Internet Appendix for "Dynamic Adverse Selection and Liquidity"

Contents

1 General Dealer Models

We consider a setup that generalizes that in Section 2 in the paper. There is a single risky asset, and time is discrete and infinite. The asset value (or fundamental value) v_t follows a general Markov process, with transition density[1]

$$f_t(w,v) \;=\; \mathsf{P}(v_{t+1} = w \mid v_t = v), \quad t \;=\; 0,1,2,\ldots$$

In the paper, v_t follows a random walk: $v_{t+1} = v_t + \varepsilon_{t+1}$, and hence the transition density is $f_t(w,v) = \mathcal{N}(w - v, 0, \sigma_v)$.

The dealer's uncertainty about the fundamental value is summarized by the *public density*, which is the density of v_t just before trading at t, conditional on all the information available at t, that is, the sequence of actions observed by the dealer at $0,1,\ldots,t-1$. Denote by $\phi_t(\cdot)$ the public density, and let μ_t be its mean (called the *public mean*) and σ_t its standard deviation (called the *public volatility*).

At each $t = 0,1,2,\ldots$ the dealer observes an action a_t that belongs to a general action space $\mathcal{A}$.[2] We assume that the action a_t depends on the asset value v_t and on some random shock η_t, and thus it is a mapping that takes (v_t, η_t) to an element of $\mathcal{A}$. Let $g_t(a_t, v_t, \eta_t)$ be the density of a_t conditional on v_t and a random shock η_t, where we omit the dependence of a_t on ϕ_t (or its moments).[3]

[1] As in the paper, we assume that all densities are rapidly decaying at infinity.

[2] In the paper, the actions are either a buy or a sell order, that is, $\mathcal{A} = \{\mathrm{B}, \mathrm{S}\}$. In models of limit order markets, the actions are elements of the set of possible orders, $\mathcal{A} = \{\mathrm{BMO}, \mathrm{BLO}, \mathrm{SLO}, \mathrm{SMO}, \mathrm{NO}\}$.

[3] Note that the action a_t typically depends also on the dealer's prices, which in turn depend on the public density ϕ_t. For instance, in the paper an informed trader who observes v_t chooses a buy order if v_t is above the ask A_t, but the ask itself depends on the public density ϕ_t.

Denote by $\psi_t(v|a) = \mathsf{P}(v_t = v \mid a_t = a)$ the posterior density of v_t after observing a_t. Using Bayes' rule, we compute:

$$\begin{aligned}
\psi_t(v|a) &= \frac{\mathsf{P}(v_t = v, a_t = a)}{\mathsf{P}(a_t = a)} = \frac{\mathsf{P}(a_t = a \mid v_t = v) \cdot \mathsf{P}(v_t = v)}{\int_v \mathsf{P}(a_t = a \mid v_t = v) \cdot \mathsf{P}(v_t = v)} \\
&= \frac{\int_\eta \mathsf{P}(a_t = a \mid v_t = v, \eta_t = \eta) \cdot \mathsf{P}(v_t = v)}{\int_{v,\eta} \mathsf{P}(a_t = a \mid v_t = v, \eta_t = \eta) \cdot \mathsf{P}(v_t = v)} = \frac{\int_\eta g_t(a, v, \eta) \cdot \phi_t(v)}{\int_{v,\eta} g_t(a, v, \eta) \cdot \phi_t(v)}.
\end{aligned} \tag{IA.1}$$

We want to compute the posterior public density $\phi_{t+1}(\cdot)$, which is the density of v_{t+1} just before trading at $t+1$. Suppose the action $a_t = a$ was observed at t. Then the posterior public density at $t+1$ is $\phi_{t+1}(w|a) = \mathsf{P}(v_{t+1} = w \mid a_t = a)$, which is equal to:

$$\begin{aligned}
\phi_{t+1}(w|a) &= \int_v \mathsf{P}(v_{t+1} = w \mid v_t = v, a_t = a) \cdot \mathsf{P}(v_t = v \mid a_t = a) \\
&= \int_v \mathsf{P}(v_{t+1} = w \mid v_t = v) \cdot \mathsf{P}(v_t = v \mid a_t = a) = \int_v f_t(w, v) \cdot \psi_t(v, a) \\
&= \frac{\int_{v,\eta} f_t(w, v) \cdot g_t(a, v, \eta) \cdot \phi_t(v)}{\int_{v,\eta} g_t(a, v, \eta) \cdot \phi_t(v)}.
\end{aligned} \tag{IA.2}$$

Note that in the second equation in (IA.2) we are implicitly assuming that the evolution of v_t is independent on the action a_t undertaken at t.

Thus, we can consider the whole evolution of the game as a Markov chain with state variable (v_t, ϕ_t). Note that the transition density $f_t(w, v)$, the action density $g_t(a_t, v_t, \eta_t)$, and the update equation (IA.2) together describe the evolution of this Markov chain.

A natural concept associated to a Markov chain is the *stationary density*, which describes how likely it is for each state to eventually occur in the game. This is a well defined concept in most cases, except when the fundamental value is not stationary, e.g., if v_t follows a random walk. In that case, nevertheless, one can show that it is still possible to define a stationary density as long as one does not require it to integrate to 1 over v. But when the fundamental value follows a stationary process, the construction of the stationary density is relatively straightforward. In Section 2 of this Internet Appendix we consider a stationary setup in which the fundamental value switches between two values according to a Poisson process. In that case, we show that the construction of the stationary density reduces to a functional equation, and we show numerically how to solve that equation.

