \tag{11}$$

After the dealer observes an order $\mathcal{O}_t$ at t, denote by $\phi_{t+1}(w|\mathcal{O}_t)$ the exact density of v_{t+1} conditional on the past order flow including $\mathcal{O}_t$, and by $\mu_{t+1,\mathcal{O}_t}$ and $\sigma_{t+1,\mathcal{O}_t}$ its mean and standard deviation, respectively. Then, before trading at $t+1$ the dealer regards v_{t+1} to be distributed as:

$$\phi_{t+1}^a(w|\mathcal{O}_t) \;=\; \mathcal{N}\big(w, \mu_{t+1,\mathcal{O}_t}, \sigma_{t+1,\mathcal{O}_t}\big). \tag{12}$$

Thus, we assume that the dealer continues to consider the public density as $\phi_t = \phi_t^a$ at each $t \in \mathbb{N}$. The accuracy of this approximation is discussed in Section 5.3.

3.2.1 Evolution of the Public Density

We introduce a new parameter, δ, which is an increasing function of the informed share ρ:

$$\delta \;=\; g^{-1}(2\rho) \;\in\; (0, \delta_{\max}), \quad \text{with} \quad \delta_{\max} \;=\; g^{-1}(2) \;\approx\; 0.647, \tag{13}$$

where $g : [0, \infty) \to [0, \infty)$ defined by $g(x) = \frac{x}{\mathcal{N}(x,0,1)}$ is one-to-one and increasing.[9]

Proposition 3 shows the evolution of the public mean and volatility, as well as of the bid–ask spread.

Proposition 3. *Suppose the public density at* $t = 0, 1, 2, \ldots$ *is* $\phi_t(v) = \mathcal{N}(v, \mu_t, \sigma_t)$*. After observing* $\mathcal{O}_t \in \{\mathrm{B}, \mathrm{S}\}$*, the posterior mean at* $t+1$ *satisfies:*

$$\mu_{t+1,\mathrm{B}} \;=\; \mu_t + \delta\sigma_t, \qquad \mu_{t+1,\mathrm{S}} \;=\; \mu_t - \delta\sigma_t. \tag{14}$$

The posterior volatility at $t+1$ *does not depend on the order* $\mathcal{O}_t$*, and satisfies:*

$$\sigma_{t+1} \;=\; \sqrt{(1-\delta^2)\sigma_t^2 + \sigma_v^2}. \tag{15}$$

The ask and bid quotes are unique and satisfy:

$$A_t \;=\; \mu_t + \delta\sigma_t, \qquad B_t \;=\; \mu_t - \delta\sigma_t, \qquad s_t \;=\; A_t - B_t \;=\; 2\delta\sigma_t. \tag{16}$$

We now investigate whether the public density reaches a steady state, in the sense that its shape converges to a particular density. As the mean μ_t evolves according to a random walk, we must demean the public density and focus on its standard deviation σ_t. Proposition 4 shows that the public volatility σ_t converges to a particular value, σ_*, regardless of the initial value σ_0.

Proposition 4. *For any* $t = 0, 1, 2, \ldots$ *the public volatility satisfies:*

$$\sigma_t^2 \;=\; \sigma_*^2 + \left(\sigma_0^2 - \sigma_*^2\right)\left(1-\delta^2\right)^t, \tag{17}$$

[9] See the proof of Proposition 3.

where:

$$\sigma_* \;=\; \frac{\sigma_v}{\delta} \;=\; \frac{\sigma_v}{g^{-1}(2\rho)}. \tag{18}$$

For any initial value σ_0 and any sequence of orders, the public volatility σ_t monotonically converges to σ_*, and the bid–ask spread monotonically converges to:

$$s_* \;=\; 2\sigma_v. \tag{19}$$

Thus, Proposition 4 shows that in the long run the equilibrium approaches a particular stationary equilibrium, which we analyze next.

3.2.2 Stationary Equilibrium

We define a "stationary equilibrium" to be an equilibrium in which the public volatility σ_t is constant. According to Proposition 4, if the initial density is $\phi_0(v) = \mathcal{N}(v, \mu_0, \sigma_*)$, then all subsequent public densities have the same volatility, namely the stationary volatility σ_*. We now analyze the properties of the stationary equilibrium.

Corollary 2. *In the stationary equilibrium, the public volatility σ_* is decreasing in the informed share ρ, while the bid–ask spread s_* does not depend on ρ. Both σ_* and s_* are increasing in the fundamental volatility σ_v.*

Intuitively, an increase in the fundamental volatility σ_v raises the public volatility as the dealer's knowledge about the fundamental value becomes more imprecise. It also increases the adverse selection overall for the dealer, hence she increases the bid–ask spread. Moreover, a decrease in the informed share ρ means that the order flow becomes less informative, and therefore the dealer's knowledge about the fundamental value is more imprecise (σ_* is large).

The surprising result is that the stationary bid–ask spread is independent of ρ. This is equivalent to the public mean update being independent of ρ. Indeed, the public mean evolves according to:

$$\mu_{t+1,\mathrm{B}} \;=\; \mu_t + \sigma_v, \qquad \mu_{t+1,\mathrm{S}} \;=\; \mu_t - \sigma_v. \tag{20}$$

Thus, the bid–ask spread is $s_* = (\mu_t + \sigma_v) - (\mu_t - \sigma_v) = 2\sigma_v$. To understand the intuition behind this result, consider the case when ρ is low. Suppose the dealer observes a buy order at t. As ρ is low, there are two effects on the size of the public mean update. The first effect is negative: the trader at t is unlikely to be informed, which decreases the size of the update. This is the traditional adverse selection effect from models such as GM85. The second effect is positive: when the trader at t *is* informed, he must have observed a large fundamental value v_t, as the uncertainty in v_t (measured by the public volatility σ_*) is also large. This we call the "dynamic efficiency effect": more informed traders create over time a more precise knowledge about the fundamental value, and thus reduce the effect of informational updates.

It turns out that the dynamic efficiency effect exactly cancels the adverse selection effect in a stationary setup, and as a result the size of the public mean updates due to order flow is independent of ρ. To understand why, consider an equilibrium which is not necessarily stationary. If there was no order flow at t, then the dealer's uncertainty (the public volatility) would increase from t to $t+1$ as the fundamental value diffuses. But there *is* order flow at t, which provides information to the dealer and hence reduces uncertainty at $t+1$. In a stationary equilibrium the public uncertainty stays constant. Thus, as the increase in uncertainty due to value diffusion is independent of the informed share ρ, the decrease in uncertainty due to order flow should also be independent of ρ. But an order flow information content that is independent of ρ translates into the size of public mean updates also being independent of ρ.

Formally, the decrease in uncertainty due to the order $\mathcal{O}_t$ at t can be evaluated by comparing the prior public density $\phi_t(v)$ and the posterior density $\psi_t(v|\mathcal{O}_t)$. One measure of the decrease in uncertainty is how much the public mean is updated after a buy or sell order (which are equally likely). But (20) implies that this update is $\pm\sigma_v$, which from the point of view of the information at t is a binary distribution, with standard deviation σ_v which is indeed independent of ρ. Note that we have also essentially proved the following result.

Corollary 3. *In the stationary equilibrium, the volatility of the change in public mean is constant and equal to σ_v.*

This result is in fact true quite generally. Indeed, in Appendix B we prove that for any filtration problem in which the variance remains constant over time the volatility of the change in public mean must equal the fundamental volatility.

3.2.3 Liquidity Dynamics

In this section we analyze the evolution of the public volatility and the bid–ask spread after a shock to either the public volatility σ_t, the fundamental volatility σ_v, or the informed share ρ. We are also interested in how quickly the equilibrium converges to the stationary equilibrium. In general, the speed of convergence of a sequence x_t that converges to a limit x_* is defined as the limit ratio:

$$S \;=\; \lim_{t\to\infty} \frac{|x_t^2 - x_*^2|}{|x_{t+1}^2 - x_*^2|}, \tag{21}$$

provided that the limit exists. Corollary 4 computes the speed of convergence for several variables of interest.

Corollary 4. *The public volatility, public variance and bid–ask spread have the same speed of convergence:*

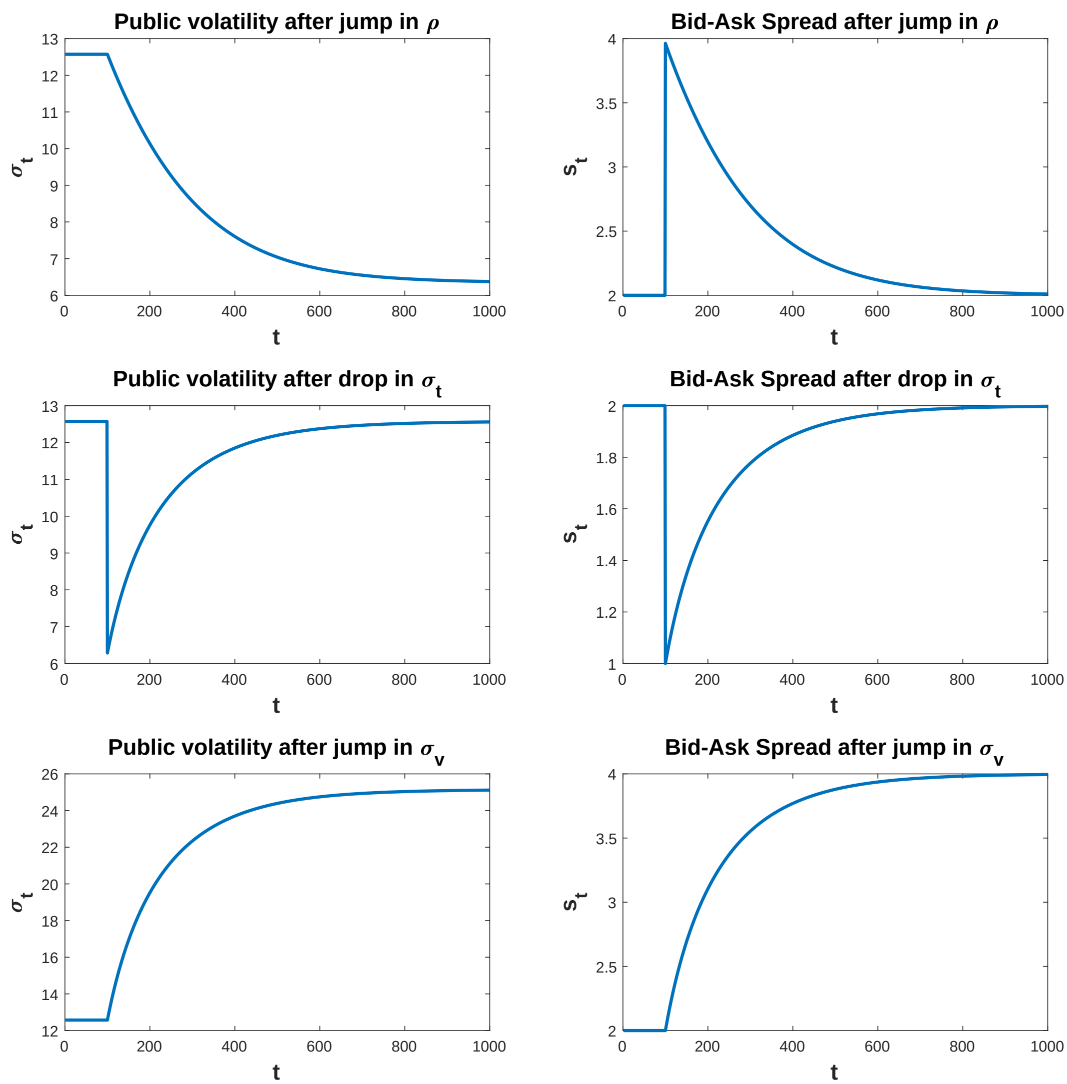
$$S \;=\; \frac{1}{1-\delta^2}. \tag{22}$$

Moreover, S is increasing in the informed share ρ.

Corollary 4 shows that the variables of interest have the same speed of convergence S, and we can thus call S simply as the "convergence speed" of the equilibrium. Another result of Corollary 4 is that a larger informed share ρ implies a faster convergence speed of the equilibrium to its stationary value. This is intuitive, as more informed trading helps the dealer make quicker dynamic inferences. Note that when $\rho = 1$, equation (13) implies that $\delta = g^{-1}(2) \approx 0.647$, thus the maximum value of δ is less than 1. Therefore, the maximum convergence speed is finite.

