THE BELLMAN PRINCIPLE OF OPTIMALITY

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As I understand, there are two approaches to dynamic optimization: the Pontrjagin (Hamiltonian) approach, and the Bellman approach. I saw several clear discussions of the Hamiltonian approach (Barro & Sala-i-Martin, Blanchard & Fischer, D. Romer), but to my surprise, I didn't see any clear treatment of the Bellman principle. What I saw so far (Duffie, Chapters 3 and 9, and the Appendix of Mas-Colell, Whinston & Green) is confusing to me. I guess I should take a look at some dynamic optimization textbook, but I'm too lazy for that. Instead, I'm going to try to figure it out on my own, hoping that my freshness on the subject can be put to use.

The first four sections are only about local conditions for having a (finite) optimum. In the last section I will discuss global conditions for optima in Bellman's framework, and give an example where I solve the problem completely. As a bonus, in Section 5, I use the Bellman method to derive the Euler–Lagrange equation of variational calculus.

1. Discrete time, certainty

We start in discrete time, and we assume perfect foresight (so no expectation will be involved). The general problem we want to solve is:

$$\begin{cases} \max\limits_{(c_t)} \sum\limits_{t=0}^{\infty} f(t, k_t, c_t) \\ \text{s.t. } k_{t+1} = g(t, k_t, c_t) \ . \end{cases} \tag{1}$$

In addition, we impose a budget constraint, which for many examples is the restriction that k_t be eventually positive (i.e. $\liminf_t k_t \geq 0$). This budget constraint excludes explosive solutions for c_t, so that we can apply the Bellman method. I won't mention the budget constraint until the last section, but we should keep in mind that without it (or some constraint like it), we might have no solution.

The usual names for the variables involved is: c_t is the *control* variable (because it is under the control of the choice maker), and k_t is the *state* variable (because it describes the state of the system at the beginning of t, when the agent makes the decision). In this paper, I call the equation $k_{t+1} = g(t, k_t, c_t)$ the "state equation", I don't know how it is called in the literature.

To get some intuition about the problem, think of k_t as capital available for production at time t, and of c_t as consumption at t. At time 0, for a starting level of capital k_0, the consumer chooses the level of consumption c_0. This determines the level of capital available for the next period, $k_1 = g(0, k_0, c_0)$. So at time 1, the consumer decides on the level of c_1, which together with k_1 determines k_2, and the cycle is repeated on and on. The infinite sum $\sum_{t=0}^{\infty} f(t, k_t, c_t)$ is to be thought of as the total "utility" of the consumer, which the latter is supposed to maximize at time 0.

Bellman's idea for solving (1) is to define a value function V at each $t = 0, 1, 2, \ldots$

$$V(t, k_t) \ = \ \max_{(c_s)} \sum_{s=t}^{\infty} f(s, k_s, c_s) \quad \text{s.t. } k_{s+1} \ = \ g(s, k_s, c_s) \ ,$$

Date: April 8, 2002, rewritten on August 14, 2024.

which represents the consumer's maximum "utility" given the initial level of k_t. Then we have the following obvious result.

Theorem 1.1. *(Bellman's principle of optimality)*
For each $t = 0, 1, 2, \ldots$, the value function satisfies:

$$V(t, k_t) \;=\; \max_{c_t} \Big[f(t, k_t, c_t) + V\big(t+1, g(t, k_t, c_t)\big) \Big] \ . \tag{2}$$

This in principle reduces an infinite-period optimization problem to a two-period optimization problem. But is this the whole story? How do we actually solve the optimization problem (1)? Here is where the textbooks I mentioned above are less clear[1]. Well, let's try to squeeze all we can out of Bellman's equation (2).