2 Dealer Model with Switching Value

2.1 Environment

The setup is as in Section 2 in the paper, except that the fundamental value is no longer following a random walk, but switches randomly between 0 and 1 with a fixed probability $\nu \in (0, \frac{1}{2})$.[4] At each $t = 0, 1, \ldots$ a trader is selected at random from a population with a fraction ρ of informed traders and a fraction $1-\rho$ of uninformed traders. The trader can buy or sell at most one unit of the asset, and trading takes place at the quotes set by a competitive risk-neutral dealer: the ask A_t and the bid B_t. If the trader at t is uninformed, then he is equally likely to buy or to sell. If the trader at t is informed and observes the value v_t, then she submits either (i) buy order if $v_t > A_t$, (ii) sell order if $v_t < B_t$.[5]

2.2 Equilibrium

The advantage of this setup is that densities on $\{0, 1\}$ are binomial, and can therefore be described by a single number, $\mu \in [0, 1]$, which is the probability that the value is 1. Thus, if we use our previous notation, the public density ϕ_t can be described simply by its mean μ_t. Proposition IA.1 shows how μ_t evolves over time, and how the ask and bid are determined.

Proposition IA.1. *Suppose the public density before trading at t has mean $\mu_t \in (0, 1)$. Then at t the ask is $A_t = \mu_{t+1,\mathrm{B}}$ and the bid is $B_t = \mu_{t+1,\mathrm{S}}$, where $\mu_{t+1,\mathcal{O}_t}$ is the mean of the public*

[4]We avoid the case $\nu \in (\frac{1}{2}, 1)$ as less realistic, as it would mean that in each period the value has more than 50% probability of shifting from 0 to 1 or vice versa.

[5]Here v_t cannot lie in between the bid and the ask: the value is either 0 or 1, while the ask and bid must lie between these two extreme values.

density at $t+1$ after an order $\mathcal{O}_t \in \{\mathrm{B}, \mathrm{S}\}$, and these numbers satisfy:

$$\begin{aligned} A_t &= \mu_{t+1,\mathrm{B}} = \frac{\nu\frac{1-\rho}{2}(1-\mu_t) + (1-\nu)\frac{1+\rho}{2}\mu_t}{\frac{1-\rho}{2} + \rho\mu_t}, \\ B_t &= \mu_{t+1,\mathrm{S}} = \frac{\nu\frac{1+\rho}{2}(1-\mu_t) + (1-\nu)\frac{1-\rho}{2}\mu_t}{\frac{1+\rho}{2} - \rho\mu_t}. \end{aligned} \tag{IA.3}$$

The bid-ask spread $s_t = A_t - B_t$ satisfies:

$$s_t = \rho(1-2\nu)\,\frac{\frac{1}{4} - \left(\frac{1}{2} - \mu_t\right)^2}{\frac{1}{4} - \rho^2\left(\frac{1}{2} - \mu_t\right)^2}. \tag{IA.4}$$

Proof. See Section 2.3 of this Internet Appendix. □

Note that as a function of μ_t the bid-ask spread s_t is symmetric around $\mu_t = \frac{1}{2}$ and has an inverted U-shape. It attains its maximum value $\rho(1-2\nu)$ when $\mu_t = \frac{1}{2}$, and it approaches 0 when μ_t is either 0 or 1. Also, the bid-ask spread is increasing in ρ for a fixed value of μ_t. But this does not mean that the *average* bid-ask spread is also increasing in ρ. In order to compute the average bid-ask spread, one must let μ_t evolve over time. When ρ is larger, one expects μ_t to be more often closer to either 0 or 1 (the possible values of v_t) due to the faster learning by the dealer; and when μ_t is closer to 0 or 1, the bid-ask spread is closer to 0. Thus, it is not obvious ex ante how the average bid-ask spread should depend on ρ. To settle this issue, we need to determine how μ_t evolves over time and how often it reaches certain values. The concept is that of the stationary density.

To formally determine the stationary density, we describe the evolution of the system as a Markov process. The state is given by the pair $(v_t, \mu_t) \in \{0,1\} \times (0,1)$. The value v_t evolves by switching with probability ν from one value to the other. The mean μ_t evolves according to the formulas (IA.3): if $v_t = 1$, with probability $\rho + \frac{1-\rho}{2} = \frac{1+\rho}{2}$ it becomes $\mu_{t+1,\mathrm{B}}$ and with probability $\frac{1-\rho}{2}$ it becomes $\mu_{t+1,\mathrm{S}}$; if $v_t = 0$, with probability $\frac{1-\rho}{2}$ it becomes $\mu_{t+1,\mathrm{B}}$ and with probability $\frac{1+\rho}{2}$ it becomes $\mu_{t+1,\mathrm{B}}$.