We now consider the effect of various types of shocks to our stationary equilibrium. In the first row of Figure 1 we show the effects of a positive shock to the informed share, meaning that ρ suddenly jumps to a higher value ρ'. This generates an increase in δ, which jumps to its new value $\delta' = g^{-1}(\rho')$, and it also generates a drop in the stationary public volatility, which is now $\sigma'_* = \sigma_v/\delta'$. Nevertheless, as there is no new information above the fundamental value, the current public volatility σ_t remains equal to its old stationary value, $\sigma_* = \sigma_v/\delta$. Proposition 4 shows that the public volatility starts decreasing monotonically toward its stationary value σ'_*. Note that according to Corollary 4 the speed of convergence to the new

Figure 1: Public Volatility and Bid–Ask Spread after Shocks.
This figure shows the effect of three types of shocks on the public volatility σ_t, and on the bid–ask spread s_t (each shock occurs at $t_0 = 100$). The initial parameters are: $\sigma_v = 1$, and $\rho = 0.1$ (hence $\delta = 0.0795$, $\sigma_* = 12.573$, $s_* = 2$). In the first row, the informed share ρ jumps from 0.1 to 0.2 (hence σ_* drops from 12.573 to 6.345). In the second row, the public volatility drops from $\sigma_* = 12.573$ to half of its value (6.286). In the third row, the fundamental volatility jumps from 1 to 2.



stationary equilibrium is $S' = 1/(1-\delta'^2)$, which is higher than the old convergence speed. We also describe the evolution of the bid–ask spread, which according to Proposition 3 satisfies $s_t = 2\delta'\sigma_t$. Initially, the bid–ask spread jumps to reflect the jump to δ'. But then, as σ_t

converges to $\sigma'_* = \sigma_v/\delta'$, the bid–ask spread starts decreasing to $s_* = 2\sigma_v$, which does not depend on ρ.

To summarize, after a positive shock to ρ, the public volatility starts decreasing monotonically to its now lower stationary value, while the bid–ask spread initially jumps and then decreases monotonically to the same stationary value (that does not depend on ρ). Intuitively, a positive shock to the informed share leads to a sudden increase in adverse selection for the dealer, reflected in an initially larger bid–ask spread, after which the bid–ask spread reverts to its fundamental value, which is independent of informed trading. At the same time, more informed trading leads to more precision for the dealer in the long run, which is reflected in a smaller public volatility.

In the second row of Figure 1 we show the effects of a negative shock to the public volatility, meaning that σ_t suddenly drops from the stationary value σ_* to a lower value. This drop can be caused for instance by public news about the value of the asset v_t. Then, according to Proposition 4, the public volatility increases monotonically back to the stationary value. The bid–ask spread is always proportional to the public density: $s_t = 2\delta\sigma_t$, hence s_t also drops initially and then increases monotonically toward the stationary value s_*. Intuitively, public news has the effect of helping the dealer initially to get a more precise understanding about the fundamental value. This brings down the bid–ask spread, as temporarily the dealer faces less adverse selection. But this decrease is only temporary, as the value diffuses and the same forces increase the public volatility and the bid–ask spread toward their corresponding stationary values, which are the same as before.

In the third row of Figure 1 we show the effects of a positive shock to the fundamental volatility, meaning that σ_v suddenly jumps to a higher value σ'_v. This implies that every value increment $v_{t+1} - v_t$ now has higher volatility, but the uncertainty in v_t, which is measured by the public volatility σ_t, stays the same.[10] Proposition 4 shows that the stationary public volatility changes to $\sigma'_* = \sigma'_v/\delta$, and the stationary bid–ask spread changes to $s'_* = 2\sigma'_v$. Therefore, the public density increases monotonically from the initial stationary value to the new stationary value, and the same is true for the bid–ask spread. Intuitively, a larger

[10]One can mix this type of shock with a shock to the public volatility σ_t, which was already analyzed.

fundamental volatility increases overall adverse selection for the dealer, and as a result both the public density and the bid–ask spread eventually increase.

4 Frequent Trading

In this section, we study the equilibrium when the trading frequency K is larger than 1, i.e., the fundamental value changes at integer times $t \in \mathbb{N}$, but trading takes time at fractional times $t \in \frac{\mathbb{N}}{K} = \{0, \frac{1}{K}, \frac{2}{K}, \dots\}$.[11] The timeline of the model at $t \in \frac{\mathbb{N}}{K}$ is as follows: (i) if t is an integer (i.e., $t \in \mathbb{N}$), the fundamental value changes to v_t; (ii) the dealer sets the ask quote A_t and the bid quote B_t; and (iii) trading takes place at the quotes set by the dealer.

Let $\phi_t(v)$ be the public density at t (i.e., the density of the value v_t just before trading at t). By assumption, the dealer approximates the public density $\phi_t(v)$ with a normal density $\phi^a_t(v) = \mathcal{N}(v, \mu_t, \sigma_t)$ such that the first two moments are correctly computed.

To find the equilibrium, we start with the ask quote A_t and the bid quote B_t. As in equation (7), the ask and bid quotes satisfy:

$$A_t = \mu_{t+\frac{1}{K},\mathrm{B}}, \quad B_t = \mu_{t+\frac{1}{K},\mathrm{S}}, \quad \text{with} \quad \mu_{t+\frac{1}{K},\mathcal{O}_t} = \int_{-\infty}^{+\infty} w\phi_{t+\frac{1}{K}}(w|\mathcal{O}_t)\mathrm{d}w, \quad \mathcal{O}_t = \{\mathrm{B},\mathrm{S}\}. \tag{23}$$

Proposition 5 shows how to update the exact public density after observing a buy or sell order.

Proposition 5. *Consider a rapidly decaying public density ϕ_t, and an ask–bid pair with $A_t > B_t$. After observing an order $\mathcal{O}_t \in \{\mathrm{B},\mathrm{S}\}$, the density of v_t is $\psi_t(v|\mathcal{O}_t)$, where:*

$$\begin{aligned} \psi_t(v|\mathrm{B}) &= \frac{\left(\rho\mathbf{1}_{v>A_t} + \frac{\rho}{2}\mathbf{1}_{v\in[B_t,A_t]} + \frac{1-\rho}{2}\right)\cdot\phi_t(v)}{\frac{\rho}{2}(1-\Phi_t(A_t)) + \frac{\rho}{2}(1-\Phi_t(B_t)) + \frac{1-\rho}{2}}, \\ \psi_t(v|\mathrm{S}) &= \frac{\left(\rho\mathbf{1}_{v<B_t} + \frac{\rho}{2}\mathbf{1}_{v\in[B_t,A_t]} + \frac{1-\rho}{2}\right)\cdot\phi_t(v)}{\frac{\rho}{2}\Phi_t(A_t) + \frac{\rho}{2}\Phi_t(B_t) + \frac{1-\rho}{2}}, \end{aligned} \tag{24}$$

[11] The case $K = 1$ coincides with the baseline model in Section 3.

where Φ_t is the cumulative density function corresponding to ϕ_t. The density at $t+\frac{1}{K}$ satisfies:

$$\phi_{t+\frac{1}{K}}(\cdot|\mathcal{O}_t) \;=\; \begin{cases} \psi_t(\cdot|\mathcal{O}_t) & \textit{if} \quad t+\frac{1}{K} \notin \mathbb{N}, \\ \psi_t(\cdot|\mathcal{O}_t) * \mathcal{N}(\cdot, 0, \sigma_v) & \textit{if} \quad t+\frac{1}{K} \in \mathbb{N}, \end{cases} \tag{25}$$

where "$*$" denotes the convolution of two densities.

Recall the definition of the parameter δ from equation (13): $\delta = g^{-1}(2\rho) \in (0, \delta_{\max})$ with $\delta_{\max} = g^{-1}(2) \approx 0.647$, and $g : [0,\infty) \to [0,\infty)$ is defined by $g(x) = \frac{x}{\mathcal{N}(x,0,1)}$, which is one-to-one and increasing.

Proposition 6 shows the evolution of the public mean and volatility, as well as of the bid–ask spread.

Proposition 6. *Suppose the public density at $t \in \frac{\mathbb{N}}{K}$ is $\phi_t(v) = \mathcal{N}(v, \mu_t, \sigma_t)$. After observing $\mathcal{O}_t \in \{\mathrm{B}, \mathrm{S}\}$, the posterior mean at $t+\frac{1}{K}$ satisfies:*

$$\mu_{t+1,\mathrm{B}} \;=\; \mu_t + \delta\sigma_t, \qquad \mu_{t+1,\mathrm{S}} \;=\; \mu_t - \delta\sigma_t. \tag{26}$$

The posterior volatility at $t+\frac{1}{K}$ does not depend on the order $\mathcal{O}_t$, and satisfies:

$$\sigma_{t+\frac{1}{K}} \;=\; \begin{cases} \sqrt{1-\delta^2}\,\sigma_t & \textit{if} \quad t+\frac{1}{K} \notin \mathbb{N}, \\ \sqrt{(1-\delta^2)\sigma_t^2 + \sigma_v^2} & \textit{if} \quad t+\frac{1}{K} \in \mathbb{N}. \end{cases} \tag{27}$$

The ask and bid quotes are unique and satisfy:

$$A_t \;=\; \mu_t + \delta\sigma_t, \qquad B_t \;=\; \mu_t - \delta\sigma_t, \qquad s_t \;=\; A_t - B_t \;=\; 2\delta\sigma_t. \tag{28}$$

We next investigate whether the public density reaches a steady state, in the sense that its shape converges to a particular density. Unlike in Proposition 4, however, we cannot expect the volatility to stay constant at noninteger times. Instead, equation (27) shows that at noninteger times the public volatility decreases exponentially, as the dealer is learning from the order flow without the offsetting effect of a change in fundamental value (which only occurs at integer times).

Thus, in this context we call an equilibrium stationary if the public volatility at integer times is constant. Proposition 7 shows that the public volatility σ_t for integer values of t converges to a unique value, σ_*, regardless of the initial value σ_0.

Proposition 7. *For any $t \in \mathbb{N}$, the public volatility σ_t satisfies:*

$$\begin{aligned} \sigma_t^2 &= \sigma_*^2 + \left(\sigma_0^2 - \sigma_*^2\right)(1-\delta^2)^{Kt}, \\ \sigma_{t+\frac{k}{K}}^2 &= \sigma_t^2 (1-\delta^2)^k \qquad \text{if} \quad k \in \{1, 2, \dots, K-1\}, \end{aligned} \tag{29}$$

where:

$$\sigma_* = \frac{\sigma_v}{\sqrt{1-(1-\delta^2)^K}}. \tag{30}$$

For any initial value σ_0 and any sequence of orders, the public volatility σ_t at integer times monotonically converges to σ_*.

Thus, Proposition 7 shows that, any equilibrium converges to a unique stationary equilibrium regardless of the initial state.

Next, we analyze the bid–ask spread in the stationary equilibrium. First note that in any equilibrium, the bid–ask spread at any $t \in \frac{\mathbb{N}}{K}$ satisfies (see equation (16)):

$$s_t = 2\delta\sigma_t. \tag{31}$$

Thus, in the stationary equilibrium described in Proposition 7, the bid–ask spread is largest at integer times, and it decreases exponentially until the next integer time, when it jumps again to the highest level. Corollary 5 shows that the highest level is equal to:

$$s_* = \frac{2\delta\sigma_v}{\sqrt{1-(1-\delta^2)^K}} = \frac{2\sigma_v}{\sqrt{1+(1-\delta^2)+\cdots+(1-\delta^2)^{K-1}}}, \tag{32}$$

and provides explicit formulas for the stationary bid–ask spread at noninteger times.[12]

Corollary 5. *In the stationary equilibrium, if $t \in \mathbb{N}$ and $k \in \{0, 1, \dots, K\}$, the bid-ask spread at $t + \frac{k}{K}$ is:*

$$s_{t+\frac{k}{K}} = s_* \left(1-\delta^2\right)^{\frac{k}{2}}, \tag{33}$$

[12]Note that when $K = 1$, the stationary value s_* is the same as the baseline value in equation (19).

where s_* is as in (32). The average bid–ask spread is:

$$\bar{s} \;=\; \bar{s}_K(\rho) \;=\; \frac{2\sigma_v}{K}\,\frac{1+(1-\delta^2)^{\frac{1}{2}}+\cdots+(1-\delta^2)^{\frac{K-1}{2}}}{\sqrt{1+(1-\delta^2)+\cdots+(1-\delta^2)^{K-1}}}. \tag{34}$$

Another formula for the average bid–ask spread is:

$$\bar{s} \;=\; \bar{s}_K(\rho) \;=\; \frac{2\sigma_v}{K}\left(\frac{(1+\alpha)(1-\alpha^K)}{(1-\alpha)(1+\alpha^k)}\right)^{\frac{1}{2}}, \quad \text{with} \quad \alpha \;=\; \sqrt{1-\delta^2}. \tag{35}$$

We next perform some comparative statics. Note that the average bid–ask spread is a function of the number of trading rounds K and the informed share ρ. Note that, as the ρ approaches 0, $\delta = g^{-1}(2\rho)$ also approaches 0. Therefore, when ρ approaches 0, the average bid–ask spread approaches:

$$\bar{s}_K(0) \;=\; \frac{2\sigma_v}{\sqrt{K}}. \tag{36}$$

Thus, a larger number of trading rounds K implies more learning by the dealer, and hence a smaller stationary bid–ask spread.