We denote partial derivatives by using subscripts. A star superscript denotes the optimum. Then the first order condition from (2) reads:

$$f_c(t, k_t, c_t^*) + V_k\big(t+1, g(t, k_t, c_t^*)\big) \cdot g_c(t, k_t, c_t^*) \;=\; 0 \ . \tag{3}$$

Looking at this formula, it is clear that we would like to be able to compute the derivative $V_k(t+1, k_{t+1})$. We can try to do that using again the formula (2). Since we are differentiating a maximum operator, we apply the envelope theorem[2] and obtain:

$$V_k(t, k_t) \;=\; f_k(t, k_t, c_t^*) + V_k\big(t+1, g(t, k_t, c_t^*)\big) \cdot g_k(t, k_t, c_t^*) \ . \tag{4}$$

From (3), we can calculate $V_k\big(t+1, g(t, k_t, c_t^*)\big)$, and substituting it in (4), we get:

$$V_k(t, k_t) \;=\; \left(f_k - \frac{f_c}{g_c} \cdot g_k \right) (t, k_t, c_t^*) \ . \tag{5}$$

Finally, substitute this formula into (3) and obtain a condition which does not depend on the value function anymore:

$$f_c(t, k_t, c_t^*) + g_c(t, k_t, c_t^*) \cdot \left(f_k - \frac{f_c}{g_c} \cdot g_k \right) \big(t+1, g(k_t, c_t^*), c_{t+1}^*\big) \;=\; 0 \ . \tag{6}$$

Notice that this formula is true for any k_t, not necessarily only for the optimal one up to that point. But in that case, c_t^* and c_{t+1}^* are the optimal choices *given* k_t. In any case, from now on we are only going to work at the optimum (t, k_t^*, c_t^*). The previous formula can be written as follows:

$$f_k(t+1) - \frac{f_c(t+1)}{g_c(t+1)} \cdot g_k(t+1) \;=\; -\frac{f_c(t)}{g_c(t)} \ . \tag{7}$$

This is the key equation that allows us to compute the optimum c_t^*, using only the initial data (f_t and g_t). I guess equation (7) should be called the Bellman equation, although in particular cases it goes by the Euler equation (see the next Example). I am going to compromise and call it the Bellman–Euler equation.

For the purposes of comparison with the continuous-time version, we construct the discrete analog of dk_t, i.e., $h(t) = k_{t+1} - k_t$. As the state equation is $k_{t+1} = g(t, k_t, c_t)$, we define the function $h(t, k, c) = g(t, k, c) - k$. Differentiating with respect to k, we obtain $g_k(t) =$

[1]MWG only discusses the existence and uniqueness of a value function, while Duffie treats only the Example mentioned above, and leaves a crucial Lemma as an exercise at the end of the chapter.

[2]To state the envelope theorem, start with a function of two variables $f(x, \theta)$, such that for every x, the maximum $\max_\theta f(x, \theta)$ is achieved at a point $\theta = \theta^*(x)$ in the interior of the θ-interval. Then

$$\frac{d}{dx} \max_\theta f(x, \theta) = \frac{\partial f}{\partial x}(x, \theta^*(x)) \ .$$

$1 + h_k(t)$, where for simplicity we omit the dependence on the variables k and c. Denote $\Delta_t \phi = \phi(t+1) - \phi(t)$. Then we rewrite the Bellman–Euler equation (7) as:

$$\Delta_t \left(\frac{f_c(t)}{h_c(t)} \right) \;=\; f_k(t+1) - \frac{f_c(t+1)}{h_c(t+1)} \cdot h_k(t+1) \;\; . \tag{8}$$

Example 1.2. In a typical dynamic optimization problem, the consumer has to maximize intertemporal utility, for which the instantaneous "felicity" is $u(c)$, with u a von Neumann–Morgenstern utility function. Thus, $f(t, k_t, c_t) = \beta^t u(c_t)$, with $\beta \in (0,1)$ the discount constant. The state equation $k_{t+1} = g(t, k_t, c_t)$ is typically:

$$e_t + \phi(k_t) = c_t + (k_{t+1} - k_t) \;\; , \tag{9}$$

where e_t is the endowment (e.g. labor income), and $\phi(k_t)$ is the production function (technology). As an example of the technology function, we have $\phi(k_t) = rk_t$. The derivative $\phi'(k_t) = r$ is then, as expected, the interest rate on capital. Notice that with the above description we have:

$$k_{t+1} \;=\; g(t, k_t, c_t) = k_t + \phi(k_t) + e_t - c_t \;\; .$$

So we get the following formulas: $\frac{\partial f_t}{\partial c_t} = u'(c_t)$, $\frac{\partial g_t}{\partial c_t} = -1$, $\frac{\partial f_t}{\partial k_t} = 0$, $\frac{\partial g_t}{\partial k_t} = 1 + r$. The Bellman–Euler equation (7) becomes:

$$u'(c_t) \;=\; \beta\big(1 + \phi'(k_t)\big) u'(c_{t+1}) \;\; ,$$

which is the usual Euler equation.

2. Discrete time, uncertainty

Now we assume everything is stochastic, and the agent solves the problem:

$$\begin{cases} \max\limits_{(c_t)} \mathbf{E}_0 \sum\limits_{t=0}^{\infty} f(t, k_t, c_t) \\ \text{s.t. } \; k_{t+1} = g(t, k_t, c_t) \;\; . \end{cases} \tag{10}$$

As usual, we denote by $\mathbf{E}_t$ the expectation given information available at time t. Then we can define the value function:

$$V(t, k_t) \;=\; \max_{(c_s)} \mathbf{E}_t \sum_{s=t}^{\infty} f(s, k_s, c_s) \qquad \text{s.t. } \; k_{s+1} = g(s, k_s, c_s) \;\; .$$

The Bellman principle of optimality (2) becomes:

$$V(t, k_t) \;=\; \max_{c_t} \Big[f(t, k_t, c_t) + \mathbf{E}_t V\big(t+1, g(t, k_t, c_t)\big) \Big] \;\; . \tag{11}$$

Now in order to derive the Euler equation with uncertainty, all we have to do is replace $V(t+1)$ in the formulas of the previous section by $\mathbf{E}_t V(t+1)$ (using, of course, the fact that differentiation commutes with expectation). We arrive at the following Bellman–Euler equation:

$$\mathbf{E}_t \left(f_k(t+1) - \frac{f_c(t+1)}{g_c(t+1)} \cdot g_k(t+1) \right) \;=\; -\frac{f_c(t)}{g_c(t)} \;\; . \tag{12}$$

For our particular Example 1.2, we get:

$$u'(c_t) \;=\; \beta(1+r) \mathbf{E}_t u'(c_{t+1}) \;\; ,$$

3. Continuous time, certainty

This is a bit trickier, but we can use the same derivation as in discrete time. The difference is that instead of the interval $[t, t+1]$ we now consider the infinitesimal interval $[t, t+dt]$.

The problem solved by the decision maker is:

$$\left\{ \begin{array}{l} \max\limits_{c_t} \displaystyle\int_0^\infty f(t, k_t, c_t) \\ \text{s.t. } \frac{dk_t}{dt} = h(t, k_t, c_t) \end{array} \right. . \tag{13}$$

The constraint can be rewritten in differential notation:

$$k_{t+dt} \;=\; k_t + h(t, k_t, c_t)dt \;\;, \tag{14}$$

thus we have a similar problem to (1) and we can solve it by an analogous method. Define the value function:

$$V(t, k_t) \;=\; \max_{(c_s)} \int_t^\infty f(s, k_s, c_s)\, ds \qquad \text{s.t. } \frac{dk_s}{ds} = h(s, k_s, c_s) \;\; .$$

The Bellman principle of optimality states that:

$$V(t, k_t) \;=\; \max_{c_t} \left[\int_t^{t+dt} f(s, k_s, c_s)\, ds + V\big(t + dt, k_t + h(t, k_t, c_t)dt\big) \right] \;\; . \tag{15}$$

As $\int_t^{t+dt} f(s, k_s, c_s)\, ds = f(t, k_t, c_t)dt$, the first order condition for a maximum is:

$$f_c(t)dt + V_k(t + dt, k_{t+dt}) \cdot h_c(t)\, dt \;=\; 0 \;\; . \tag{16}$$

This is equivalent to:

$$V_k(t + dt) \;=\; -\frac{f_c(t)}{h_c(t)} \;\; . \tag{17}$$

As in Section 1, we apply the envelope theorem to derive:

$$V_k(t) \;=\; f_k(t)dt + V_k(t + dt) \cdot \big(1 + h_k(t)\, dt\big) \;\; . \tag{18}$$