The stationary density is a triple $(\theta, \lambda_0, \lambda_1)$, where $\theta \in (0,1)$ is a number, and λ_0, λ_1 are densities on $(0, 1)$:

$$\theta = \mathsf{P}(v_t = 1), \qquad \lambda_0(x) = \mathsf{P}(\mu_t = x | v_t = 0), \qquad \lambda_1(x) = \mathsf{P}(\mu_t = x | v_t = 1). \qquad \text{(IA.5)}$$

We show how to compute the stationary density. Let $\varepsilon \in \{+1, -1\}$. Define the functions m_+ and m_- for $x \in (0, 1)$ by:

$$m_\varepsilon(x) = \frac{\nu \frac{1-\varepsilon\rho}{2}(1-x) + (1-\nu)\frac{1+\varepsilon\rho}{2}\, x}{\frac{1-\varepsilon\rho}{2} + \varepsilon\rho x}, \quad \varepsilon \in \{-1, +1\}, \qquad \text{(IA.6)}$$

where for simplicity we denote $m_+ = m_{+1}$ and $m_- = m_{-1}$. Lemma IA.1 summarizes the properties of m_ε.

Lemma IA.1. *The functions m_+ and m_- are increasing and bijective between $(0, 1)$ and $(\nu, 1-\nu)$, and satisfy $m_+(x) > m_-(x)$ for $x \in (0, 1)$. Each m_ε has a unique fixed point $x_\varepsilon \in (0, 1)$, with:*

$$x_+ = \frac{\rho - \nu + \sqrt{\rho^2(1-2\nu) + \nu^2}}{2\rho}, \qquad x_- = \frac{\rho + \nu - \sqrt{\rho^2(1-2\nu) + \nu^2}}{2\rho}. \qquad \text{(IA.7)}$$

Moreover, $x_+ + x_- = 1$, and the difference $x_+ - x_- = (1-2\nu)/\big((1-2\nu+\frac{\nu^2}{\rho^2})^{1/2} + \frac{\nu}{\rho}\big)$ is positive and increasing in ρ. For any $x \in (0, 1)$, the sequence defined by $x_0 = x$ and $x_{k+1} = m_\varepsilon(x_k)$ converges to x_ε. The inverse functions n_+ and n_- and their derivatives are, respectively,

$$n_\varepsilon(y) = \frac{(1-\varepsilon\rho)(y-\nu)}{1-2\nu+\varepsilon\rho(1-2y)}, \qquad n_\varepsilon'(y) = \frac{(1-\rho^2)(1-2\nu)}{\big(1-2\nu+\varepsilon\rho(1-2y)\big)^2}. \qquad \text{(IA.8)}$$

The inverse functions n_ε are increasing and bijective between $(\nu, 1-\nu)$ and $(0, 1)$. Each of the functions m_ε or n_ε takes the interval $[x_-, x_+]$ into itself.

Proof. See Section 2.3 of this Internet Appendix. □

To understand why the image of m_ε is in $(\nu, 1-\nu)$, we provide intuition why $m_\varepsilon(x) > \nu$ for x very close to 0. First, note that Proposition IA.1 implies that the public mean evolves according to $\mu_{t+1} = m_\varepsilon(\mu_t)$, with $\varepsilon = +1$ after a buy order or $\varepsilon = -1$ after a sell order. Thus, even if μ_t is very close to 0, with probability ν the fundamental value at $t+1$ switches from 0 to 1, hence we expect μ_{t+1} to be larger than ν.

The stationary density essentially describes the long run distribution of μ_t, and thus we are interested in the properties of sequences of the form $m_{\varepsilon_1}\big(m_{\varepsilon_2}\big(\cdots m_{\varepsilon_t}(\mu_0)\cdots\big)\big)$ for very large t. But the image of m_ε is $(\nu, 1-\nu)$, hence the support of the stationary densities λ_0 and λ_1 must be included in $(\nu, 1-\nu)$. As $m_+ > m_-$ on $(0,1)$, the support of λ_0 and λ_1 must be also included in $\big(m_-(\nu), m_+(1-\nu)\big)$. Repeating this argument after a sufficiently large sequence of m_ε, since m_ε has a unique point x_ε it follows that the support of λ_0 and λ_1 must be included in $[x_-, x_+]$. Proposition IA.2 shows that this is indeed the case.

Proposition IA.2. *Let* $\theta = \frac{1}{2}$, *and let* λ_0 *and* λ_1 *be two two functions defined on* $[x_-, x_+]$ *and extended by 0 elsewhere on* $(0,1)$ *that satisfy the system:*

$$\begin{aligned}
\lambda_0(y) \;&=\; (1-\nu)\,\frac{1-\rho}{2}\,\lambda_0\big(n_+(y)\big)\,n'_+(y) + (1-\nu)\,\frac{1+\rho}{2}\,\lambda_0\big(n_-(y)\big)\,n'_-(y) \\
&\quad + \nu\,\frac{1+\rho}{2}\,\lambda_1\big(n_+(y)\big)\,n'_+(y) + \nu\,\frac{1-\rho}{2}\,\lambda_1\big(n_-(y)\big)\,n'_-(y), \\
\lambda_1(y) \;&=\; \nu\,\frac{1-\rho}{2}\,\lambda_0\big(n_+(y)\big)\,n'_+(y) + \nu\,\frac{1+\rho}{2}\,\lambda_0\big(n_-(y)\big)\,n'_-(y) \\
&\quad + (1-\nu)\,\frac{1+\rho}{2}\,\lambda_1\big(n_+(y)\big)\,n'_-(y) + (1-\nu)\,\frac{1-\rho}{2}\,\lambda_1\big(n_-(y)\big)\,n'_-(y).
\end{aligned} \tag{IA.9}$$