To study the effect of the informed share ρ on the average bid–ask spread, we normalize the average bid–ask spread $\bar{s}_K(\rho)$ by dividing by $\bar{s}_K(0)$ (which does not depend on ρ):

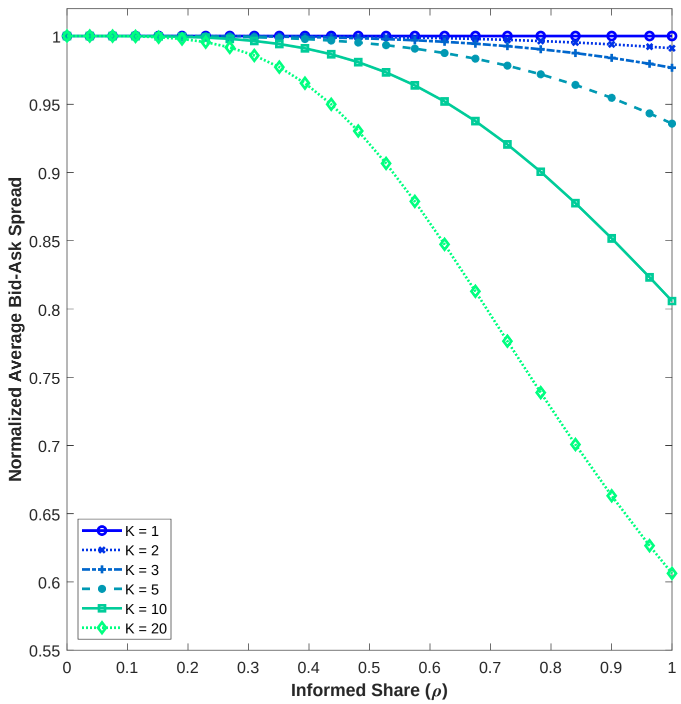
$$\frac{\bar{s}_K(\rho)}{\bar{s}_K(0)} \;=\; \frac{1+(1-\delta^2)^{\frac{1}{2}}+\cdots+(1-\delta^2)^{\frac{K-1}{2}}}{\sqrt{K}\,\sqrt{1+(1-\delta^2)+\cdots+(1-\delta^2)^{K-1}}}. \tag{37}$$

Proposition 8 shows that the average bid–ask spread is always decreasing in the informed share.

Proposition 8. *The average bid–ask spread $\bar{s}_K(\rho)$ is strictly decreasing in ρ for all $K > 1$.*

Figure 2 shows this ratio for several values of $K \in \mathbb{N}_+$. We note that the (normalized) average bid–ask spread does not depend on ρ (see Corollary 2), while for $K > 1$ the average bid–ask spread is decreasing in ρ. The intuition of this result is that the dynamic efficiency effect dominates the adverse selection effect when there is more than one trading round. Indeed, in the presence of more informed traders, dealers learn more quickly, and the bid-ask spreads decrease even faster in trading rounds when the fundamental value does not change.

Figure 2: Average Bid–Ask Spread and the Informed Share.
This figure shows the normalized average bid–ask spread, given by the ratio $\bar{s}_K(\rho)/\bar{s}_K(0)$ from equation (37), where $K \in \{1, 2, 3, 5, 10\}$ is the number of trading rounds, and $\rho \in (0, 1)$ is the informed share.



A more precise intuition behind Proposition 8 is subtle, however, as the effect of stationarity on the equilibrium is not obvious. Indeed, Appendix B shows that the equilibrium is stationary only if the equation $\mathsf{Var}(v_{t+1} - v_t) = \mathsf{Var}(\mu_{t+1} - \mu_t)$ is true. Proposition 6 implies that the public mean change is: $\mu_{t+1} - \mu_t = \pm h_t \pm h_{t+\frac{1}{K}} \pm \cdots \pm h_{t+\frac{K-1}{K}}$, where $h_\tau = \frac{s_\tau}{2}$ is the half spread at $\tau \in \frac{\mathbb{N}}{K}$. As $\mathsf{Var}(v_{t+1} - v_t) = \sigma_v^2$, in a stationary equilibrium we have:

$$\sigma_v^2 \;=\; \mathsf{Var}(\mu_{t+1} - \mu_t) \;=\; h_t^2 + h_{t+\frac{1}{K}}^2 + \cdots + h_{t+\frac{K-1}{K}}^2, \quad t \in \mathbb{N}. \tag{38}$$

The Cauchy–Schwartz inequality implies:

$$\sigma_v^2 \;\geq\; K\bar{h}^2, \quad \text{with} \quad \bar{h} \;=\; \frac{h_t + h_{t+\frac{1}{K}} + \cdots + h_{t+\frac{K-1}{K}}}{K} \;=\; \frac{\bar{s}}{2}, \tag{39}$$

where $\bar{h}$ is the average half spread. This is an equality only if $h_t = h_{t+\frac{1}{K}} = \cdots = h_{t+\frac{K-1}{K}}$, which occurs in the limit when ρ approaches 0. Intuitively, when the informed share ρ is larger, the half spread $h_{t+\frac{k}{K}}$ decays more quickly over time due to faster learning, and therefore the inequality $\sigma_v^2 > K\bar{h}^2$ is stronger, i.e., the average half spread $\bar{h}$ is smaller.

5 Robustness and Extensions

In this section we discuss various assumptions of the model, as well as the robustness of our main result, that the dynamic efficiency effect is strong enough to overcome the adverse selection effect.

5.1 Informed and Uninformed Traders

In this section, we discuss the agents' motivation to trade in the baseline model when the trading frequency is $K = 1$. A standard way to endogenize uninformed trading is to assume that these traders possess a relative private valuation u and trade only when u is above the cost of trading. In this setup, the cost of trading is equal to half the bid–ask spread, which according to Proposition 4 is equal to the fundamental volatility σ_v. Thus, if we set $u = \sigma_v$, the uninformed traders are indifferent between trading and not trading. E.g., if a trader expects the value to be μ_t, the trader is indifferent between doing nothing (and getting a utility of 0) and buying at the ask price, which costs $A_t - \mu_t = \sigma_v$ but yields an offsetting private valuation $u = \sigma_v$.

To endogenize informed trading, we assume that traders that become informed a cost c of acquiring information which must be paid at t before observing the value v_t. Recall that in the stationary equilibrium $(v_t - \mu_t)/\sigma_*$ has the standard normal distribution. Also, equation (18) implies that $\sigma_v/\sigma_* = \delta$, where $\delta = \delta(\rho)$ is the increasing function of ρ from equation (13).

Thus, the expected profit of an informed trader before observing the value at t is:

$$
\begin{aligned}
\Pi(\rho) &= \mathsf{E}_t(v_t - A_t | v_t > A_t) \cdot \mathsf{P}(v_t > A_t) + \mathsf{E}_t(B_t - v_t | v_t < A_t) \cdot \mathsf{P}(v_t < B_t) \\
&= 2\sigma_* \, \mathsf{E}_t\Big(\frac{v_t - \mu_t}{\sigma_*} - \frac{A_t - \mu_t}{\sigma_*} \;\Big|\; v_t > A_t\Big) \cdot \mathsf{P}\Big(\frac{v_t - \mu_t}{\sigma_*} > \frac{A_t - \mu_t}{\sigma_*}\Big) \\
&= 2\,\frac{\sigma_v}{\delta}\,\Big(\int_\delta^\infty x\phi(x)\mathrm{d}x - \delta\Big) \cdot \big(1 - \Phi(\delta)\big) \;=\; \frac{2\sigma_v(\phi(\delta) - \delta)\Phi(-\delta)}{\delta},
\end{aligned}
\tag{40}
$$

where $\phi(\cdot)$ is the standard normal density and $\Phi(\cdot)$ its cumulative density function. It is straightforward to check that $\Phi(\rho)$ is decreasing in ρ, which intuitively means that a larger fraction of informed traders leads to a smaller ex ante profit for each informed trader. Therefore, with free entry, if the informed traders' ex ante profit $\Phi(\rho)$ is higher than the information acquisition cost c, the fraction of informed traders should rise until $\Phi(\rho) = c$, when the informed traders become indifferent between trading and not trading.

The discussion above shows that the informed share ρ is in one-to-one correspondence of the information acquisition cost c. We can therefore interpret a change in ρ as originating from an exogenous change in the cost c. This is useful, e.g., when interpreting the results in Section 3.2.3, where we analyze the effects on liquidity of an unexpected change in the informed share ρ.

Note that the parameters of the model can be set such that all traders, informed or not, are indifferent between trading and not trading. This provides microfoundations for orderly trade in our model.

Finally, we discuss what happens when an informed trader observes a fundamental value v_t between the ask and the bid. In that case, by assumption an uninformed trader jumps in and trades at t. This assumption is made for convenience, since otherwise it is possible to have no trade at t, which would inform the dealer that v_t lies in the interval $[B_t, A_t]$ and would considerably lower the public volatility. One way to avoid this somewhat artificial situation is to note that ρ is likely to be small in practice. In that case, a calculation as in equation (40) shows that the probability that v_t is in the interval $[B_t, A_t]$ is equal to $\Phi(\delta) - \Phi(-\delta) \approx 2\phi(0)\delta$, which is also small, and therefore unlikely to influence much the overall results.

5.2 Fundamental Value and Learning

In this section, we analyze the robustness of our main result under different specifications for the fundamental value and for the dealer's learning process. Recall the intuition for why the dynamic efficiency effect offsets the adverse selection effect in the baseline model when the trading frequency is $K = 1$ (see Section 3.2.2): Suppose ρ is low and the dealer observes a buy order at t. A low ρ means that the buyer is unlikely to be informed, which implies that the update of public mean (and hence the dealer's bid–ask spread) should be small. This is the adverse selection effect. However, in the rare case when the buyer *is* actually informed, he must have observed a large fundamental value v_t, since the uncertainty in v_t (measured by the public volatility σ_*) is also large. This is the dynamic efficiency effect.

Note that for this offsetting argument to fully work, the public volatility σ_* must be very large when ρ is very small. This is possible only if the range of the fundamental value is not restricted to become very large. Such restrictions can occur in two ways:

(i) The fundamental value is directly assumed to be bounded;

(ii) The signals received by the dealer essentially bound the dealer's uncertainty.

5.2.1 Alternating Value

The situation (i) above occurs if we require the fundamental value to lie in a bounded interval such as $[0, 1]$.[13] We analyze such a model in Section 2 in the Internet Appendix, where the fundamental value is either 0 or 1 (as in Glosten and Milgrom, 1985), and it switches every period between these two values with probability $\nu < 1/2$. In that case, the dynamic efficiency effect no longer offsets the adverse selection effect. Nevertheless, even if ρ is small, as long as ρ is large relative to the switching parameter ν, the dynamic efficiency effect is relatively strong, and as a result the dependence of the average bid–ask spread on ρ is weaker, and the equilibrium approaches the one in the diffusing-value model where the average bid–ask spread is independent of ρ.

A similar result is likely to hold in a model in which the fundamental value follows a

[13] This setup of course cannot occur if the fundamental value follows a random walk.

mean-reverting process, as the fundamental value is, in some sense, more bounded in the mean-reverting case than in the random walk case. Note, however, that the random walk case is more plausible under the interpretation of the fundamental value v_t as the expected liquidation value using all possible available information (including private information) at t: Indeed, in that case, the value increments are independent with respect to all previous information, and therefore v_t is the sum of many independent increments, which is essentially the definition of a random walk.

5.2.2 Public News

The situation (ii) above occurs if the dealer receives at every t signals about the level v_t. Note that this is not by itself enough to bound the dealer's uncertainty about v_t. Indeed, we next consider an extension of our model in which, in addition to observing the order flow, the dealer receives at every t signals about the increment $v_t - v_{t-1}$. In that case, the main result goes through. The reason is that the dealer's uncertainty about the level v_t becomes large when ρ is small: the dealer only learns about v_t once, after which she learns only about future value increments. If, by contrast, although perhaps less realistically, the dealer received at every t signals about the level v_t, then the uncertainty would remain bounded even when ρ was very small, and the main result would no longer hold.

Suppose that before each $t \in \mathbb{N}$, the dealer receives a signal $\Delta s_t = s_t - s_{t-1}$ about the increment $\Delta v_t = v_t - v_{t-1}$, of the form:

$$\Delta s_t \;=\; \Delta v_t + \Delta \eta_t, \quad \text{with} \quad \Delta \eta_t \;=\; \eta_t - \eta_{t-1} \overset{IID}{\sim} \mathcal{N}\big(\cdot, 0, \sigma_\eta\big). \tag{41}$$

Let μ_t and σ_t be respectively the public mean and public volatility just before trading at $t \in \mathbb{N}$ (but after the signal Δs_t is observed). Note that this extension generalizes the model in Section 2: when σ_η approaches infinity, it is as if the dealer receives no signal at t. Proposition 9 shows the evolution of the public mean and volatility, as well as the stationary volatility and bid–ask spread.

Proposition 9. *For any $t = 0, 1, 2, \ldots$ the public mean and volatility satisfy:*

$$\mu_{t+1} \;=\; \mu_t \pm \delta\sigma_t + \frac{\sigma_v^2}{\sigma_v^2 + \sigma_\eta^2}\Delta s_{t+1}, \qquad \sigma_t^2 \;=\; \sigma_*^2 + \left(\sigma_0^2 - \sigma_*^2\right)\left(1 - \delta^2\right)^t, \tag{42}$$

where $\delta = g^{-1}(2\rho)$, as in equation (13), and

$$\sigma_* \;=\; \frac{\sigma_{v\eta}}{\delta} \quad \textit{with} \quad \sigma_{v\eta} \;=\; \frac{\sigma_v \sigma_\eta}{\sqrt{\sigma_v^2 + \sigma_\eta^2}}. \tag{43}$$

For any initial value σ_0 and any sequence of orders, the public volatility σ_t monotonically converges to σ_*, and the bid–ask spread monotonically converges to:

$$s_* \;=\; 2\sigma_{v\eta}. \tag{44}$$

Proposition 9 is essentially the same result as Proposition 4, except that the fundamental volatility σ_v is replaced here by $\sigma_{v\eta}$. The parameter $\sigma_{v\eta}$ represents the increase in dealer uncertainty from t to $t+1$, conditional on her receiving the signal Δs_{t+1}.[14] When σ_η is 0, the dealer learns perfectly the increment Δv, hence even if the dealer does not know the initial value v_0, she ends up by learning v_t almost perfectly (she also learns about v_t from the order flow). When σ_η approaches infinity, the dealer receives uninformative signals, $\sigma_{v\eta}$ approaches σ_v, and the equilibrium behavior is described as in Proposition 4.