We substitute (17) into (18) to obtain:

$$V_k(t) \;=\; -\frac{f_c(t)}{h_c(t)} + \left(f_k(t) - \frac{f_c(t)}{h_c(t)} \cdot h_k(t) \right) dt \;\; .$$

If ϕ is a differentiable function, then $\phi(t + dt)dt = \phi(t)dt$. Thus, we obtain:

$$V_k(t + dt) \;=\; -\frac{f_c(t + dt)}{h_c(t + dt)} + \left(f_k(t) - \frac{f_c(t)}{h_c(t)} \cdot h_k(t) \right) dt \;\; . \tag{19}$$

Combining equations (17) and (19), we get:

$$\frac{f_c(t + dt)}{h_c(t + dt)} - \frac{f_c(t)}{h_c(t)} \;=\; \left(f_k(t) - \frac{f_c(t)}{h_c(t)} \cdot h_k(t) \right) dt \;\; .$$

Using the formula $\phi(t + dt) - \phi(t) = \frac{d\phi}{dt}\, dt$, we rewrite the above formula as:

$$\frac{d}{dt}\left(\frac{f_c(t)}{h_c(t)} \right) \;=\; f_k(t) - \frac{f_c(t)}{h_c(t)} \cdot h_k(t) \tag{20}$$

This is the Bellman–Euler equation in continuous-time. We note that it is quite similar to our equation (8) in discrete time.

To obtain more explicit formulas in continuous time, we need to be more careful. Note that, just as in Section 1, I have omitted the dependence of the function $f_c(t)$ of the arguments k

and c, and what I really mean is $f_c(t, k_t, c_t)$, with c_t calculated at the optimum. For simplicity, however, I will omit the star superscript for c_t. Then we calculate:

$$\frac{d}{dt}(f_c) = f_{tc} + f_{kc} \cdot h + f_{cc} \cdot \frac{dc}{dt} \ .$$

Thus, we rewrite the Bellman–Euler equation (20) as follows:

$$- \left(f_{tc} + f_{kc} \cdot h + f_{cc} \cdot \frac{dc}{dt} \right) \ = \ - \frac{f_c}{h_c} \left(h_{tc} + h_{kc} \cdot h + h_{cc} \cdot \frac{dc}{dt} - h_c \cdot h_k \right) - h_c \cdot f_k \ . \tag{21}$$

In general, in order to solve this, we rewrite (21) as $dc_t/dt = \lambda(t, k_t, c_t)$, so the optimum is given by the following system of ODEs:

$$\begin{cases} dc_t/dt \ = \ \lambda(t, k_t, c_t) \\ dk_t/dt \ = \ h(t, k_t, c_t) \ . \end{cases} \tag{22}$$

Example 3.1. Applying the previous analysis to the setup in Example 1.2, we have $f(t, k_t, c_t) = e^{-\rho t} u(c_t)$ and $dk_t/dt = h(t, k_t, c_t) = e_t + \phi(k_t) - c_t$. The Euler equation (20) becomes:

$$\frac{d}{dt}\Big(-e^{-\rho t} u'(c_t) \Big) \ = \ e^{-\rho t} u'(c_t) \phi'(k_t) \ ,$$

or equivalently:

$$\left[-\frac{u''(c_t)c_t}{u'(c_t)} \right] \cdot \frac{dc_t/dt}{c_t} \ = \ \phi'(k_t) - \rho \ . \tag{23}$$

Notice that we get the same equation as (7') from Blanchard and Fischer (Chapter 2, p. 40), so we are on the right track.

4. Continuous time, uncertainty

In the first part of this section, I assume that the uncertainty comes from the function h, e.g., if h depends on an uncertain endowment e_t. In the second part of this section, I will also consider a stochastic constraint.