Then, $(\theta, \lambda_0, \lambda_1)$ *constitute a stationary density of the model.*

Proof. See Section 2.3 of this Internet Appendix. □

Proposition IA.2 shows that the existence of the stationary density reduces to a functional system of equations that can be solved numerically. This can be done by starting with arbitrary densities λ_0 and λ_1, and iterating the equations in (IA.9) until convergence is achieved. Because of symmetry, we expect that $\lambda_1(y) = \lambda_0(1-y)$. It turns out that the cumulative density $\Lambda_i(y) = \int_0^y \lambda_i(u)\mathrm{d}u$, $i = 1, 2$, satisfies a simpler equation, which is the same as (IA.9) but with no derivative of n_ε in the formula (see equation (IA.19 below). Thus, we work directly to obtain Λ_0, and then compute λ_0 as its numerical derivative.[6]

[6] To obtain better numerical results, we smooth the resulting cumulative density Λ_0 by using a moving average with parameter $N/10$, where N is the number of division points (we choose $N = 10^6$). The results are qualitatively similar when we do not smooth the results, but then the density λ_0 appears quite discontinuous, especially since for large values of ρ the derivative of λ_0 at x_+ and x_- approaches infinity.

Figure IA.1: bid-ask Spread with Switching Value.
This figure shows the average bid-ask spread, computed using the stationary density, against the fraction of informed trading (ρ), for two values of the switching probability parameter $\nu \in \{0.01, 0.02\}$.

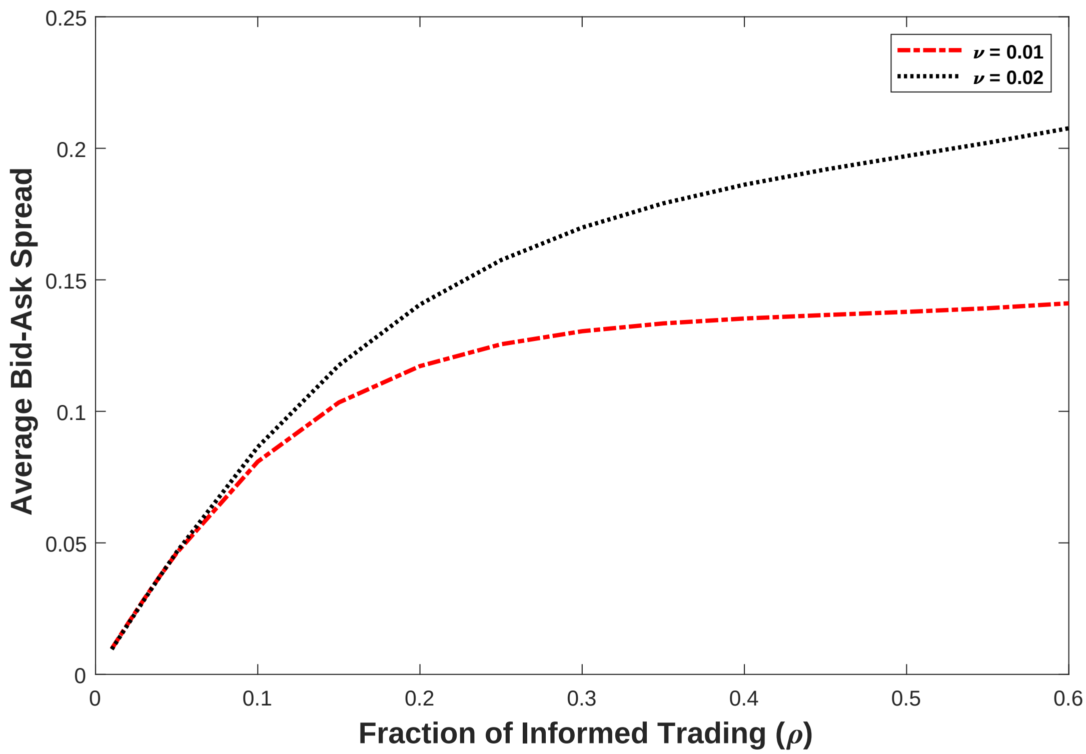

Figure IA.1 computes the average bid-ask spread as a function of the fraction of informed trading ρ. For each value of ρ, we compute the stationary density using the numerical method described above, and then compute the average bid-ask spread using the weights given by the stationary density: $s_{\text{ave}} = \int_0^1 s(x)\lambda_0(x)\mathrm{d}x$ (by symmetry, the same result is produced if we use λ_1 instead). We do not use values of ρ above 0.6 as the numerical procedure does not work well in that case (see Footnote 6).

As values for the switching parameter we choose $\nu \in \{0.01, 0.02\}$. We are interested in small values of ν relative to ρ for two reasons. First, it is more realistic to assume that the value does not change very frequently. Second, one of the goals of this exercise is to compare the effect of informed trading in the switching-value model and compare it with the diffusing-value model. But when ν is large (relative to ρ), the main uncertainty about the fundamental value in the switching-value model arises from the value frequently switching between 0 and 1, and not so much from the action of informed traders. In particular, in the diffusing-value model the public volatility depends strongly on ρ, while this dependence is small in the switching-value model when ν is large.