The stationary bid–ask spread s_* is twice the parameter $\sigma_{v\eta}$. Thus, the bid–ask spread is increasing in the news uncertainty parameter σ_η, and ranges from 0 (when $\sigma_\eta = 0$) to $2\sigma_v$ (when $\sigma_\eta = \infty$). The relation between the bid–ask spread and σ_η is intuitive: with more imprecise news, the dealer is more uncertain about the asset value, and sets a larger stationary bid–ask spread.

Note that even in this more general context the stationary bid–ask spread s_* does not depend on the informed share ρ. The intuition is the same as for Proposition 4, and is discussed at the end of the proof of Proposition 9. This intuition is based on the general result (proved in Appendix B) that for any filtration problem in which the variance remains

[14] Indeed, its square $\sigma_{v\eta}^2$ is equal to the conditional variance $\mathsf{Var}\left(\Delta v_{t+1} | \Delta s_{t+1}\right)$.

constant over time, the variance of the change in public mean must equal the fundamental variance. But the latter variance is independent of ρ, as is the variance of the signal Δs, hence the bid–ask spread is also independent of ρ.

5.3 Comparison with the Exact Equilibrium

In this section, we analyze in more detail the evolution of the public density ϕ_t when the dealer is fully Bayesian. In particular, we are interested in computing the average shape of the public density over all possible future paths of the game.[15] Note that when computing the average shape of a density, we consider the average of the various densities after demeaning them. We then show numerically that this average exists and we compare it to the stationary public density described in Section 3.2.2, which is a normal density with mean 0 and standard deviation equal to $\sigma_* = \sigma_v/\delta$.

We thus demean the variables and densities involved in the previous formulas, and in addition we normalize them by σ_*:

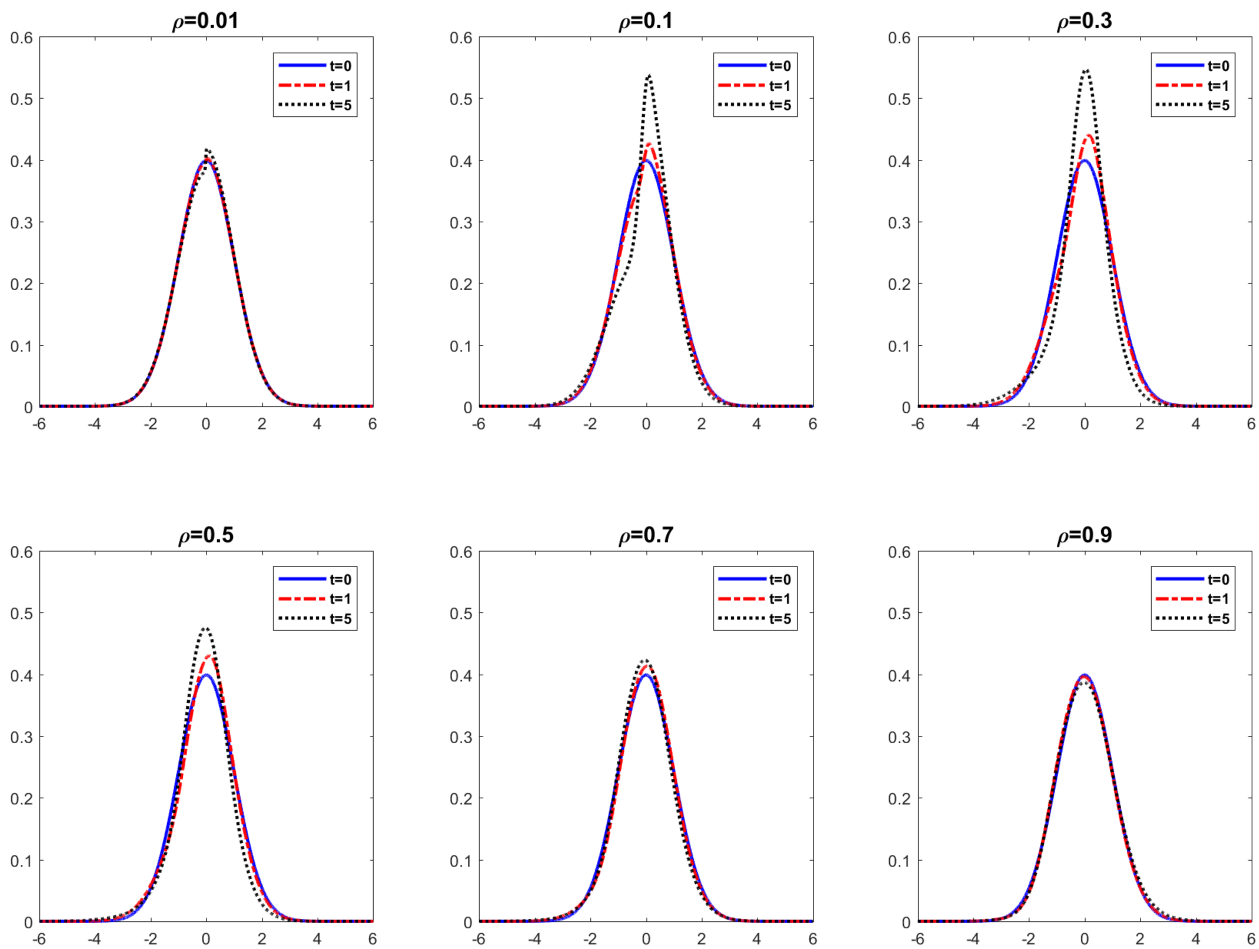
$$\begin{aligned} \tilde{A}_t &= \frac{A_t - \mu_t}{\sigma_*}, \qquad \tilde{B}_t = \frac{B_t - \mu_t}{\sigma_*}, \qquad \tilde{v}_t = \frac{v_t - \mu_t}{\sigma_*}, \qquad \tilde{\phi}_t(\tilde{v}) = \sigma_* \phi_t(\mu_t + \sigma_* \tilde{v}), \\ \tilde{\Phi}_t(\tilde{v}) &= \int_{-\infty}^{\tilde{v}} \tilde{\phi}_t(w)\mathrm{d}w, \qquad \tilde{\psi}_t(\tilde{v}|\mathcal{O}_t) = \sigma_* \psi_t(\mu_t + \sigma_* \tilde{v}|\mathcal{O}_t). \end{aligned} \tag{45}$$

With this new notation, the equations (4) and (5) from Proposition 1 imply the following result.

Corollary 6. *Consider a rapidly decaying public density ϕ_t with normalization $\tilde{\phi}_t$, and an ask–bid pair (A_t, B_t) with normalization $(\tilde{A}_t, \tilde{B}_t)$. After observing an order $\mathcal{O}_t \in \{\mathrm{B}, \mathrm{S}\}$, the*

[15] As we see in Internet Appendix (Sections 1 and 2), one expects a well defined stationary density for the continuous time Markov chain associated to our game. The only problem is that the fundamental value v_t is no longer stationary in our case, but follows a random walk. One can show that it is still possible to define a stationary density as long as one does not require it to integrate to 1 over v. But we are interested in the simpler problem of computing the marginal stationary density of $v_t - \mu_t$, which we solve numerically.

Figure 3: Exact Normalized Public Density after Series of Buy Orders.
Each of the 6 graphs represents the evolution of the normalized public density $\tilde{\phi}_t$ after $t = 0$, $t = 1$ and $t = 5$ buy orders. The initial normalized public density in all cases (at $t = 0$) is the standard normal density with mean 0 and volatility 1. The 6 graphs correspond to the informed share $\rho \in \{0.01, 0.1, 0.3, 0.5, 0.7, 0.9\}$.



normalized density at $t+1$ *is* $\tilde{\phi}_{t+1}(\tilde{w}|\mathcal{O}_t)$*, where:*

$$\begin{aligned}
\tilde{\phi}_{t+1}(\tilde{w}|\mathrm{B}) &= \int_{-\infty}^{+\infty} \mathcal{N}\Big(\frac{\tilde{w}-\tilde{v}+\tilde{A}_t}{\delta}\Big) \frac{\big(\rho \mathbf{1}_{\tilde{v}>\tilde{A}_t} + \frac{\rho}{2}\mathbf{1}_{\tilde{v}\in[\tilde{B}_t,\tilde{A}_t]} + \frac{1-\rho}{2}\big)\cdot \tilde{\phi}_t(\tilde{v})}{\frac{\rho}{2}(1-\tilde{\Phi}_t(\tilde{A}_t)) + \frac{\rho}{2}(1-\tilde{\Phi}_t(\tilde{B}_t)) + \frac{1-\rho}{2}}\,\mathrm{d}\tilde{v}, \\
\tilde{\phi}_{t+1}(\tilde{w}|S) &= \int_{-\infty}^{+\infty} \mathcal{N}\Big(\frac{\tilde{w}-\tilde{v}+\tilde{B}_t}{\delta}\Big) \frac{\big(\rho \mathbf{1}_{\tilde{v}<\tilde{B}_t} + \frac{\rho}{2}\mathbf{1}_{\tilde{v}\in[\tilde{B}_t,\tilde{A}_t]} + \frac{1-\rho}{2}\big)\cdot \tilde{\phi}_t(\tilde{v})}{\frac{\rho}{2}\tilde{\Phi}_t(\tilde{A}_t) + \frac{\rho}{2}\tilde{\Phi}_t(\tilde{B}_t) + \frac{1-\rho}{2}}\,\mathrm{d}\tilde{v}.
\end{aligned} \tag{46}$$

Figure 3 displays the normalized public density after $t = 0$, $t = 1$, and $t = 5$ buy orders for various values of the informed share ρ. We notice by visual inspection that the normalized public density is close to the standard normal density even after a sequence of 5 buy orders (this sequence happens with probability 2^{-5}, which is approximately 3.13%). The deviation of the normalized public densities from the standard normal density is at its smallest level when the informed share ρ is either small or large, and it peaks for an intermediate value ρ near 0.2. When ρ is small, the order flow is uninformative, hence the posterior is not far from the prior. When ρ is large, the order flow is very informative, hence the posterior depends strongly on the increment, which is normally distributed.

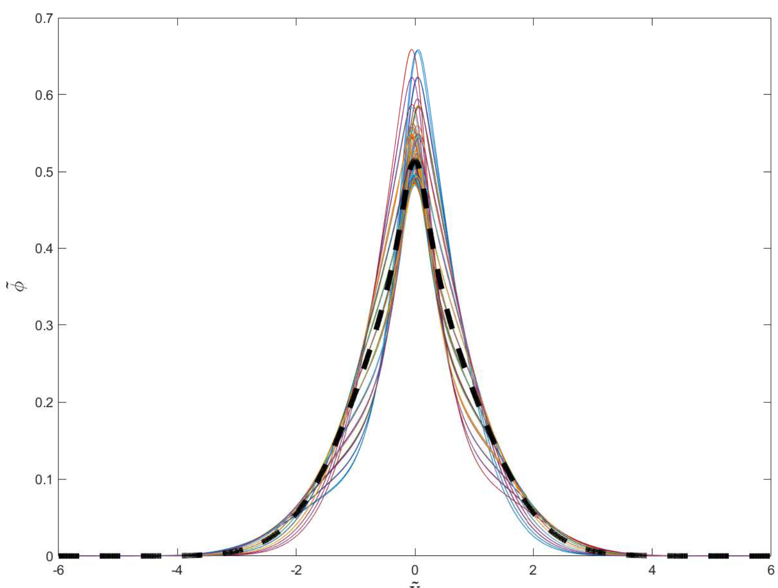
Table 1: Average Normalized Public Density after Series of Random Orders. For each informed share $\rho \in \{0.01, 0.1, 0.3, 0.5, 0.7, 0.9\}$, consider 200 random series of 20 orders chosen among buy or sell with equal probability, and denote by $\tilde{\phi}_S$ the normalized public density computed after observing the series $S = 1, 2, \ldots, 200$. The table displays four estimated moments of the average $\psi = \frac{\tilde{\phi}_1 + \tilde{\phi}_2 + \cdots + \tilde{\phi}_{200}}{200}$: the mean $\mu = \int_{-\infty}^{+\infty} x\psi(x)\mathrm{d}x$, the standard deviation $\sigma = \left(\int_{-\infty}^{+\infty} (x-\mu)^2 \psi(x)\mathrm{d}x\right)^{1/2}$, the skewness $\int_{-\infty}^{+\infty} \left(\frac{x-\mu}{\sigma}\right)^3 \psi(x)\mathrm{d}x$, and the kurtosis $\int_{-\infty}^{+\infty} \left(\frac{x-\mu}{\sigma}\right)^4 \psi(x)\mathrm{d}x$. It also displays the average bid–ask spread normalized by $s_* = 2\sigma_v$ (N.Spread).