When the uncertainty comes from the function h, the agent solves the problem:

$$\begin{cases} \max_{c_t} \mathbf{E}_0 \int_0^\infty f(t, k_t, c_t) \\ \text{s.t. } \frac{dk_t}{dt} = h(t, k_t, c_t) \ . \end{cases} \tag{24}$$

The value function takes the form:

$$V(t, k_t) \ = \ \max_{(c_s)} \int_t^\infty f(s, k_s, c_s)\, ds \quad \text{s.t. } \frac{dk_s}{ds} \ = \ h(s, k_s, c_s) \ ,$$

and the Bellman principle of optimality (15) becomes:

$$V(t, k_t) \ = \ \max_{c_t} \left[\int_t^{t+dt} f(s, k_s, c_s)\, ds + \mathbf{E}_t V\big(t + dt, k_t + h(t, k_t, c_t)dt\big) \right] \ . \tag{25}$$

As in Section 3, we obtain the following Bellman–Euler equation:

$$\mathbf{E}_t \frac{d}{dt}\left(\frac{f_c}{h_c} \right) \ = \ f_k - \frac{f_c}{h_c} \cdot h_k \ . \tag{26}$$

For the setup in Example 1.2, we get:

$$\left[-\frac{u''(c_t)c_t}{u'(c_t)} \right] \cdot \mathbf{E}_t \frac{dc_t/dt}{c_t} \ = \ \phi'(k_t) - \rho \ . \tag{27}$$

Now I assume everything is stochastic, and the agent solves the problem:

$$(28) \qquad \begin{cases} \max_{c_t} \mathbf{E}_0 \int_0^\infty f(t, k_t, c_t) \\ \text{s.t. } dk_t = \alpha(t, k_t, c_t)dt + \beta(t, k_t, c_t)dW_t \ , \end{cases}$$

where W_t is a one-dimensional Wiener process (Brownian motion), and α, β are deterministic funtions. If we define the value function $V(t, k_t)$ as above, the Bellman principle of optimality implies:

$$(29) \qquad V(t, k_t) \ = \ \max_{c_t} \Big[f(t, k_t, c_t)\, dt + \mathbf{E}_t V(t + dt, k_{t+dt}) \Big] \ .$$

Now $V(t+dt, k_{t+dt}) = V(t+dt, k_t + \alpha dt + \beta dW_t) = V(t, k_t) + V_t dt + V_k(\alpha dt + \beta dW_t) + \frac{1}{2} V_{kk} \beta^2 dt$, where the last equality comes from Itô's lemma. Taking expectation at t, it follows that:

$$(30) \qquad \mathbf{E}_t V(t + dt, k_{t+dt}) \ = \ V(t, k_t) + V_t\, dt + V_k \alpha\, dt + \tfrac{1}{2} V_{kk} \beta^2\, dt \ .$$

The Bellman principle (29) can be written equivalently as:

$$\sup_{c_t} \Big(-V(t, k_t) + f(t, k_t, c_t)dt + \mathbf{E}_t V(t + dt, k_{t+dt}) \Big) \ \le \ 0 \ ,$$

with equality at the optimum c_t. Combining this equation with equation (30), we obtain:

$$(31) \qquad \sup_a \mathcal{D}^a V(t, y) + f(t, y, a) \ = \ 0 \ ,$$

where $\mathcal{D}^a$ is the partial differential operator defined by:

$$(32) \qquad \mathcal{D}^a V(t, y) \ = \ V_t(t, y) + V_y(t, y)\alpha(t, y, a) + \tfrac{1}{2} V_{yy} \beta(t, y, a)^2 \ .$$

Equation (31) is also known as the Hamilton–Jacobi–Bellman equation. (We got the same equation as Duffie, chapter 9A, so we're fine.) This is not quite a PDE yet, because we have a supremum operator before it. However, the first order condition for (31) does give a PDE. The boundary condition comes from some transversality condition that we have to impose on V at infinity (see Duffie).

Notice that for the stochastic case we took a different route than before. This is because now we cannot eliminate the value function anymore (the reason is that we get an extra term in the first order condition coming from the dW_t-term, and that term depends on V as well). So this approach first looks at a value function which satisfies the Hamilton–Jacobi–Bellman equation, and then derives the optimal consumption c_t and capital k_t.