By inspecting Figure IA.1 we see that the average bid-ask spread increases with ρ, but also that the slope becomes less steep for larger values of ρ. The reason is that the dynamic efficiency effect (which in the diffusing-value model exactly cancels the adverse selection effect) here is dampened by the constraint that the value be only 0 or 1. Indeed, when ρ is low we normally would expect very imprecise knowledge of the dealer, but this imprecision is not allowed to go above 1. Nevertheless, when ρ is large, the dynamic efficiency effect becomes stronger, and as a result the dependence of the average bid-ask spread on ρ is weaker, and the equilibrium approaches the one in the diffusing-value model where the average bid-ask spread is independent of ρ.

2.3 Proofs

Proof of Proposition IA.1. The transition density is binomial: for all $v \in \{0, 1\}$, we have:

$$f_t(w, v) \;=\; (1-\nu)\mathbf{1}_{w=v} + \nu\mathbf{1}_{w\neq v}. \tag{IA.10}$$

Next, we describe the density $g_t(a, v) = \mathsf{P}(a_t = a \mid v_t = v)$ which is used in equation (IA.1).[7] Suppose that $A_t, B_t \in (0, 1)$. Then, if $v \in \{0, 1\}$ the condition $v > A_t$ is equivalent to $v = 1$, and the condition $v < B_t$ is equivalent to $v = 0$. We compute:

$$g_t(\mathrm{B}, v) \;=\; \rho\mathbf{1}_{v=1} + (1-\rho)\tfrac{1}{2}, \qquad g_t(\mathrm{S}, v) \;=\; \rho\mathbf{1}_{v=0} + (1-\rho)\tfrac{1}{2}. \tag{IA.11}$$

[7] Technically, g_t should also depend on the public density ϕ_t, which in the current setup is summarized by its mean μ_t. But as we see in (IA.11) below, $g_t(a, v)$ does not depend on μ_t.

Suppose the public density $\phi_t(v)$ is binomial with coefficient μ_t. As in equation (IA.1), the posterior density of v_t conditional on the order $\mathcal{O}_t \in \{\mathrm{B}, \mathrm{S}\}$ satisfies:

$$\begin{aligned}
\psi_t(v|\mathrm{B}) &= \frac{\left(\rho \mathbf{1}_{v>A_t} + \frac{1-\rho}{2}\right) \cdot \phi_t(v)}{\int_v \left(\rho \mathbf{1}_{v>A_t} + \frac{1-\rho}{2}\right) \cdot \phi_t(v)} = \frac{\left(\rho \mathbf{1}_{v=1} + \frac{1-\rho}{2}\right) \cdot \left((1-\mu_t)\mathbf{1}_{v=0} + \mu_t \mathbf{1}_{v=1}\right)}{\frac{1-\rho}{2}(1-\mu_t) + \frac{1+\rho}{2}\,\mu_t}, \\
\psi_t(v, \mathrm{S}) &= \frac{\left(\rho \mathbf{1}_{v<B_t} + \frac{1-\rho}{2}\right) \cdot \phi_t(v)}{\int_v \left(\rho \mathbf{1}_{v<B_t} + \frac{1-\rho}{2}\right) \cdot \phi_t(v)} = \frac{\left(\rho \mathbf{1}_{v=0} + \frac{1-\rho}{2}\right) \cdot \left((1-\mu_t)\mathbf{1}_{v=0} + \mu_t \mathbf{1}_{v=1}\right)}{\frac{1+\rho}{2}(1-\mu_t) + \frac{1-\rho}{2}\,\mu_t}.
\end{aligned} \tag{IA.12}$$

We now compute the posterior public density $\phi_{t+1}(w|\mathcal{O}_t) = \int_v f_t(w, v) \cdot \psi_t(v|\mathcal{O}_t)$:

$$\begin{aligned}
\phi_{t+1}(w|\mathrm{B}) &= \sum_{v\in\{0,1\}} \left((1-\nu)\mathbf{1}_{w=v} + \nu \mathbf{1}_{w\neq v}\right) \cdot \frac{\left(\rho \mathbf{1}_{v=1} + \frac{1-\rho}{2}\right) \cdot \left((1-\mu_t)\mathbf{1}_{v=0} + \mu_t \mathbf{1}_{v=1}\right)}{\frac{1-\rho}{2} + \rho\mu_t}, \\
\phi_{t+1}(w|\mathrm{S}) &= \sum_{v\in\{0,1\}} \left((1-\nu)\mathbf{1}_{w=v} + \nu \mathbf{1}_{w\neq v}\right) \cdot \frac{\left(\rho \mathbf{1}_{v=0} + \frac{1-\rho}{2}\right) \cdot \left((1-\mu_t)\mathbf{1}_{v=0} + \mu_t \mathbf{1}_{v=1}\right)}{\frac{1+\rho}{2} - \rho\mu_t}.
\end{aligned} \tag{IA.13}$$