ρ	0.01	0.1	0.3	0.5	0.7	0.9
Mean	-0.000	0.000	-0.001	-0.001	-0.010	-0.002
St.Dev.	1.000	1.002	1.039	1.056	1.041	1.009
Skewness	-0.001	0.018	0.016	-0.003	-0.012	-0.009
Kurtosis	3.005	3.419	4.587	4.597	4.089	3.343
N.Spread	1.003	0.966	0.959	0.988	1.014	1.004

It turns out, however, that the stationary shape of the public density is not precisely normal, but it has "fat tails," that is, its fourth centralized moment (kurtosis) is larger than 3. Table 1 displays, for each $\rho \in \{0.01, 0.1, 0.3, 0.5, 0.7, 0.9\}$, several moments of the average normalized density computed after 200 different random paths. As the starting density (at $t = 0$) is standard normal for all the different ρ, we need to make sure that we choose a path length long enough for the average density to stabilize. Numerically, we see that it is enough to choose $t = 20$.[16] Thus, in Table 1 we display the first four centralized moments for the average normalized public density at $t = 20$, computed over 200 random paths.

The first three moments of the average density at $t = 20$ are similar to the moments of the standard normal density: the mean and the skewness (centralized third moment) are close to 0, and the standard deviation is close to 1. The kurtosis, however, is larger than 3, indicating that the stationary public density has indeed fat tails. Nevertheless, the deviation from the standard normal density is not large, especially when ρ is small or large. Moreover, the last row in Table 1 implies that the average bid–ask spread in each case is quite close to $s_* = 2\sigma_v$, which is the stationary value in the approximate Bayesian case: see equation (19). Thus, we argue that the normal approximation made in Section 3.2 is reasonable, especially when it comes to our main liquidity measure, the bid–ask spread.

Figure 4: Normalized Public Density after Series of Random Orders.
For an informed share $\rho = 0.1$, consider 200 random series of 20 orders chosen among buy or sell with equal probability, and denote by $\tilde{\phi}_S$ the normalized public density computed after observing the series $S = 1, 2, \ldots, 200$. The table displays the densities $\tilde{\phi}_S$, as well as their average $\psi = \frac{\tilde{\phi}_1+\tilde{\phi}_2+\cdots+\tilde{\phi}_{200}}{200}$. The average density is displayed with a thick dashed line.



The question remains how different the normalized public density can be from the average

[16] We have checked that the average density at $t = 20$ is in absolute value less than 0.01 apart from the average density at $t = 25$ or $t = 30$.

density. This question is already discussed tangentially in Figure 3, where we observe the normalized public density after five buy orders. But to understand this issue in more detail, we choose one particular value of the informed share, $\rho = 0.1$, for which the normalized public density after five buy orders appears more different than the normal density. Figure 4 displays the normalized public density after each of the 200 random series of 20 orders, along with the average density. Then, the results in Table 1 and Figure 4 can be summarized by observing that the normalized public density does not deviate too far from its average value, and in turn this average value does not deviate too far from the standard normal density.

More important for our purposes, however, is to compare the average bid–ask spread in the exact equilibrium with the stationary bid–ask spread $s_* = 2\sigma_v$ from equation (19). Result 1 shows that these two values are numerically the same for all the informed share we considered.

Result 2. When the trading frequency is $K = 1$, the average bid–ask spread is constant and equal to $s_* = 2\sigma_v$. When $K > 1$, the average bid–ask spread is decreasing in the informed share ρ.

We discuss the numerical verification of this result in the Appendix.

6 Conclusion

In this paper we have presented a dealer model in which the asset value follows a random walk. The stationary equilibrium of the model has novel properties, and is affected by two opposite effects: First, under the traditional adverse selection effect, the dealer sets higher bid–ask spreads to protect from a larger number of informed traders. Second, under the dynamic efficiency effect, the dealer learns faster from the order flow when there are more informed traders, and this reduces the bid–ask spread.

Our main finding is that the dynamic efficiency effect is strong enough to offset the adverse selection in the baseline case, when the trading frequency is equal to 1. In that case, the stationary bid–ask spread no longer depends on the informed share (the fraction of traders that are informed). If the trading frequency is larger than 1, the dynamic efficiency effect dominates the adverse selection effect, and the average stationary bid-ask spread is decreasing in the informed share. The strength of this dependence is increasing in the trading frequency, as dynamic efficiency has more time to reduce the bid-ask spread.

The nonstationary equilibria converge to the stationary equilibrium, regardless of the initial state. The evolution of the nonstationary equilibrium after various types of shocks provides additional testable implications of our model. For instance, after a positive shock to the informed share (e.g., if more informed investors start trading in that stock) the bid–ask spread jumps but then it decreases again to its stationary level. This type of liquidity resilience occurs purely for informational reasons, without any additional market maker jumping in to provide liquidity.

Appendix A. Proofs of Results

Proof of Proposition 1. Using Bayes' rule, the posterior density of v_t after observing $\mathcal{O}$ is:

$$\psi_t(v|\mathcal{O}) \;=\; \frac{\mathsf{P}(\mathcal{O}_t = \mathcal{O} \mid v_t = v) \cdot \mathsf{P}(v_t = v)}{\int_v \mathsf{P}(\mathcal{O}_t = \mathcal{O} \mid v_t = v) \cdot \mathsf{P}(v_t = v)} \;=\; \frac{g_t(\mathcal{O}, v) \cdot \phi_t(v)}{\int_v g_t(\mathcal{O}, v) \cdot \phi_t(v)}, \tag{A1}$$

where $\int_v F(v)$ is shorthand for $\int_{-\infty}^{+\infty} F(v)\mathrm{d}v$. Substituting $g_t(\mathcal{O}, v)$ from (2) and (3) in the above equation, we obtain (4).

Let $f(w, v) = \mathsf{P}(v_{t+1} = w | v_t = v) = \mathcal{N}(w - v, 0, \sigma_v)$ be the transition density of v_t. To compute the posterior density of v_t after observing $\mathcal{O}_t = \mathcal{O}$, note that:

$$\begin{aligned}\phi_{t+1}(w|\mathcal{O}) \;&=\; \int_v \mathsf{P}(v_{t+1} = w \mid v_t = v, \mathcal{O}_t = \mathcal{O}) \cdot \mathsf{P}(v_t = v \mid \mathcal{O}_t = \mathcal{O}) \\ &=\; \int_v \mathsf{P}(v_{t+1} = w \mid v_t = v) \cdot \mathsf{P}(v_t = v \mid \mathcal{O}_t = \mathcal{O}) \;=\; \int_v f(w, v) \cdot \psi_t(v|\mathcal{O}),\end{aligned} \tag{A2}$$

which proves (5).

To simplify notation, we omit conditioning on the order $\mathcal{O}_t$. From (4), it follows that the posterior density ψ_t is equal to ϕ_t multiplied by a piecewise constant function. The prior density ϕ_t is rapidly decaying, hence it is bounded. Therefore ψ_t is also bounded and continuous, although it is no longer smooth. Nevertheless, when we convolute $\psi_t(\cdot)$ with $\mathcal{N}(\cdot, 0, \sigma_v)$ the result ϕ_{t+1} becomes smooth. Indeed, the N'th derivative $\mathrm{d}^N\phi_{t+1}(w)/\mathrm{d}w^N$ involves differentiating the smooth function $\mathcal{N}(w-v, 0, \sigma_v)$ under the integral sign. As the remaining term $\psi_t(v)$ is bounded, the integrals are well defined, and hence ϕ_{t+1} is a smooth function. The fact that ϕ_{t+1} is also rapidly decaying can be seen in the same way, using again the fact that ψ_t is bounded. □

Proof of Corollary 1. By definition of the ask–bid pair, A_t is the mean of the posterior density of v_t after observing a buy order at t. But the increment $v_{t+1} - v_t$ has zero mean and is independent of the previous variables until t. Therefore, A_t is also the mean of the posterior density of v_{t+1} after observing a buy order at t. Similarly, B_t is the mean of the posterior density of v_{t+1} after observing a sell order at t. This proves the equations in (7). □

Proof of Proposition 2. Define the following function:[17]

$$H_t(v) \;=\; \int_{-\infty}^{v} w\phi_t(w)\mathrm{d}w \;=\; v\Phi_t(v) - \int_{-\infty}^{v} \Phi_t(w)\mathrm{d}w. \tag{A3}$$

Note that $H_t(-\infty) = 0$ and $H_t(+\infty) = \int_{-\infty}^{\infty} w\phi_t(w)\mathrm{d}w = \mu_t$. Also, note that:

$$\Theta_t(v) \;=\; \mu_t\Phi_t(v) - H_t(v). \tag{A4}$$

To prove the desired equivalence, start with an ask–bid pair (A_t, B_t). This pair must satisfy the dealer's pricing conditions: A_t is the mean of $\psi_t(\cdot|\mathrm{B})$, and B_t is the mean of $\psi_t(\cdot|\mathrm{S})$. Using the formulas in (4) for $\psi_t(v|\mathcal{O})$, we compute:

$$\begin{aligned} A_t \;&=\; \frac{\rho\big(\mu_t - H_t(A_t)\big) + \frac{\rho}{2}\big(H_t(A_t) - H_t(B_t)\big) + \frac{1-\rho}{2}\mu_t}{\frac{\rho}{2}(1-\Phi_t(A_t)) + \frac{\rho}{2}(1-\Phi_t(B_t)) + \frac{1-\rho}{2}}, \\ B_t \;&=\; \frac{\rho H_t(B_t) + \frac{\rho}{2}\big(H_t(A_t) - H_t(B_t)\big) + \frac{1-\rho}{2}\mu_t}{\frac{\rho}{2}\Phi_t(A_t) + \frac{\rho}{2}\Phi_t(B_t) + \frac{1-\rho}{2}}. \end{aligned} \tag{A5}$$

[17] In the formula for H_t we use integration by parts, and also the fact that $\lim_{v\to-\infty} v\Phi_t(v) = 0$. To prove this last fact, suppose $v = -x$ with $x > 0$. Since ϕ_t is rapidly decaying, $\phi_t(-x) < Cx^{-3}$ for some constant C. Then $x\Phi_t(-x) = x\int_{-\infty}^{-x} \phi_t(w)\mathrm{d}w < x\frac{Cx^{-2}}{2}$, which implies $\lim_{x\to\infty} x\Phi_t(-x) = 0$.

Using (A4), we compute the following differences:

$$\begin{aligned} A_t - \mu_t &= \frac{\frac{\rho}{2}\Theta_t(A_t) + \frac{\rho}{2}\Theta_t(B_t)}{\rho(1-\Phi_t(A_t)) + \frac{\rho}{2}(\Phi_t(A_t) - \Phi_t(B_t)) + \frac{1-\rho}{2}}, \\ \mu_t - B_t &= \frac{\frac{\rho}{2}\Theta_t(A_t) + \frac{\rho}{2}\Theta_t(B_t)}{\rho\Phi_t(B_t) + \frac{\rho}{2}(\Phi_t(A_t) - \Phi_t(B_t)) + \frac{1-\rho}{2}}. \end{aligned} \tag{A6}$$

As Θ_t is strictly positive everywhere (see Footnote 8), we have the following inequalities: $A_t > \mu_t > B_t$, or equivalently $A_t \in (\mu_t, +\infty)$ and $B_t \in (-\infty, \mu_t)$. The equations (A6) can be written as:

$$F(A_t, B_t) \;=\; 0, \qquad G(A_t, B_t) \;=\; 0, \tag{A7}$$

where the functions F and G are defined in (8). Conversely, suppose we have a solution (A_t, B_t) of (A7), with $A_t > \mu_t > B_t$. Then, this pair satisfies the equations in (A6), which are the dealer's pricing conditions. Thus, (A_t, B_t) is an ask–bid pair.

We now show that a solution of (A7) exists. The partial derivatives of F and G are:

$$\begin{aligned} \frac{\partial F}{\partial A} &= -\frac{\Theta_t(A) + \Theta_t(B)}{(A-\mu_t)^2}, & \frac{\partial F}{\partial B} &= \frac{A-B}{A-\mu_t}\,\phi_t(B), \\ \frac{\partial G}{\partial A} &= -\frac{A-B}{\mu_t - B}\,\phi_t(A), & \frac{\partial G}{\partial B} &= \frac{\Theta_t(A) + \Theta_t(B)}{(\mu_t - B)^2}. \end{aligned} \tag{A8}$$

From (8) we see that $F(A,B)$ has well defined limits at $B = \pm\infty$, which follows from the formulas: $\Theta_t(\pm\infty) = 0$, $\Phi_t(-\infty) = 0$, and $\Phi_t(+\infty) = 1$. Thus we extend the definition of F for all $B \in \bar{\mathbb{R}} = [-\infty, +\infty]$. Now fix $B \in \bar{\mathbb{R}}$. We show that there is a unique solution $A = \alpha(B)$ of the equation $F(A,B) = 0$. From (A8) we see that $\frac{\partial F}{\partial A} < 0$ for all $A \in (\mu_t, \infty)$. From (8) we see that when $A \searrow \mu_t$, $F(A,B) \nearrow \infty$; while when $A \nearrow \infty$, $F(A,B) \searrow -\frac{1+\rho}{\rho} + 1 + \Phi_t(B) = -\frac{1}{\rho} + \Phi_t(B) < 0$ (recall that $\rho \in (0,1)$). Thus, for any B there is a unique solution of $F(A,B) = 0$ for $A \in (\mu_t, \infty)$. Denote this unique solution by $\alpha(B)$. Differentiating the equation $F\big(\alpha(B), B\big) = 0$ implies that for all B the derivative of $\alpha(B)$ is $\alpha'(B) = -\frac{\partial F}{\partial B}(\alpha(B), B) / \frac{\partial F}{\partial A}(\alpha(B), B) > 0$. Define $\underline{A} = \alpha(-\infty)$ and $\overline{A} = \alpha(\mu_t)$. The results above imply that both $\underline{A}$ and $\overline{A}$ belong to (μ_t, ∞), and α is a bijective function between $[-\infty, \mu_t]$ and $[\underline{A}, \overline{A}]$.