5. The Euler–Lagrange equation

The reason I am treating this variational problem here is that the Bellman method seems very well suited to solve it. The classical Euler–Lagrange equation is:

$$(33) \qquad \frac{d}{dt} \left(\frac{\partial F}{\partial \dot{c}} \right) = \frac{\partial F}{\partial c}$$

and it solves the following problem:

$$(34) \qquad \begin{cases} \max_{c(t)} \int_a^b F(t, c, \dot{c})dt \\ \text{s.t. } c(a) = P \text{ and } c(b) = Q \ . \end{cases}$$

Suppose we know the optimum curve c only up to some point $x = c(t)$. Then we define the value function:

$$V(t, x) \ = \ \max_{c(s)} \int_t^b F(s, c, \dot{c})ds \quad \text{s.t. } c(t) = x \text{ and } c(b) = Q \ .$$

In the discussion that follows, we denote by λ a direction in $\mathbb{R}^n$ (the curve c also has values in $\mathbb{R}^n$). The Bellman principle of optimality implies:

$$V(t,x) \;=\; \max_{\lambda} \int_t^{t+dt} F(t,x,\lambda)dt + V_{t+dt}(x+\lambda dt) \;, \tag{35}$$

where the last term comes from the identity $c(t+dt) = c(t) + \dot{c}(t)dt = x + \lambda dt$. The first order condition for this maximum (after dividing through by dt) is:

$$\frac{dV_{t+dt}}{dc} \;=\; -F_{\dot{c}} \;.$$

The envelope theorem applied to equation (35) implies:

$$\frac{dV_t}{dc} \;=\; F_c dt + \frac{dV_{t+dt}}{dc} \;=\; F_c dt - F_{\dot{c}} \;.$$

If we replace t by $t+dt$ in the above equation, we get:

$$\frac{dV_{t+dt}}{dc} \;=\; F_c dt - F_{\dot{c}}(t+dt) \;.$$

Combining the two previous formulas for dV_{t+dt}/dt, we obtain:

$$F_{\dot{c}}(t+dt) - F_{\dot{c}}(t) \;=\; F_c dt \;.$$

This implies:

$$\left(\frac{d}{dt}F_{\dot{c}}\right) dt \;=\; F_c dt \;,$$

which, after dividing by dt, is the desired Euler–Lagrange equation (34).

6. Conditions for a global optimum

There are two main issues in the existence of the solution to the Bellman equation. One is whether the value function is finite or not. This can be resolved by the budget constraint, as we see below. The other issue is whether or not the maximum in the Bellman principle of optimality is attained at an interior point (in order to get the first order condition to hold). This last issue should be resolved once we analyze the solution, and check that we have indeed an interior maximum. Of course, after we know that the Bellman–Euler equation holds, we need to do a little bit of extra work to see which of the possible solutions fits the initial data of the problem.

Since I do not plan to develop the general theory for this (one can look it up in a texbook), I will just analyze our initial Example 1.2. Recall that we are solving the problem:

$$\begin{cases} \max\limits_{(c_t)} \sum\limits_{t=0}^{\infty} \beta^t u(c_t) \\ \text{s.t. } k_{t+1} = (1+r)k_t + e_t - c_t \;. \end{cases}$$

I assume that the endowment is always positive and has a finite present value $\sum_t \beta^t e_t < \infty$. I impose the requirement that the capital k_t be eventually positive, i.e. that $\liminf_t k_t \geq 0$. For simplicity, I assume also that $\beta(1+r) = 1$.

We start with an optimum sequence of consumption (c_t). Such a sequence clearly exists, since we are looking for a (c_t) that achieves the maximum utility at zero (finite or not). Note that we do not assume anything particular about it; for example we do not assume that the value function corresponding to it is finite, or that it attains an interior optimum. Now we start to use our assumptions and show that (c_t) does satisfy all those properties.