As $\phi_{t+1}(w|\mathrm{B})$ is a density, we have $\phi_{t+1}(w=1|\mathrm{B}) + \phi_{t+1}(w=0|\mathrm{B}) = 1$, and therefore we only need to compute:

$$\phi_{t+1}(w=1|\mathrm{B}) = \nu \cdot \frac{\frac{1-\rho}{2}(1-\mu_t)}{\frac{1-\rho}{2} + \rho\mu_t} + (1-\nu) \cdot \frac{\frac{1+\rho}{2}\,\mu_t}{\frac{1-\rho}{2} + \rho\mu_t}. \tag{IA.14}$$

Similarly, for $\phi_{t+1}(w|\mathrm{S})$ we only need to compute:

$$\phi_{t+1}(w=1|\mathrm{S}) = \nu \cdot \frac{\frac{1+\rho}{2}(1-\mu_t)}{\frac{1+\rho}{2} - \rho\mu_t} + (1-\nu) \cdot \frac{\frac{1-\rho}{2}\,\mu_t}{\frac{1+\rho}{2} - \rho\mu_t}. \tag{IA.15}$$

The density $\phi_{t+1}(\cdot|\mathcal{O}_t)$ is binomial, hence its mean $\mu_{t+1,\mathcal{O}_t}$ is simply $\phi_{t+1}(w=1|\mathcal{O}_t)$. Thus, if we rewrite the formulas for $\phi_{t+1}(w=1|\mathrm{B})$ and $\phi_{t+1}(w=1|\mathrm{S})$, we obtain:

$$\mu_{t+1,\mathrm{B}} = \frac{\nu\frac{1-\rho}{2}(1-\mu_t) + (1-\nu)\frac{1+\rho}{2}\mu_t}{\frac{1-\rho}{2} + \rho\mu_t}, \quad \mu_{t+1,\mathrm{S}} = \frac{\nu\frac{1+\rho}{2}(1-\mu_t) + (1-\nu)\frac{1-\rho}{2}\mu_t}{\frac{1+\rho}{2} - \rho\mu_t}. \tag{IA.16}$$

The dealer's pricing conditions are that $A_t = \mu_{t+1,\mathrm{B}}$ and $B_t = \mu_{t+1,\mathrm{S}}$, which proves the equations in (IA.3). Using these equations, one verifies directly that the difference $s_t = A_t - B_t$ satisfies (IA.4). □

Proof of Lemma IA.1. The derivative of m_ε is $m'_\varepsilon(x) = \frac{(1-\rho^2)(1-2\nu)}{(1-\varepsilon\rho+2\varepsilon\rho x)^2} > 0$, and one checks that $m_\varepsilon(0) = \nu$ and $m_\varepsilon(1) = 1-\nu$. Hence, m_ε is increasing and bijective between $(0,1)$ and $(\nu, 1-\nu)$. We compute the difference $m_+(x) - m_-(x) = \rho(1-2\nu)\frac{1-(1-2x)^2}{1-\rho^2(1-2x)^2} > 0$, hence $m_+(x) > m_-(x)$ for all $x \in (0,1)$. Also, $x_+ - x_- = \frac{1-2\nu}{(1-2\nu+\frac{\nu^2}{\rho^2})^{1/2}+\frac{\nu}{\rho}}$, which is increasing in ρ.

We compute $m_\varepsilon(x) - x = \frac{\nu(1-\varepsilon\rho)+2(\varepsilon\rho-\nu)x-2\varepsilon\rho x^2}{1-\varepsilon\rho+2\varepsilon\rho x}$. Hence, $m_+(x) = x$ has two solutions $x = \frac{\rho-\nu\pm\sqrt{\Delta}}{2\rho}$, with $\Delta = \rho^2(1-2\nu)+\nu^2$. Similarly, $m_-(x) = x$ has two solutions: $x = \frac{\rho+\nu\pm\sqrt{\Delta}}{2\rho}$. We see that $x_+ + x_- = \frac{\rho-\nu+\sqrt{\Delta}}{2\rho} + \frac{\rho+\nu-\sqrt{\Delta}}{2\rho} = 1$, and one verifies that $x_+, x_- > 0$, $\frac{\rho-\nu-\sqrt{\Delta}}{2\rho} < 0$, and $\frac{\rho+\nu+\sqrt{\Delta}}{2\rho} > 1$. This proves that x_ε is the unique fixed point of m_ε in $(0,1)$. Also, as $m_\varepsilon(0) = \nu > 0$, we have that $m_\varepsilon(x) - x$ is positive on $(0, x_\varepsilon)$ and negative on $(x_\varepsilon, 1)$.

Consider the sequence given by $x_{k+1} = m_\varepsilon(x_k)$ for some value $x_0 > x_\varepsilon$. As m_ε is increasing and x_ε is a fixed point of m_ε, we have $x_1 = m_\varepsilon(x_0) > x_\varepsilon$, and by induction $x_k > x_\varepsilon$ for all n. As $m_\varepsilon(x) - x$ is negative on $(x_\varepsilon, 1)$, it follows that $x_{k+1} - x_k < 0$ for all n, that is, the sequence x_k is decreasing. Thus, x_k is convergent and its limit is x_ε. A similar argument works for the case when $x_0 < x_\varepsilon$, and it follows that x_k converges to x_ε for all initial values $x_0 \in (0,1)$.