A similar analysis shows that for all $A \in \bar{\mathbb{R}}$, there is a unique solution $B = \beta(A)$ of the equation $G(A,B)=0$. Moreover, the function β is increasing, and if we define $\underline{B}=\beta(\mu_t)$ and $\overline{B}=\beta(\infty)$, it follows that both $\underline{B}$ and $\overline{B}$ belong to $(-\infty,\mu_t)$, and the function α is bijective between $[\mu_t,\infty]$ and $[\underline{B},\overline{B}]$.

Next, define the function $f:\bar{\mathbb{R}}\to\bar{\mathbb{R}}$ by:

$$f(A) \;=\; \alpha\big(\beta(A)\big). \tag{A9}$$

Consider the set:

$$S \;=\; \{(A,B)\mid A-f(A) \;=\; 0\;,\; B \;=\; \beta(A)\}. \tag{A10}$$

It is straightforward to show that S coincides with the set of all ask–bid pairs. Indeed, $(A,B)\in S$ is equivalent to $A=\alpha(B)$ and $B=\beta(A)$, which, from the discussion above, is equivalent to $F(A,B)=0$ and $G(A,B)=0$. Therefore, the existence of an ask–bid pair is equivalent to there being at least one solution of $A-f(A)=0$.

We now show that the equation $A-f(A)=0$ has at least one solution. The function $f(A)$ is increasing and bijective between $[\mu_t,\infty]$ and $[\alpha(\underline{B}),\alpha(\overline{B})]$. As $\underline{B},\overline{B}\in(-\infty,\mu_t)$, it follows that $[\alpha(\underline{B}),\alpha(\overline{B})]\subset(\underline{A},\overline{A})\subset(\mu_t,\infty)$. When $A\searrow\mu_t$, $A-f(A)\to\mu_t-\alpha(\underline{B})<0$, while when $A\nearrow\infty$, $A-f(A)\to\infty-\alpha(\overline{B})>0$. Thus, there exists a solution of $A-f(A)=0$ on (μ_t,∞). $\square$

Verification of Result 1. If $f(A)=\alpha(\beta(A))$ as in equation (A9), the proof of Proposition 2 shows that every ask–bid pair (A,B) corresponds to a solution of the equation $A-f(A)=0$. Thus, to prove uniqueness it is sufficient to prove that the derivative of f is less than 1 everywhere. Numerically, we verified this inequality for each $\rho\in\{0.1,0.2,\ldots,0.9\}$, and 1000 random sequences of orders of length up to 200. (Recall that ϕ_0 is assumed to be the stationary density $\mathcal{N}(\cdot,0,\sigma_*)$.) $\square$

Proof of Proposition 3. Denote by $\phi(\cdot)=\mathcal{N}(\cdot,0,1)$ the standard normal density, and by $\Phi(\cdot)$ its cumulative density. Consider the function $g:[0,\infty)\to[0,\infty)$ defined by $g(x)=\frac{x}{\phi(x)}$. As the derivative of ϕ is $\phi'(x)=-x\phi(x)$, the derivative of g is $g'(x)=\frac{1+x^2}{\phi(x)}>0$ for all x. Moreover, $g(0)=0$ and $\lim_{x\to\infty}g(x)=\infty$, hence g is increasing and a one-to-one and mapping of $[0,\infty)$.

Consider $t \in \mathbb{N}$. As usual, $\phi_t(\cdot)$ denotes the density of v_t just before trading at t, and $\psi_t(\cdot|\mathcal{O}_t)$ denotes the density of v_t after trading at t. By assumption, ϕ_t is normal with mean μ_t and volatility σ_t:

$$\phi_t(v) \;=\; \frac{1}{\sigma_t}\,\phi\left(\frac{v-\mu_t}{\sigma_t}\right) \;=\; \mathcal{N}(v,\mu_t,\sigma_t). \tag{A11}$$

Define the normalized ask and bid, respectively, by:

$$a_t \;=\; \frac{A_t-\mu_t}{\sigma_t}, \qquad b_t \;=\; \frac{B_t-\mu_t}{\sigma_t}. \tag{A12}$$

We now compute the mean and volatility of $\phi_{t+1}(\cdot|\mathcal{O}_t)$. As the increment $v_{t+1}-v_t \sim \mathcal{N}(0,\sigma_v^2)$ is independent of past variables, the mean and volatility of $\phi_{t+1}(\cdot|\mathcal{O}_t)$ satisfy:

$$\mu_{t+1,\mathcal{O}_t} \;=\; \int_v v\,\psi_t(v|\mathcal{O}_t), \qquad \sigma^2_{t+1,\mathcal{O}_t} \;=\; \sigma_v^2 \;+\; \int_v \big(v-\mu_{t+1,\mathcal{O}_t}\big)^2\psi_t(v|\mathcal{O}_t). \tag{A13}$$

From (4), we have $\psi_t(v|\mathrm{B}) = \frac{\rho\mathbf{1}_{v>A_t}+\frac{\rho}{2}\mathbf{1}_{v\in[B_t,A_t]}+\frac{1-\rho}{2}}{\frac{\rho}{2}(1-\Phi_t(A_t))+\frac{\rho}{2}(1-\Phi_t(B_t))+\frac{1-\rho}{2}}\,\phi_t(v)$. With the change of variables $z=\frac{v-\mu_t}{\sigma_t}$, we compute the posterior mean conditional on a buy order:

$$\begin{aligned}
\mu_{t+1,\mathrm{B}} \;&=\; \mu_t + \int_{-\infty}^{+\infty}(v-\mu_t)\,\frac{\rho\mathbf{1}_{v>A_t}+\frac{\rho}{2}\mathbf{1}_{v\in[B_t,A_t]}+\frac{1-\rho}{2}}{\frac{\rho}{2}(1-\Phi_t(A_t))+\frac{\rho}{2}(1-\Phi_t(B_t))+\frac{1-\rho}{2}}\,\frac{1}{\sigma_t}\phi\big(\tfrac{v-\mu_t}{\sigma_t}\big)\,\mathrm{d}v \\
&=\; \mu_t+\sigma_t\int_{-\infty}^{+\infty} z\,\frac{\rho\mathbf{1}_{z>a_t}+\frac{\rho}{2}\mathbf{1}_{z\in[b_t,a_t]}+\frac{1-\rho}{2}}{\frac{\rho}{2}(1-\Phi(a_t))+\frac{\rho}{2}(1-\Phi(b_t))+\frac{1-\rho}{2}}\,\phi(z)\,\mathrm{d}z \\
&=\; \mu_t+\sigma_t\,\frac{\phi(-a_t)+\phi(-b_t)}{\Phi(-a_t)+\Phi(-b_t)+\frac{1-\rho}{\rho}}.
\end{aligned} \tag{A14}$$

Similarly, the posterior mean conditional on a sell order is:

$$\mu_{t+1,\mathrm{S}} \;=\; \mu_t-\sigma_t\,\frac{\phi(a_t)+\phi(b_t)}{\Phi(a_t)+\Phi(b_t)+\frac{1-\rho}{\rho}}, \tag{A15}$$

To compute $\sigma^2_{t+1,\mathcal{O}_t}$, we notice that:

$$\int_v\big(v-\mu_t\big)^2\psi_t(v|\mathcal{O}_t) \;=\; \int_v\big(v-\mu_{t+1,\mathcal{O}_t}\big)^2\psi_t(v|\mathcal{O}_t) \;+\; \big(\mu_{t+1,\mathcal{O}_t}-\mu_t\big)^2, \tag{A16}$$

where we use the fact that $\int_v (v - \mu_{t+1,\mathcal{O}_t})\psi_t(v|\mathcal{O}_t) = 0$. Using (A13) and (A16), a similar calculation as in (A14) implies that the posterior variance conditional on a buy order satisfies:

$$
\begin{aligned}
\sigma_{t+1,\mathrm{B}}^2 \; - \; \sigma_v^2 \; + \; \left(\mu_{t+1,\mathrm{B}} - \mu_t\right)^2 \; &= \; \int_v \left(v - \mu_t\right)^2 \psi_t(v|\mathrm{B}) \\
&= \; \sigma_t^2 \int_{-\infty}^{+\infty} z^2 \, \frac{\rho \mathbf{1}_{z > a_t} + \frac{\rho}{2} \mathbf{1}_{z \in [b_t, a_t]} + \frac{1-\rho}{2}}{\frac{\rho}{2}(1 - \Phi(a_t)) + \frac{\rho}{2}(1 - \Phi(b_t)) + \frac{1-\rho}{2}} \, \phi(z) \, \mathrm{d}z \\
&= \; \sigma_t^2 \left(1 \; + \; \frac{a_t \phi(a_t) + b_t \phi(b_t)}{\Phi(-a_t) + \Phi(-b_t) + \frac{1-\rho}{\rho}} \right).
\end{aligned}
\tag{A17}
$$

Similarly, the posterior variance conditional on a sell order satisfies:

$$
\sigma_{t+1,\mathrm{S}}^2 \; - \; \sigma_v^2 \; + \; \left(\mu_{t+1,\mathrm{S}} - \mu_t\right)^2 \; = \; \sigma_t^2 \left(1 \; - \; \frac{a_t \phi(a_t) + b_t \phi(b_t)}{\Phi(a_t) + \Phi(b_t) + \frac{1-\rho}{\rho}} \right). \tag{A18}
$$

We now use the fact that a_t and b_t are the normalized ask and bid. Equation (7) implies that the ask is $A_t = \mu_{t+1,\mathrm{B}}$ and the bid is $B_t = \mu_{t+1,\mathrm{S}}$. If we normalize these equations, we have $a_t = \frac{\mu_{t+1,\mathrm{B}} - \mu_t}{\sigma_t}$ and $b_t = \frac{\mu_{t+1,\mathrm{S}} - \mu_t}{\sigma_t}$. Using (A14) and (A15), we obtain:

$$
a_t \; = \; \frac{\phi(-a_t) + \phi(-b_t)}{\Phi(-a_t) + \Phi(-b_t) + \frac{1-\rho}{\rho}}, \qquad b_t \; = \; -\frac{\phi(a_t) + \phi(b_t)}{\Phi(a_t) + \Phi(b_t) + \frac{1-\rho}{\rho}}, \tag{A19}
$$

We show that this system has a unique solution. We use the notation from the proof of Proposition 2, adapted to this particular case. For $(a, b) \in (0, \infty) \times (-\infty, 0)$, define $F(a, b) = \frac{\phi(a)+\phi(b)}{a} - \Phi(-a) - \Phi(-b) - \frac{1-\rho}{\rho}$ and $G(a, b) = \frac{\phi(a)+\phi(b)}{-b} - \Phi(a) - \Phi(b) - \frac{1-\rho}{\rho}$. As in the proof of Proposition 2, for $b \in [-\infty, 0]$ define $\alpha(b)$ as the unique solution of $F(\alpha(b), b) = 0$; and for $a \in [0, \infty]$ define $\beta(a)$ as the unique solution of $G(a, \beta(a)) = 0$. For $a \in (0, \infty)$, define $f(a) = \alpha(\beta(a))$. We show that any solution a of the equation $a - f(a) = 0$ must satisfy $a < 1$. Let $b = \beta(a)$. Since $a = \alpha(b)$, by definition $F(a, b) = 0$. As in the proof of Proposition 2, one shows that F is decreasing in a, and that $F(0, b) = +\infty > 0$. As $b < 0$, $F(1, b) = \phi(1) - \Phi(-1) + \phi(b) - \Phi(-b) - \frac{1-\rho}{\rho} < \phi(1) - \Phi(-1) + \phi(0) - \Phi(0) \approx -0.0177 < 0$. As $F(a, b) = 0$ and F is decreasing in a (and a is positive), we have just proved that $a \in (0, 1)$. A similar argument (adapted to the function G) shows that $b = \beta(a) \in (-1, 0)$.