Denote by $PV_t(c) = \sum_{s \geq t} \beta^{s-t} c_s$, the present value of consumption at time t, and similarly for k and e. Using the state equation, we deduce that for t sufficiently large (so that k_{t+1} becomes positive, hence bigger than -1),

$$c_t \ \leq \ (1+r)k_t + e_t + 1 \ .$$

By writing this inequality for all $s > t$, multiplying through by β^{s-t}, and adding up the previous inequalities for $s \geq t$, we get:

$$PV_t(c) \ \leq \ (1+r)PV_t(k) + PV_t(e) + \frac{1}{1-\beta} \ . \tag{36}$$

By assumption, $PV_t(e) < \infty$. We show that $PV_t(c) < \infty$. If $PV_t(k) < \infty$, we are done. If $PV_t(k) = \infty$, then we can consume out of this capital until $PV_t(k)$ becomes finite. Certainly, by doing this, $PV_t(c)$ will increase, yet it will still satisfy (36). All the terms on the right hand side of (36) are finite, hence so is $PV_t(c)$, at the new level of consumption. That means that our original $PV_t(c)$ must be finite. Moreover, we show by contradiction that $PV_t(k)$ is finite. Suppose it is not. Then, we proceed as above and increase $PV_t(c)$, while still keeping it finite. But this is in contradiction with the fact that (c_t) was chosen to be an optimum. (I have to admit that this part is tricky, but we cannot avoid it if we want to be rigorous!) Since u is monotone and concave, $PV_t\big(u(c)\big)$ is also finite. That means that the Bellman value function $V(t, k_t)$ is finite, which resolves our first concern.

We now consider the Bellman principle of optimality:

$$V(t, k_t) \ = \ \max_{c_t} \Big[\beta^t u(c_t) + V\big(t+1, (1+r)k_t + e_t - c_t)\big) \Big] \ .$$

Here is a subtle idea: If we save one small quantity ϵ today, tomorrow it becomes $(1+r)\epsilon$, and we can either consume it, or leave it for later. This latter decision has to be made in an optimum fashion, as this is the definition of V_{t+1}, so we are indifferent between consuming $(1+r)\epsilon$ tomorrow and saving it for later on. Thus, we might as well assume that we are consuming it tomorrow, so when we calculate V_{t+1}, only tomorrow's utility is affected. Therefore, for marginal purposes we can regard V_{t+1} as equal to $\beta^{t+1} u(c_{t+1})$. Then, to analyze the Bellman optimum locally is the same as analyzing

$$\max_{\epsilon} \Big[\beta^t u(c_t - \epsilon) + \beta^{t+1} u\big(c_{t+1} + (1+r)\epsilon\big) \Big]$$

locally around the optimum $\epsilon = 0$. Since $\beta(1+r) = 1$, the Bellman–Euler equation is:

$$u'(c_t) \ = \ u'(c_{t+1}) \quad \text{or equivalently} \quad c_t \ = \ c_{t+1} \ = \ c \ .$$

As u is concave, it is easy to see that $\epsilon = 0$ is a local maximum. This resolves our second concern.

We notice that consumption should be smoothed completely, so that $c_t = c$ for all t. Recall the state equation is $e_t - c_t = k_{t+1} - (1+r)k_t$. Multiplying this by $\beta^t = 1/(1+r)^t$ and summing over all t, we get:

$$\sum_t \beta^t (e_t - c_t) \ = \ \sum_t \frac{1}{(1+r)^t} \big(k_{t+1} - k_t\big) \ = \ \lim_{t \to \infty} \beta^t k_{t+1} - (1+r)k_0 \ .$$

During the discussion of the finiteness of the value function, we noticed that $PV_t(k) < \infty$. In particular, this implies that $\lim_t \beta^t k_t = 0$. So we get the formula $\sum_t \beta^t e_t - \frac{1}{1-\beta} c = -(1+r)k_0$, which implies that:

$$c \ = \ r\beta \sum_{t=0}^{\infty} \beta^t e_t + r k_0 \ . \tag{37}$$

AS there is only one optimum, it must be a global optimum, which ends the solution of our example.