The computation of the inverse function n_ε and its derivative is straightforward, as is the fact that n_ε is increasing and bijective between $(\nu, 1-\nu)$ and $(0,1)$. Finally, we have $x_+ = m_+(x_+) > m_+(x_-) > m_-(x_-) = x_-$ and $x_+ = m_+(x_+) > m_-(x_+) > m_-(x_-) = x_-$, which implies that $m_\varepsilon\big([x_-, x_+]\big) \subseteq [x_-, x_+]$, and the same argument works for the inverse function n_ε. $\square$

Proof of Proposition IA.2. The stationary density is determined by one number: $\theta = \mathsf{P}(v_t = 1)$, and two densities: $\lambda_0(x) = \mathsf{P}(\mu_t = x | v_t = 0)$ and $\lambda_1(x) = \mathsf{P}(\mu_t = x | v_t = 1)$. To find θ, we impose the stationarity condition $\theta = \mathsf{P}(v_{t+1} = 1)$. Recall from (IA.10) that the transition density from v_t to v_{t+1} is $f_t(w, v) = \mathsf{P}(v_{t+1} = w | v_t = v) = (1-\nu)\mathbf{1}_{w=v} + \nu\mathbf{1}_{w\neq v}$. We compute $\mathsf{P}(v_{t+1} = 1) = \sum_{v\in\{0,1\}} \mathsf{P}(v_{t+1} = 1 | v_t = v)\,\mathsf{P}(v_t = v) = \nu(1-\theta) + (1-\nu)\theta = \theta + \nu(1-2\theta)$. As this last term is equal to θ, the unique solution is $\theta = \frac{1}{2}$. This equality also implies that $\lambda_0(x)\mathrm{d}x = \mathsf{P}(\mu_t = x, v_t = 0)/P(v_t = 0) = 2\,\mathsf{P}(v_t = 0, \mu_t = x)$, hence $\mathsf{P}(v_t = 0, \mu_t = x) = \frac{1}{2}\lambda_0(x)\mathrm{d}x$. Similarly, $\mathsf{P}(v_t = 1, \mu_t = x) = \frac{1}{2}\lambda_1(x)\mathrm{d}x$.

Recall that the state variable is the pair $K_t = (v_t, \mu_t)$. The stationarity condition requires that $\mathsf{P}(K_{t+1}=k') = \int_k \mathsf{P}(K_{t+1}=k'|K_t=k)\,\mathsf{P}(K_t=k)$, therefore:

$$\begin{aligned}
\tfrac{1}{2}\lambda_0(y)\mathrm{d}y &= \int_{\substack{v\in\{0,1\}\\ x\in(0,1)}} \mathsf{P}\big(v_{t+1}=0,\mu_{t+1}=y \mid v_t=v,\mu_t=x\big)\ \mathsf{P}(v_t=v,\mu_t=x) \\
&= \int_{x\in(0,1)} \begin{array}{l} \mathsf{P}\big(v_{t+1}=0,\mu_{t+1}=y \mid v_t=0,\mu_t=x\big)\ \tfrac{1}{2}\lambda_0(x)\mathrm{d}x \\ +\mathsf{P}\big(v_{t+1}=0,\mu_{t+1}=y \mid v_t=1,\mu_t=x\big)\ \tfrac{1}{2}\lambda_1(x)\mathrm{d}x \end{array}.
\end{aligned} \tag{IA.17}$$

From (IA.11), the density $g_t(\mathcal{O}_t, v, \mu_t) = \mathsf{P}(\mathcal{O}_t \mid v_t = v, \mu_t = x)$ does not depend on μ_t and satisfies $g_t(\mathrm{B}, v) = \rho\mathbf{1}_{v=1} + (1-\rho)\frac{1}{2}$ and $g_t(\mathrm{S}, v) = \rho\mathbf{1}_{v=0} + (1-\rho)\frac{1}{2}$. In general, $\mathsf{P}\big(v_{t+1} = w, \mu_{t+1} = y \mid v_t = v, \mu_t = x\big) = \int_{\mathcal{O}_t\in\{\mathrm{B},\mathrm{S}\}} \mathsf{P}\big(v_{t+1} = w, \mu_{t+1} = y \mid \mathcal{O}_t, v_t = v, \mu_t = x\big) g_t(\mathcal{O}_t, v)$. From (IA.16), we have $\mu_{t+1,\mathrm{B}} = m_+(\mu_t)$ and $\mu_{t+1,\mathrm{S}} = m_-(\mu_t)$, where m_+ and m_- are defined in (IA.6). Recall that n_+ and n_- are the inverses of m_+ and m_-, respectively. For $y \in (0,1)$ we compute:

$$\begin{aligned}
\lambda_0(y)\mathrm{d}y &= \int_{\substack{x\in(0,1)\\ \mathcal{O}_t\in\{\mathrm{B},\mathrm{S}\}}} \begin{array}{l} \mathsf{P}\big(v_{t+1}=0, m_{\mathcal{O}_t}(x)=y \mid \mathcal{O}_t, v_t=0,\mu_t=x\big)\ g_t(\mathcal{O}_t, v_t=0)\ \lambda_0(x)\mathrm{d}x \\ +\mathsf{P}\big(v_{t+1}=0, m_{\mathcal{O}_t}(x)=y \mid \mathcal{O}_t, v_t=1,\mu_t=x\big)\ g_t(\mathcal{O}_t, v_t=1)\ \lambda_1(x)\mathrm{d}x \end{array} \\
&= \int_{x\in(0,1)} \mathsf{P}\big(v_{t+1}=0 \mid v_t=0\big)\ \mathsf{P}\big(m_+(x)=y\big)\ g_t(\mathrm{B}, v_t=0)\ \lambda_0(x)\mathrm{d}x \\
&\quad + \int_{x\in(0,1)} \mathsf{P}\big(v_{t+1}=0 \mid v_t=0\big)\ \mathsf{P}\big(m_-(x)=y\big)\ g_t(\mathrm{S}, v_t=0)\ \lambda_0(x)\mathrm{d}x \\
&\quad + \int_{x\in(0,1)} \mathsf{P}\big(v_{t+1}=0 \mid v_t=1\big)\ \mathsf{P}\big(m_+(x)=y\big)\ g_t(\mathrm{B}, v_t=1)\ \lambda_1(x)\mathrm{d}x \\
&\quad + \int_{x\in(0,1)} \mathsf{P}\big(v_{t+1}=0 \mid v_t=1\big)\ \mathsf{P}\big(m_-(x)=y\big)\ g_t(\mathrm{S}, v_t=1)\ \lambda_1(x)\mathrm{d}x \\
&= (1-\nu)\,\frac{1-\rho}{2}\,\lambda_0\big(n_+(y)\big)\,\mathrm{d}\big(n_+(y)\big) + (1-\nu)\,\frac{1+\rho}{2}\,\lambda_0\big(n_-(y)\big)\,\mathrm{d}\big(n_-(y)\big) \\
&\quad + \nu\,\frac{1+\rho}{2}\,\lambda_1\big(n_+(y))\big)\,\mathrm{d}\big(n_+(y)\big) + \nu\,\frac{1-\rho}{2}\,\lambda_1\big(n_-(y))\big)\,\mathrm{d}\big(n_-(y)\big),
\end{aligned} \tag{IA.18}$$

where by convention $\lambda_{0,1}\big(n_\varepsilon(y)\big)$ is set to 0 when y is outside of $(\nu, 1-\nu)$, the definition interval for n_ε. A similar computation produces the recursive equation for $\lambda_1(y)$. Putting together the equations for λ_0 and λ_1, we get (IA.9). We also obtain that $\lambda_0(y) = \lambda_1(y) = 0$ when y is outside of $(\nu, 1-\nu)$.

It remains to prove that the support of λ_0 and λ_1 is included in the interval $[x_-, x_+]$, where x_ε is the unique fixed point of m_ε in $(0,1)$. Consider the sequence a_k defined by $a_0 = 0$ and $a_{k+1} = m_-(a_k)$, and the sequence b_k defined by $b_0 = 1$ and $b_{k+1} = m_+(b_k)$. From the proof of Lemma IA.1, we know that the a_k is increasing and converges to x_-, and b_k is decreasing and converges to x_+. Note that $a_1 = \nu$ and $b_1 = 1 - \nu$. We have proved above that $\lambda_i(y) = 0$ when y is outside of (a_1, b_1) and $i \in \{0,1\}$. Suppose we have already proved that $\lambda_i(y) = 0$ when y is outside of (a_k, b_k) for some integer $k \geq 1$. In the induction step, we show that $\lambda_i(y) = 0$ when $y \in (a_k, a_{k+1}]$ (and a similar argument works for $y \in [b_{k+1}, b_k)$). We have $n_-(y) \in (n_-(a_k), n_-(a_{k+1})] = (a_{k-1}, a_k]$, hence $n_-(y) \leq a_k$. But $n_+(y) < n_-(y)$, hence $n_+(y) \leq a_k$ as well.[8] As $n_\varepsilon(y)$ lie outside (a_k, b_k) for $\varepsilon \in \{-1, +1\}$, we have $\lambda_i\big(n_\varepsilon(y)\big) = 0$ for y outside of (a_{k+1}, b_{k+1}). Thus, equation (IA.18) shows that $\lambda_i(y) = 0$ for y outside of (a_{k+1}, b_{k+1}), and the induction step is finished. Since the intersection of the intervals (a_k, b_k) is $[x_-, x_+]$, it follows that $\lambda_i(y) = 0$ for y outside $[x_-, x_+]$.

Note that by integrating equation (IA.18) from 0 to y, we obtain the following equation for the cumulative stationary density $\Lambda_i(y) = \int_0^y \lambda_i(u)\mathrm{d}u$, $i \in \{1,2\}$:

$$\begin{aligned} \Lambda_0(y) \;&=\; (1-\nu)\,\frac{1-\rho}{2}\,\Lambda_0\big(n_+(y)\big) + (1-\nu)\,\frac{1+\rho}{2}\,\Lambda_0\big(n_-(y)\big) \\ &\quad + \nu\,\frac{1+\rho}{2}\,\Lambda_1\big(n_+(y)\big) + \nu\,\frac{1-\rho}{2}\,\Lambda_1\big(n_-(y)\big), \end{aligned} \tag{IA.19}$$

and a similar equation for Λ_1. □

[8] As $m_+(x) > m_-(x)$ for all $x \in (0,1)$, the inverse functions satisfy the opposite inequality: $n_+(y) < n_-(y)$ for all $y \in (\nu, 1-\nu)$.