As in equation (A9), define $f(a) = \alpha\big(\beta(a)\big)$. We need to show that the equation $a - f(a) =$

0 has a unique solution in $(0,\infty)$. By contradiction, suppose there are at least two solutions $a_1 < a_2$, and suppose a_1 is the smallest such solution and a_2 the largest. As f is continuous and takes values in some compact interval $[\underline{b},\overline{b}]$ (see the proof of Proposition 2), a_1 and a_2 are well defined. Also, since f is increasing, f is a bijection of $[a_1,a_2]$. The argument above then shows that both a_1 and a_2 are in $(0,1)$. If we prove that $f' < 1$ on $[a_1,a_2]$, it follows that $a - f(a)$ is increasing on $[a_1,a_2]$ and cannot therefore be equal to 0 at both ends. This contradiction therefore proves uniqueness, as long as we show that indeed $f' < 1$ on $(0,1)$. Let $a \in (0,1)$ and denote $b = \beta(a)$ and $a' = \alpha(b)$. Then by the chain rule $f'(a) = \alpha'(b)\beta'(a)$. Differentiating the equations $F(\alpha(b),b) = 0$ and $G(a,\beta(a) = 0$, we have $\alpha'(b) = \frac{\phi(b)(a'-b)a'}{\phi(a')+\phi(b)}$ and $\beta'(b) = \frac{\phi(a)(a-b)(-b)}{\phi(a)+\phi(b)}$. Both these derivatives are of the form $\frac{\phi(x_1)(x_1+x_2)x_1}{\phi(x_1)+\phi(x_2)}$ with $x_1,x_2 \in (0,1)$. This function is increasing in x_2, hence it is smaller than $\frac{\phi(x_1)(x_1+1)x_1}{\phi(x_1)+\phi(1)}$, which is increasing in x_1, hence smaller than 1, which is the value corresponding to $x_1 = 1$. Thus, $f' < 1$ on $(0,1)$ and the uniqueness is proved.

To find the unique solution, note that by symmetry we expect $a_t = -b_t$. If we impose this condition, we have $\Phi(a_t) + \Phi(b_t) = \Phi(-a_t) + \Phi(-b_t) = 1$. Therefore, we need to solve the equation $a_t = 2\rho\phi(a_t)$ for $a_t > 0$, or equivalently $g(a_t) = \frac{a_t}{\phi(a_t)} = 2\rho$. But equation (13) implies that $g(\delta) = 2\rho$, hence $a_t = \delta$. The solution of (A19) is then:

$$a_t \;=\; -b_t \;=\; \delta, \qquad \text{or} \qquad A_t \;=\; \mu_t + \delta\sigma_t, \quad B_t \;=\; \mu_t - \delta\sigma_t, \tag{A20}$$

which proves equation (16). Therefore, the posterior mean satisfies:

$$\mu_{t+1,\mathrm{B}} \;=\; \mu_t + \delta\sigma_t, \qquad \mu_{t+1,\mathrm{S}} \;=\; \mu_t - \delta\sigma_t, \tag{A21}$$

which proves equation (14).

Equation (A21) also implies that $(\mu_{t+1,\mathcal{O}_t} - \mu_t)^2 = \delta^2\sigma_t^2$ for $\mathcal{O}_t \in \{\mathrm{B},\mathrm{S}\}$. As $a_t = -b_t$, equations (A17) and (A18) imply that $\sigma_{t+1,\mathcal{O}_t}^2 - \sigma_v^2 + \delta^2\sigma_t^2 = \sigma_t^2$. Thus, the posterior variance satisfies:

$$\sigma_{t+1,\mathrm{B}}^2 \;=\; \sigma_{t+1,\mathrm{S}}^2 \;=\; (1-\delta^2)\sigma_t^2 + \sigma_v^2, \tag{A22}$$

which proves equation (15). $\square$

Proof of Proposition 4. From (13), it follows that $\delta < \delta_{\max} \approx 0.647$, hence $\delta < 1$. Equation (15) implies that the public variance evolves according to $\sigma_t^2 = (1-\delta^2)\sigma_{t-1}^2 + \sigma_v^2$ for any $t \geq 0$ (by convention, $\sigma_{-1} = 0$). Iterating this equation, we obtain $\sigma_t^2 = (1-\delta^2)^t\sigma_0^2 + \frac{1-(1-\delta^2)^t}{\delta^2}\sigma_v^2$. Using $\sigma_* = \frac{\sigma_v}{\delta}$, we obtain $\sigma_t^2 = \sigma_*^2 + (1-\delta^2)^t(\sigma_0^2 - \sigma_*^2)$, which proves (17). As $\delta \in (0,1)$, it is clear that σ_t^2 converges monotonically to σ_*^2 for any initial value σ_0. The bid–ask spread satisfies $s_t = 2\sigma_t\delta$, hence it converges to $2\sigma_*\delta = 2\frac{\sigma_v}{\delta}\delta = 2\sigma_v = s_*$. □

Proof of Corollary 2. Following the proof of Proposition 4, recall that g is increasing on $(0,\infty)$. Its inverse g^{-1} is therefore also increasing, and $\sigma_* = \sigma_v / g^{-1}(2\rho)$ is decreasing in ρ. The dependence on σ_v is straightforward. □

Proof of Corollary 3. Conditional on the information at t, each order (buy or sell) is equally likely. Therefore, the change in the public mean $\mu_{t+1,\mathcal{O}_t} - \mu_t$ has a binary distribution with probability 1/2, which has standard deviation equal to σ_v, which is the fundamental volatility. □

Proof of Corollary 4. Equation (15) shows that the public variance σ_t^2 evolves according to $\sigma_{t+1}^2 = (1-\delta^2)\sigma_t^2 + \sigma_v^2$. Taking the limit on both sides, we get $\sigma_*^2 = (1-\delta^2)\sigma_*^2 + \sigma_v^2$. Subtracting the two equations above, we get $\sigma_{t+1}^2 - \sigma_*^2 = (1-\delta^2)(\sigma_t^2 - \sigma_*^2)$, which proves the speed of convergence formula (22) for the public variance. As $\sigma_t^2 - \sigma_*^2 = (\sigma_t - \sigma_*)(\sigma_t + \sigma_*)$, the formula (22) is true for the public volatility as well. Finally, the bid–ask spread is $s_t = 2\delta\sigma_t$, which proves (22) for the bid–ask spread. □

Proof of Proposition 5. The proof mirrors the proof of Proposition 1. □

Proof of Proposition 6. We follow the proof of Proposition 3. Consider $t \in \frac{\mathbb{N}}{K}$. As usual, $\phi_t(\cdot)$ denotes the density of v_t just before trading at t, and $\psi_t(\cdot|\mathcal{O}_t)$ denotes the density of v_t after trading at t. By assumption, ϕ_t is normal with mean μ_t and volatility σ_t:

$$\phi_t(v) \;=\; \frac{1}{\sigma_t}\,\phi\left(\frac{v-\mu_t}{\sigma_t}\right) \;=\; \mathcal{N}(v,\mu_t,\sigma_t), \tag{A23}$$

where $\phi(\cdot) = \mathcal{N}(\cdot, 0, 1)$ be the standard normal density. From (25), we have:

$$\phi_{t+\frac{1}{K}}(\cdot|\mathcal{O}_t) \;=\; \begin{cases} \psi_t(\cdot|\mathcal{O}_t) & \text{if} \quad t+\frac{1}{K} \notin \mathbb{N}, \\ \psi_t(\cdot|\mathcal{O}_t) * \mathcal{N}(\cdot, 0, \sigma_v) & \text{if} \quad t+\frac{1}{K} \in \mathbb{N}, \end{cases} \tag{A24}$$

For future use, define the normalized ask and bid quotes by:

$$a_t \;=\; \frac{A_t - \mu_t}{\sigma_t}, \qquad b_t \;=\; \frac{B_t - \mu_t}{\sigma_t}. \tag{A25}$$

Consider the case in which the fundamental value does not change at $t+\frac{1}{K}$ (i.e., $t+\frac{1}{K} \notin \mathbb{N}$). Then, the mean and volatility of $\phi_{t+\frac{1}{K}}(\cdot|\mathcal{O}_t) = \psi_t(\cdot|\mathcal{O}_t)$ are:

$$\mu_{t+\frac{1}{K},\mathcal{O}_t} \;=\; \int_v v\, \psi_t(v|\mathcal{O}_t), \qquad \sigma^2_{t+\frac{1}{K},\mathcal{O}_t} \;=\; \int_v \big(v - \mu_{t+1,\mathcal{O}_t}\big)^2 \psi_t(v|\mathcal{O}_t). \tag{A26}$$

From (24), we have $\psi_t(v|\mathrm{B}) = \frac{\rho \mathbf{1}_{v>A_t} + \frac{\rho}{2}\mathbf{1}_{v\in[B_t,A_t]} + \frac{1-\rho}{2}}{\frac{\rho}{2}(1-\Phi_t(A_t)) + \frac{\rho}{2}(1-\Phi_t(B_t)) + \frac{1-\rho}{2}}\, \phi_t(v)$. With the change of variables $z = \frac{v-\mu_t}{\sigma_t}$, we compute the posterior mean conditional on a buy order:

$$\begin{aligned} \mu_{t+\frac{1}{K},\mathrm{B}} \;&=\; \mu_t + \int_{-\infty}^{+\infty} (v-\mu_t)\, \frac{\rho \mathbf{1}_{v>A_t} + \frac{\rho}{2}\mathbf{1}_{v\in[B_t,A_t]} + \frac{1-\rho}{2}}{\frac{\rho}{2}(1-\Phi_t(A_t)) + \frac{\rho}{2}(1-\Phi_t(B_t)) + \frac{1-\rho}{2}}\, \frac{1}{\sigma_t}\phi\big(\tfrac{v-\mu_t}{\sigma_t}\big)\, \mathrm{d}v \\ &=\; \mu_t + \sigma_t \int_{-\infty}^{+\infty} z\, \frac{\rho \mathbf{1}_{z>a_t} + \frac{\rho}{2}\mathbf{1}_{z\in[b_t,a_t]} + \frac{1-\rho}{2}}{\frac{\rho}{2}(1-\Phi(a_t)) + \frac{\rho}{2}(1-\Phi(b_t)) + \frac{1-\rho}{2}}\, \phi(z)\, \mathrm{d}z \\ &=\; \mu_t + \sigma_t\, \frac{\phi(-a_t) + \phi(-b_t)}{\Phi(-a_t) + \Phi(-b_t) + \frac{1-\rho}{\rho}}, \end{aligned} \tag{A27}$$

where $\Phi(\cdot)$ its cumulative standard normal density. Similarly, the posterior mean conditional on a sell order is:

$$\mu_{t+\frac{1}{K},\mathrm{S}} \;=\; \mu_t - \sigma_t\, \frac{\phi(a_t) + \phi(b_t)}{\Phi(a_t) + \Phi(b_t) + \frac{1-\rho}{\rho}}, \tag{A28}$$

We now impose the condition that a_t and b_t are, respectively, the normalized ask and bid quotes. As in the proof of Proposition 3, this condition implies that:

$$a_t \;=\; -b_t \;=\; \delta, \qquad \text{or, equivalently,} \quad A_t \;=\; \mu_t + \delta\sigma_t, \quad B_t \;=\; \mu_t - \delta\sigma_t, \tag{A29}$$

which proves equation (28). Therefore, the posterior mean satisfies:

$$\mu_{t+\frac{1}{K},\mathrm{B}} \;=\; \mu_t + \delta\sigma_t, \qquad \mu_{t+\frac{1}{K},\mathrm{S}} \;=\; \mu_t - \delta\sigma_t, \tag{A30}$$

which proves equation (26). To compute $\sigma^2_{t+\frac{1}{K},\mathcal{O}_t}$, we notice that:

$$\int_v \big(v-\mu_t\big)^2 \psi_t(v|\mathcal{O}_t) \;=\; \int_v \big(v-\mu_{t+\frac{1}{K},\mathcal{O}_t}\big)^2 \psi_t(v|\mathcal{O}_t) \;+\; \big(\mu_{t+\frac{1}{K},\mathcal{O}_t}-\mu_t\big)^2, \tag{A31}$$

where we use the fact that $\int_v \big(v-\mu_{t+\frac{1}{K},\mathcal{O}_t}\big)\psi_t(v|\mathcal{O}_t) = 0$. Using (A26) and (A31), a similar calculation as in (A27) implies that the posterior variance conditional on a buy order satisfies:

$$\begin{aligned}
\sigma^2_{t+\frac{1}{K},\mathrm{B}} \;+\; \big(\mu_{t+\frac{1}{K},\mathrm{B}}-\mu_t\big)^2 \;&=\; \int_v \big(v-\mu_t\big)^2 \psi_t(v|\mathrm{B}) \\
&=\; \sigma_t^2 \int_{-\infty}^{+\infty} z^2 \, \frac{\rho \mathbf{1}_{z>a_t} + \frac{\rho}{2}\mathbf{1}_{z\in[b_t,a_t]} + \frac{1-\rho}{2}}{\frac{\rho}{2}(1-\Phi(a_t)) + \frac{\rho}{2}(1-\Phi(b_t)) + \frac{1-\rho}{2}} \, \phi(z)\,\mathrm{d}z \\
&=\; \sigma_t^2 \left(1 \;+\; \frac{a_t\phi(a_t) + b_t\phi(b_t)}{\Phi(-a_t)+\Phi(-b_t)+\frac{1-\rho}{\rho}}\right).
\end{aligned} \tag{A32}$$

Similarly, the posterior variance conditional on a sell order satisfies:

$$\sigma^2_{t+\frac{1}{K},\mathrm{S}} \;+\; \big(\mu_{t+\frac{1}{K},\mathrm{S}}-\mu_t\big)^2 \;=\; \sigma_t^2 \left(1 \;-\; \frac{a_t\phi(a_t) + b_t\phi(b_t)}{\Phi(a_t)+\Phi(b_t)+\frac{1-\rho}{\rho}}\right). \tag{A33}$$

Equation (A30) also implies that for $\mathcal{O}_t \in \{\mathrm{B},\mathrm{S}\}$, we have:

$$\big(\mu_{t+\frac{1}{K},\mathcal{O}_t}-\mu_t\big)^2 \;=\; \delta^2\sigma_t^2. \tag{A34}$$

As $a_t = -b_t$, equations (A32) and (A33) imply that $\sigma^2_{t+\frac{1}{K},\mathcal{O}_t} + \delta^2\sigma_t^2 = \sigma_t^2$. Thus, the posterior variance satisfies:

$$\sigma^2_{t+\frac{1}{K},\mathrm{B}} \;=\; \sigma^2_{t+\frac{1}{K},\mathrm{S}} \;=\; (1-\delta^2)\sigma_t^2, \tag{A35}$$

which proves equation (27) for the case $t+\frac{1}{K} \notin \mathbb{N}$.

Finally, consider the case in which the fundamental value changes at $t+\frac{1}{K}$ (i.e., $t+\frac{1}{K} \in \mathbb{N}$). Then, we obtain the same results, except that $\phi_{t+\frac{1}{K}}(\cdot|\mathcal{O}_t) = \psi_t(\cdot|\mathcal{O}_t) * \mathcal{N}(\cdot,0,\sigma_v)$. As

$\mathcal{N}(\cdot, 0, \sigma_v)$ has zero mean, the analysis for the posterior mean and for the ask and bid quotes leads to the same equations as before. The only difference occurs in the posterior variance, which is higher by σ_v^2. Thus, we obtain:

$$\sigma^2_{t+\frac{1}{K},\mathrm{B}} \;=\; \sigma^2_{t+\frac{1}{K},\mathrm{S}} \;=\; (1-\delta^2)\sigma^2_{t+\frac{1}{2}} + \sigma_v^2, \tag{A36}$$

which proves equation (27) for the case $t + \frac{1}{K} \in \mathbb{N}$. □

Proof of Proposition 7. Equation (27) implies that $\sigma^2_{t+\frac{1}{K}} = (1-\delta^2)\sigma_t^2$ if $t + \frac{1}{K} \notin \mathbb{N}$. Thus, for any $t \in \mathbb{N}$ and $k \in \{1, 2, \dots, K-1\}$, we have $\sigma^2_{t+\frac{k}{K}} = \sigma_t^2(1-\delta^2)^k$, which proves the second part of (29).

If $t \in \mathbb{N}$, equation (27) at $t' = t + \frac{K-1}{K}$ implies that $\sigma^2_{t'+\frac{1}{K}} = (1-\delta^2)\sigma^2_{t'} + \sigma_v^2 = (1-\delta^2)(1-\delta^2)^{K-1}\sigma_t^2 + \sigma_v^2$. Thus, for any $t \in \mathbb{N}$, we have:

$$\sigma^2_{t+1} \;=\; \beta\sigma_t^2 + \sigma_v^2, \quad \text{with} \quad \beta \;=\; (1-\delta^2)^K. \tag{A37}$$

Using this equation from 0 to $t-1$, we get $\sigma_t^2 = \beta^t\sigma_0^2 + (\beta^{t-1} + \beta^{t-2} \cdots + 1)\sigma_v^2 = \beta^t\sigma_0^2 + \frac{1-\beta^t}{1-\beta}\sigma_v^2$, which implies $\sigma_t^2 - \frac{\sigma_v^2}{1-\beta} = \beta^t\big(\sigma_0^2 - \frac{\sigma_v^2}{1-\beta}\big)$. As $\sigma_* = \frac{\sigma_v^2}{1-\beta}$ and $\beta = (1-\delta^2)^K$, it follows that for all $t \in \mathbb{N}$:

$$\sigma_t^2 - \sigma_*^2 \;=\; (1-\delta^2)^{Kt}(\sigma_0^2 - \sigma_*^2), \tag{A38}$$

which proves the first part of (29). As $\delta \in (0,1)$, the sequence $(\sigma_t^2)_{t\in\mathbb{N}}$ converges monotonically to σ_*^2 for any initial value of σ_0. □

Proof of Corollary 5. This follows by putting together equations (29), (30), and (31). □

Proof of Proposition 8. Let $K > 1$. Using the formula (35) for $\bar{s}_K(\rho)$, we must prove that the function:

$$f(\alpha) \;=\; \frac{(1+\alpha)(1-\alpha^K)}{(1-\alpha)(1+\alpha^k)} \tag{A39}$$

is strictly increasing in $\alpha = \sqrt{1-\delta^2}$. Some algebraic manipulation shows that the derivative of f is:

$$f'(\alpha) = \frac{2\big(1-\alpha^{2K}-K\alpha^{K-1}(1-\alpha^2)\big)}{(1-\alpha)^2(1+\alpha^K)^2}$$
$$= \frac{2\alpha(1-\alpha^2)}{(1-\alpha)^2(1+\alpha^K)^2}\sum_{k=0}^{\tilde{K}}\alpha^{2k}\big(1-\alpha^{K-1-2k}\big)^2, \quad \text{with} \quad \tilde{K} = \lfloor\tfrac{K-1}{2}\rfloor. \tag{A40}$$

This shows that f is stricly increasing in α, which completes the proof. □

Proof of Proposition 9. The only difference from the setup of Section 2 is that after trading at t (but before trading at $t+1$) the dealer receives a signal $\Delta s_{t+1} = \Delta v_{t+1} + \Delta\eta_{t+1}$. By notation, just before trading at t, v_t is distributed as $\mathcal{N}(\cdot, \mu_t, \sigma_t)$. We thus follow the proof of Propositions 3 and 4, and infer that after trading at t the dealer regards v_t to be distributed as $\mathcal{N}(\cdot, \mu'_t, \sigma'_t)$, where $\mu'_t = \mu_t \pm \delta\sigma_t$ and $\sigma'^2_t = (1-\delta^2)\sigma^2_t$.[18] After observing $\Delta s_{t+1} = \Delta v_{t+1} + \Delta\eta_{t+1}$, the dealer computes $\mathsf{E}\big(\Delta v_{t+1}|\Delta s_{t+1}\big) = \frac{\sigma_v^2}{\sigma_v^2+\sigma_\eta^2}\Delta s_{t+1}$ and $\mathsf{Var}\big(\Delta v_{t+1}|\Delta s_{t+1}\big) = \frac{\sigma_v^2\sigma_\eta^2}{\sigma_v^2+\sigma_\eta^2} = \sigma^2_{v\eta}$. Hence, after observing the signal, the dealer regards v_{t+1} to be distributed as $\mathcal{N}(\cdot, \mu_{t+1}, \sigma_{t+1})$, with:

$$\mu_{t+1} = \mu'_t + \frac{\sigma_v^2}{\sigma_v^2+\sigma_\eta^2}\Delta s_{t+1}, \qquad \sigma^2_{t+1} = \sigma'^2_t + \sigma^2_{v\eta} = (1-\delta^2)\sigma_t + \sigma^2_{v\eta}. \tag{A41}$$

The recursive equation for σ_t is the same as (15), except that instead of σ_v we now have $\sigma_{v\eta}$. Then, the same proof as in Propositions 3 and 4 can be used to derive all the desired results.

Note that equation (42) implies that the change in public mean is $\Delta\mu_{t+1} = \pm\delta\sigma_t + \frac{\sigma_v^2}{\sigma_v^2+\sigma_\eta^2}\Delta s_{t+1}$. Thus, in the stationary equilibrium, $\mathsf{Var}(\Delta\mu_{t+1}) = \delta^2\sigma^2_* + \frac{\sigma_v^4}{(\sigma_v^2+\sigma_\eta^2)^2}(\sigma_v^2+\sigma_\eta^2) = \sigma^2_{v\eta} + \frac{\sigma_v^4}{\sigma_v^2+\sigma_\eta^2} = \sigma_v^2 = \mathsf{Var}(\Delta v_{t+1})$. This verifies the result in Appendix B that in any stationary filtration problem the variance of the change in public mean must equal the fundamental variance. Moreover, the half spread is equal to $\delta\sigma_* = \sigma_{v\eta}$, which does not depend on the informed share ρ. □

Proof of Corollary 6. This follows directly from equations (4) and (5) from Proposition 1, making the change of variables from equation (45). □

[18] The sign $\pm$ is plus if a buy order is submitted at t, and minus if a sell order is submitted at t.

Verification of Result 2. To compute the average bid–ask spread in the exact equilibrium, we consider each value of the informed share $\rho \in \{0.1, 0.2, \dots, 0.9\}$, and consider $M = 1000$ random sequences of orders of length up to 200. We start with the stationary public density $\phi_0 = \mathcal{N}(\cdot, 0, \sigma_*)$, and for each $m \in \{1, 2, \dots, 1000\}$ we compute the bid–ask corresponding to the final density $\phi_{t=200}$. The length parameter (200) is taken large enough so that it gives enough time for the exact equilibrium to be stationary. (We verify this is by checking that the average public density for $t > 100$ does not change.) We then compute the average bid–ask spread s_ρ, along with its standard deviation, and then construct a 95% confidence band around the mean by using the standard deviation of the bid–ask spread series. When the trading frequency is $K = 1$, we verify that for all the values of ρ considered, the theoretical value $s_* = 2\sigma_v$ is within the error band. For the case $K > 1$, we consider the particular cases $K = 2$ and $K = 3$, and we verify that the average bid–ask spread is increasing in the informed share ρ. Moreover, the equilibrium values of the average bid–ask spread from Corollary 5 are within the error band. □

Appendix B. Stationary Filtering

We show that in a filtration problem that is stationary (in a sense to be defined below) the variance of value changes is the same as the variance of the public mean changes. Let v_t be a discrete time random walk process with constant volatility σ_v. Suppose each period the market gets (public) information about v_t. Let $\mathcal{I}_t$ be the public information set available at time t. Denote by $\mu_t = \mathsf{E}(v_t | \mathcal{I}_t) = \mathsf{E}_t(v_t)$ the public mean at time t, i.e., the expected asset value given all public information. This filtration problem is called stationary if the public variance is constant over time:

$$\mathsf{Var}_t(v_t) = \mathsf{Var}_{t+1}(v_{t+1}). \tag{B1}$$

Proposition 10 gives a necessary and sufficient condition for the filtration problem to be stationary.

Proposition 10. *The filtration problem is stationary if and only if:*

$$\mathsf{Var}(v_{t+1} - v_t) \;=\; \mathsf{Var}(\mu_{t+1} - \mu_t). \tag{B2}$$

Proof. Since $\mu_t = \mathsf{E}_t(v_t)$, we have the decomposition $v_t = \mu_t + \eta_t$, where η_t is orthogonal on the information set $\mathcal{I}_t$. Moreover, $\mathsf{Var}(\eta_t) = \mathsf{Var}_t(v_t)$. Similarly, $v_{t+1} = \mu_{t+1} + \eta_{t+1}$, and $\mathsf{Var}(\eta_{t+1}) = \mathsf{Var}_{t+1}(v_{t+1})$. Thus, the stationary condition reads $\mathsf{Var}(v_{t+1} - \mu_{t+1}) = \mathsf{Var}(v_t - \mu_t)$. We can decompose $v_{t+1} - \mu_t$ in two ways:

$$\begin{aligned} v_{t+1} - \mu_t \;&=\; (v_{t+1} - \mu_{t+1}) + (\mu_{t+1} - \mu_t) \\ &=\; (v_{t+1} - v_t) + (v_t - \mu_t). \end{aligned} \tag{B3}$$

We verify that these are orthogonal decompositions. The first condition is that $\mathsf{cov}(v_{t+1} - \mu_{t+1}, \mu_{t+1} - \mu_t) = 0$, i.e., that $\mathsf{cov}(\eta_{t+1}, \mu_{t+1} - \mu_t) = 0$. But η_{t+1} is orthogonal on $\mathcal{I}_{t+1}$, which contains μ_{t+1} and μ_t. The second condition is that $\mathsf{cov}(v_{t+1} - v_t, v_t - \mu_t) = 0$. But v_t has independent increments, so $v_{t+1} - v_t$ is independent of v_t and anything contained in the information set at time t. (This is true as long as the market does not get at t information about the asset value at a future time.)

The total variance of the two orthogonal decompositions in (B3) must be the same, hence $\mathsf{Var}(v_{t+1} - \mu_{t+1}) + \mathsf{Var}(\mu_{t+1} - \mu_t) = \mathsf{Var}(v_{t+1} - v_t) + \mathsf{Var}(v_t - \mu_t)$. But being stationary is equivalent to $\mathsf{Var}(v_{t+1} - \mu_{t+1}) = \mathsf{Var}(v_t - \mu_t)$, which is then equivalent to $\mathsf{Var}(v_{t+1} - v_t) = \mathsf{Var}(\mu_{t+1} - \mu_t)$. $\square$

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